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Scalar Wave Diffraction by a Perfectly Soft Infinitely Thin Circular Ring

Fatih DİKMEN, Ertuğrul KARAÇUHA
Gebze Institute of Technology, Department of Electronics Engineering, Gebze, Kocaeli-TURKEY
Yury Alexanderovich TUCHKIN
Gebze Institute of Technology, Department of Electronics Engineering, Gebze, Kocaeli-TURKEY
Institute of Radiophysics and Electronics, The Ukrainian National Academy of Sciences, 12, Proskury st., Kharkov, 310085, The UKRAINE

Abstract

A new strong mathematically rigorous and numerically effective method for solving the boundary value problem of scalar (for example acoustic) wave diffraction by a perfectly soft (Dirichlet boundary condition) infinitely thin circular ring is proposed. The method is based on the combination of the Orthogonal Polynomials Approach, and on the ideas of the methods of analytical regularization. As a result of the suggested regularization procedure, the initial boundary value problem is equivalently reduced to the infinite system of the linear algebraic equations of the second kind, i.e., to an equation of the type $(I+H)x=b$ *in the space of* ℓ_2 *square summable sequences. This equation can be solved numerically by means of the truncation method with, in principle, any required accuracy.*

1. Introduction

Modern diffraction theory and its applied branches, such as the design of antenna systems, give special importance to the investigation of obstacles such as the structure that the paper will consider within. The known methods do not always provide the results with accuracy and reliability, fitting the requirements of practical goals. The objective of this paper is to fill this gap for the flat circular ring structure, which belongs to a class of geometrical structures finding a wide range of applications, in flat antenna structures and satellite antenna systems [1-4].

This paper presents a new mathematically rigorous and numerically efficient method for solving the problem of scalar wave diffraction by a perfectly soft infinitely thin circular ring (Fig. 1) in the case when scalar waves satisfy the Helmholtz equation.

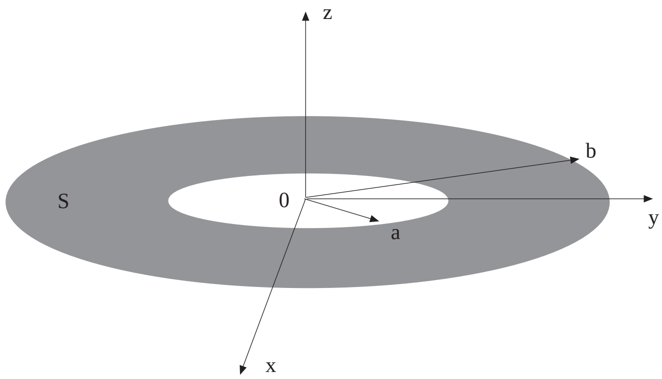

Figure 1. The geometry of the considered problem

Over the past three decades, the attention of many authors has been directed towards diffraction problems for open structures, formed by unclosed thin screens, including the considered ring, which can be considered one of the canonical structures of modern diffraction theory. This means that different universal or specialized methods of diffraction theory have been applied to this structure for comparison of their relative efficiency and accuracy.

All these methods can be separated into two major groups, namely, direct methods, such as Moment Method, Finite Difference Method, and special, so-called numerical-analytical, methods based on the variables separation method with correspondent integral transformations, etc.

The methods of the first group reduce, as a rule, the initial boundary value problem to the functional equation or algebraic system of the first kind. The main drawback of such systems is well known. The corresponding numerical process is unstable for rather large systems, giving strong limitations on the accuracy and possibility of the method [10-11].

Methods of the second group are free, as a rule, of such drawbacks and produce the functional equation of the second kind. But such methods often look like very specialized ones just for the screens structure under consideration (see [10-11]).

The main goal of our work was the development of a method (or class of methods) which is rather universal, like the methods of the first group, but produces the final algebraic system with properties that are characteristic of those of the second group.

The suggested method is based on the Orthogonal Polynomials Method, going back to the G. Ya. Popov's papers (see, for example [5,6]), and on the ideas of the Analytical Regularization Method [7-10]. It has to be pointed out that the technique of analytical regularization in the form of the so-called semi-inversion procedure was applied at first in the diffraction theory in the paper of Z. S. Agranovich, V .A. Marchenko and V.P. Shestopulov [11].

During the last three decades, the Orthogonal Polynomials Approach and Analytical Regularization Method have been subjects under intensive consideration, including their applications in the theory of elasticity and the diffraction theory (see, for example, monographs [12-14] and [15-17] respectively and references there in). Various methods that are very similar to the Orthogonal Polynomials Approach have been used by many authors (usually without referring to G. Ya. Popov or his followers) – see, for example, [18]. Most of the ideas of these investigations can be described as classical and well verified, powerful tools in mathematical physics and its numerical implementations.

The method proposed herein is devoted to the construction of a solving procedure for the above-mentioned canonical problem, but it is based on the combination of the direct methods with further analytical regularization of the correspondent infinite algebraic system. This last step is the characteristic feature of numerical-analytical methods, but the technique that has been used is a rather universal one, and from our description of the method it clearly follows that this method can easily be generalized for more complicated structures. For example, the generalization of the method can be used without any losses of efficiency and accuracy for the case of a few rings [19] or of arbitrary shaped axially symmetrical toroidal screens [20], or for electromagnetic wave diffraction by perfectly conductive ring [21].

The combination of direct and numerical-analytical methods used in our approach can be briefly described as follows. We start with the Dirichlet diffraction boundary value problem for the Helmholtz equation. By means of the Green's third formula we obtain the integral representation for scattering field. The representation's integral over the obstacle surface S consists of the product of the Green's function of three-dimensional free space and unknown function, which must be the jump-function of the limiting values of the scattering field normal derivatives on surface S. Substitution of this integral representation into the boundary condition gives the relevant integral equation of the first kind.

Fourier Transformation reduces this two-dimensional integral equation over surface S to the infinite set of an independent (non-interacting) one-dimensional integral equation of the first kind, including logarithmically singular kernels. The Orthogonal Polynomials Method enables us to represent the unknown functions each of these one-dimensional integral equations as infinite Fourier-Chebyshev series, involving Chebyshev polynomials of the first kind, with unknown coefficients. Using the orthogonal property and completeness of the system of Chebyshev polynomials, one reduces every one-dimensional integral equation to the infinite algebraic system of the first kind of the type

$$Ax = b, \tag{1}$$

which is the final stage of the conventional Orthogonal Polynomials Method.

The constructed analytical regularization procedure for this algebraic system has the following form. First we define the new vector-column of unknowns $y = R^{-1}x$ by means of some linear invertible operator R, and second, act on the both sides of equation (1) by some linear invertible operator L. As a result, equation (1) is equivalently transformed into another one of the form

$$LARy = Lb. \tag{2}$$

The most important thing is the possibility of constructing the double-sided regularizator (L, R) as a pair of diagonal matrices and, this product has the form

$$LAR = I + H, \tag{3}$$

where H is the compact operator in the Hilbert ℓ_2 space of square summable sequences and I is the identical operator. This means that every above mentioned one-dimensional integral equation has been equivalently reduced to the equation of the second kind of the type

$$(I + H)y = c, \quad y, c \in \ell_2 \tag{4}$$

with compact in ℓ_2 operator H.

The characteristic properties of such equations are:

1) the solutions y^N of truncated systems

$$(I + H^N)y = c^N \tag{5}$$

tend to the solution y^∞ of infinite system (4):

$$||y^\infty - y^N|| \to 0 \quad \text{for} \quad N \to \infty, \tag{6}$$

where H^N and c^N are sets of finite dimensional operators and vectors-columns respectively and

$$||H - H^N|| \to 0,\ ||c - c^N|| \to 0 \ \text{ for } \ N \to \infty; \tag{7}$$

2) condition numbers v_N of truncated system:

$$v_N = ||I + H^N|| \cdot ||(I + H^N)^{-1}|| \tag{8}$$

are uniformly bounded and have the finite limit

$$v_N \leq \ const, v_N \to v_\infty = ||I + H|| \cdot ||(I + H)^{-1}||. \tag{9}$$

That is why the real numerical process of solving of these truncated system is stable relatively to the round off errors for any arbitrary large N. Consequently, our algorithm has a unique quality: the solution of initial diffraction problem can be obtained numerically with, in principle, any required accuracy.

2. The Diffraction Problem and Its Reduction to the Integral Equation of the First Kind

Let us consider an infinitely thin circular ring-shaped screen, i.e. surface S on $z = 0$ plane between two circles having inner and outer radii a and b, respectively (see Fig. 1), and suppose that scalar wave $u^i(p)$, which satisfies the homogeneous Helmholtz equation in a vicinity of contour S, is incident on the ring. It is necessary to find the scattering field $u^s(p)$ that satisfies the homogeneous Helmholtz equation:

$$(\Delta + k^2)u^s(p) = 0, \quad p \in R^3 \setminus S, \tag{10}$$

Sommerfeld radiation conditions:

$$\lim_{|p| \to +0} |p| \left(\frac{\partial u^s(p)}{\partial |p|} - iku^s(p) \right) = 0 \tag{11}$$

and the Dirichlet boundary condition, the exact form of which needs additional details, which are discussed below.

Because of the existence of infinitely thin edge of the ring, we could not use the classic posing of Dirichlet boundary problem (see [22]). That is why, at first, we define surface S as an open set and $\partial S^{(a)}$ as inner and $\partial(S)^{(b)}$ as outer contours, that form the boundary of the manifold S:

$$S = \{(\rho, \varphi, z) : \rho = (a, b), \varphi \in [-\pi, \pi], z = 0\} \tag{12}$$

$$\partial S^{(a)} = \{(\rho, \varphi, z), \rho = a, \varphi \in [-\pi, \pi], z = 0\} \tag{13a}$$

$$\partial S^{(b)} = \{(\rho, \varphi, z), \rho = b, \varphi \in [-\pi, \pi], z = 0\} \tag{13b}$$

where (ρ, φ, z) are the coordinates of the points in the corresponding cylindrical coordinate system (see Fig. 1).

Then grad $u^s(p)$ demands to have the following representation:

$$grad\ u^s(p) = [dist(p, \partial S^{(a)}) \cdot\ dist(p, \partial S^{(b)})]^{-1/2}\Phi(p),\ p \in R^3 \setminus \overline{S}, \tag{14}$$

where $\overline{S} = S \cup \partial S^{(+)} \cup \partial S^{(-)}$, and dist $(p, \partial S^{(\pm)})$ is distance from point p to the corresponding contour $\partial S^{(+)}$ or $\partial S^{(-)}$. As we suppose, the following limits exist:

$$u^{s(\pm)}(p) = \lim_{h\to+0} u^s(p \pm hn_p) \quad p \in \dot{S}; \tag{15}$$

$$\Phi^{(\pm)}(p) = \lim_{h\to+0} \Phi(p \pm hn_p) \quad p \in \overline{P}, \tag{16}$$

which are uniform ones respectively to all points on $\overline{S}$, and the limiting functions $u^{s(\pm)}(p), \Phi^{(\pm)}(p)\ p \in S$, belong to the Hölder class of functions ([22]). Here n_p is unit outward normal to surface $\overline{S}$ in the point $p \in \overline{P}$, i.e. $n_p = e_z$, where e_z is unit ort in z-direction. According to formula (10), we suppose, of course, that

$$u^s(p) \in C^2(R^3 \setminus S). \tag{17}$$

Now we are able to define the Dirichlet boundary condition as the following:

$$u^{s(+)}(p) + u^i(p) = u^{s(-)}(p) + u^i(p) = 0, \quad p \in S. \tag{18}$$

It is easy to understand that formulas (14)-(16) perform mathematical description of the Meixner conditions, well known in physics literature, [24, 25] near the thin edge.

Using the standard technique (see [22]), one can prove that the solution $u^s(q)$ of the posed diffraction boundary value problem is unique, if it exists.

Let us consider now the integral

$$W(q) = \int_S G(q,p)Z(p)ds_p, \quad q \in R^3 \setminus \overline{S}, \tag{19}$$

where $G(q,p)$ is Green's function of free space R^3 ([24]):

$$G(q,p) = -\frac{1}{4\pi}\frac{e^{ik|q-p|}}{|p-p|} \tag{20}$$

and $Z(p)$ is a function of the type

$$Z(p) = [dist(p\partial S^{(a)}) \cdot dist(p, \partial S^{(b)})]^{-1/2}h(p), \quad p \in S, \tag{21}$$

where $h(p)$ is a function, which belongs to Hölder class on $\overline{S}$.

Integral (19) can be considered as generalized potential of a single layer ([22]) and because of property (21) of function $Z(p)$, it can be proved that this potential has the following properties:

1) there exist limits uniformon $\overline{S}$ (see formulas (15),(16))

$$W^{(\pm)}(q) = \int_S G(q,p)Z(p)ds_p \quad q \in \overline{S} \tag{22}$$

$$\frac{\partial W^{(\pm)}(q)}{\partial n} = \int_S \frac{\partial G(q,p)}{\partial n_q} Z(p)ds_p \pm \frac{1}{2}Z(q), \quad q \in S; \tag{23}$$

2) $W^{(\pm)}(q)$ are differentiable functions on surface S and their surface gradient can be represented in the form (21) with corresponding vector-function $h(p)$ belonging to Hölder class on surface $\overline{S}$;

3) The integral in formula (23) exists in the sense of the Cauchy principal value integral ([22]) and it is a function of Hölder class on surface $\overline{S}$.

From formula (23) it evidently follows that

$$\frac{\partial W^{(+)(q)}}{\partial n} - \frac{\partial W^{(-)}(q)}{\partial n} = Z(p), \quad q \in S. \tag{24}$$

By means of Green's formula technique ([22]) it can be proved that the solution $u^s(q)$ of the considered diffraction problem (10)-(18), if it exists, has the following representation:

$$u^s(q) = \int_S G(q,p) \cdot \delta \frac{\partial u^s(p)}{\partial n} ds_p, \quad q \in R^3 \setminus \overline{S}, \tag{25}$$

where

$$\delta \frac{\partial u^s(p)}{\partial n} = \frac{\partial u^s(+)(p)}{\partial n} - \frac{\partial u^{s(-)}(p)}{\partial n}, \quad p \in S. \tag{26}$$

From formulas (14) and (16), it clearly follows that function $\delta \frac{\partial u^s}{\partial n}$ has a representation of form (21) and, consequently, the integral in (25) can be considered as generalized potential of a single layer of the kind (19), (21). Substitution of the right hand side of formula (25) into boundary conditions (18) gives, according to formula (22), the following integral relationship:

$$\int_S G(q,p)\delta \frac{\partial u^s(p)}{\partial n} ds_p = -u^i(q), \quad q \in S. \tag{27}$$

Now let us consider the function

$$Z(p) = \delta \frac{\partial u^s(p)}{\partial n} \tag{28}$$

as unknown function, and relationship (27) as integral equation

$$\int_S G(q,p)Z(p)ds_p = -u^i(q), \quad q \in S \tag{29}$$

with unknown function $Z(p)$, which has a representation of form (21).

It can easily be proved that integral (19) with density $Z(p)$ satisfying equation (29), is a solution of the considered diffraction problem (10)-(18).

Let us suppose now that equation (29) has two solutions, say $Z_1(p)$ and $Z_2(p)$, and will prove that $Z_1(p) = Z_2(p),\ p \in R$. Really, function $Z_0(p) = Z_1(p) - Z_2(p)$ is a solution of homogeneous equation (29). Let us consider integral $W_0(q)$, defined by formula (19) with $Z(p) = Z_0(p)$. Because $Z_0(p)$ is a solution of homogeneous equation (29), from formula (22) it follows that $W_0(q)$ is a solution of the homogeneous boundary value problem (10)-(18) (with $u^i(p) \equiv 0$). As we mentioned above, the solution of the boundary value problem is unique, if it exists. That is why $W_0(q) \equiv 0, q \in R^3$. From this fact and formula (24) immediately follows $Z_0(p) = 0, p \in S$. That is why the solution of integral equation (29) in class (21) is unique, if it exists.

Thus, the diffraction boundary value problem is equivalently reduced to the integral equation of the first kind of type (29).

3. Reduction of Two-Dimensional Integral Equation to The Infinite Set of One-Dimensional Ones

Let us rewrite equation (29) in the above mentioned cylindrical coordinates ρ, φ, z. For this goal, we define the following functions:

$$R(\rho_q, \rho_p; \varphi) = \{(\rho_q - \rho_p)^2 + 4\rho_q\rho_p \sin^2(\varphi/2)\}^{1/2}; \quad \varphi \in [-\pi, \pi]; \quad \rho_q, \rho_p \in [a, b] \tag{30}$$

$$\tilde{G}(\rho_q, \rho_p; \varphi) = -\frac{e^{ikR(\rho_q, \rho_p; \varphi)}}{4\pi R(\rho_q, \rho_p; \varphi)}; \tag{31}$$

$$\tilde{g}(\rho_q, \varphi_q) = -u^i(q), q = (\rho_q, \varphi_q, 0) \in S; \tag{32}$$

$$\tilde{Z}(\rho_p, \varphi_p) = Z(p), \quad p = (\rho_p, \varphi_p, z_p) \in S. \tag{33}$$

Because the distance $|q - p|$ between two points $q, p \in S,$ where and $q = (\rho_q, \varphi_q, z_q)$, in $p = (\rho_p, \varphi_p, z_p)$ the cylindrical coordinate system is:

$$|q - p| = R(\rho_q, \rho_p, \varphi_q - \varphi_p). \tag{34}$$

integral equation (29) can easily be rewritten as following:

$$\int_{-\pi}^{\pi}\int_{a}^{b} \tilde{Z}(\rho_p, \varphi_p).\tilde{G}(\rho_q, \rho_p; \varphi)\rho_p d\rho_p d\varphi_p = \tilde{g}(\rho_q, \varphi_q); a \le \rho_q \le b, f_q \in [-p, p] \tag{35}$$

with unknown function $\tilde{Z}(\rho_p, \varphi_p)$.

Functions $\tilde{G}(\rho_q, \rho_p; \varphi)$, $\tilde{Z}(\rho_p, \varphi_p)$ and $\tilde{g}(\rho_q, \varphi_p)$ can be represented as their Fourier series:

$$\tilde{G}(\rho_q, \rho_p; \varphi) = \sum_{m=-\infty}^{\infty} G_m(\rho_q, \rho_p) e^{im\varphi}, \quad \varphi \in [-\pi, \pi] \quad \rho_q, \rho_p \in [a, b]; \tag{36}$$

$$\tilde{Z}(\rho_p,\varphi)=\sum_{m=-\infty}^{\infty} Z_m(\rho_p)e^{im\varphi},\quad \varphi\in[-\pi,\pi]\quad \rho_p\in[a,b]; \tag{37}$$

$$\tilde{g}(\rho_q,\varphi)=\sum_{m=-\infty}^{\infty} g_m(\rho_q)e^{im\varphi},\quad \varphi\in[-\pi,\pi]\quad \rho_q\in[a,b], \tag{38}$$

where functions $G_m(\rho_q,\rho_p)$, $Z_m(\rho_p)$ and $g_m(\rho_q)$ are equal to

$$G_m(\rho_q,\rho_p)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde{G}(\rho_q,\rho_p;\varphi)e^{-im\varphi}d\varphi,\quad \rho_p,\rho_q\in[a,b]; \tag{39}$$

$$Z_m(\rho_p)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde{Z}(\rho_p,\varphi)e^{-im\varphi}d\varphi,\quad \rho_p\in[a,b]; \tag{40}$$

$$g_m(\rho_q)=\frac{1}{2\pi}\int_{2\pi}^{\pi}\tilde{g}(\rho_q,\varphi)e^{-im\varphi}d\varphi,\quad \rho_q\in[a,b] \tag{41}$$

The substitution of formulas (36)-(38) into formula (35) gives the equality of the form

$$\int_{-a}^{b}\int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty}Z_n(\rho_p)e^{in\varphi_p}\sum_{m=-\infty}^{\infty}G_m(\rho_q,\rho_p)e^{im(\varphi_q-\varphi_p)}d\varphi_p\rho_p d\rho_p=\sum_{m=-\infty}^{\infty}g_m(\rho_q)e^{im\varphi_q},$$
$$\rho_q\in[a,b],\quad \varphi_q\in[-\pi,\pi] \tag{42}$$

After changing the order of integration and term by term integration of the series with respect to φ_p and using orthogonal properties of the functions system $\{e^{im\varphi}\}_{m=-\infty}^{\infty}$, one obtains the equality of two Fourier series and, consequently, the equality of the correspondent Fourier coefficients, i.e.,

$$\int_{a}^{b}Z_m(\rho_p)G_m(\rho_q,\rho_p)\rho_p d\rho_p=g_m(\rho_q),\quad \rho_q\in[a,b],\quad m=0,\pm1,\pm2,\pm3,\ldots \tag{43}$$

Let us suppose that smooth function $\eta(t), t\in[-1,1]$ which is a one-to-one parameterization of interval $[a,b]$ by means of points of the interval [-1,1] is given. Such a function $\eta(t)$ in general must have a positive derivative, $\eta'>0$. The following special parameterization is used in the calculations presented below:

$$\eta(t)=\frac{b-a}{2}t+\frac{a+b}{2};\quad b>a,\quad t\in[-1,1] \tag{44}$$

Considering arbitrary $\eta(t)$, with its properties in general, equation (43) can be rewritten as ($\rho_q=\eta(u),\rho_p=\eta(v)$);

$$\int_{-1}^{1} \tilde{Z}_m(v)\tilde{G}_m(u,v)\eta(v)\eta'(v)d\eta(v) = \tilde{g}_m(u). \tag{45}$$

Thus, integral equation (29) has been reduced to the infinite set of non-inter-connected integral equation (45) of the first kind with kernels defined by (30), (31), (39) and (44) with unknown functions $\tilde{Z}_m(v)$, which evidently has representation of the form (see (21), (33), (40), (44))

$$\tilde{Z}_m(v) = (1-v^2)^{-1/2}h_m(v), \quad v \in (-1,1), \tag{46}$$

where $h_m(v)$ are functions of Hölder class on the interval $[-1,1]$.

Before constructing a solving procedure for equation (45), it is necessary to understand the singular structure and properties of smoothness of kernel $\tilde{G}_m(u,v)$. It is clear that $\tilde{G}_m(u,v)$ is an infinitely smooth function at any point $u \neq v$, only in the vicinity of the points $u = v$ requires detailed investigation. The analysis of integral (39) with function $\tilde{G}(\rho_q,\rho_p,\varphi)$ shows that function $\tilde{G}_m(u,v)$ has a representation of the form

$$\tilde{G}_m(u,v) = \frac{1}{4\pi^2\sqrt{\eta(u)\eta(v)}}\left[\ln|u-v|\left\{1+\sum_{n=2}^{3}A_n^m(u)|u-v|^n\right\} + H_3^m(u,v)\right] \tag{47}$$

where $H_3^m(u,v)$ has continuous derivatives of the third order and has only a logarithmic singularity at its fourth order derivative; $A_2^m(u)$ and $A_3^m(u)$ are infinitely smooth functions which have following representations:

$$A_2^m(u) = \frac{[k\eta'(u)]^2}{4} - \frac{m^2-1/4}{4}\left(\frac{\eta'(u)}{\eta(u)}\right). \tag{48a}$$

$$A_3^m(u) = \frac{m^2-1/4}{4}\left(\frac{\eta'(u)}{\eta(u)}\right)^3 \tag{48b}$$

Representation (47) gives the complete, for our purposes, description of the singular behavior of function $\tilde{G}_m(u,v)$. Let us consider new kernel defined as

$$\hat{G}_m(u,v) = 4\pi^2\sqrt{\eta(u)\eta(v)}\tilde{G}_m(u,v) \tag{49a}$$

and functions $K_m(u,v)$ by means of the following equality

$$-\frac{1}{\pi}\hat{G}_m(u,v) = -\frac{1}{\pi}\ln|u-v| + K_m(u,v), \quad m = 0,\pm1,\pm2,\pm3,\ldots\ . \tag{49b}$$

From formulas (47)-(49) it follows that functions $K_m(u,v)$ have the following representation

$$K_m(u,v) = -\frac{1}{\pi}\left\{\ln|u-v|\sum_{n=2}^{3}A_n^m(u)|u-v|^n\right\} - \frac{1}{\pi}H_3^m(u,v) \tag{50}$$

with the same coefficients A_2^m, A_3^m and function $H_3^m(u,v)$, from formula (49).

Using formula (49) and corresponding properties of the terms inside the equation, one can rewrite this equation follows:

$$\int_{-1}^{1} \left\{ -\frac{1}{\pi} \ln |u-v| + K_m(u-v) \right\} \hat{Z}_m(v) dv = \hat{g}_m(u), \quad u \in [-1,1], \quad m = 0, \pm 1, \pm 2, \ldots \tag{51a}$$

with new forms of unknown function $\hat{Z}_m(v)$, which, of course, has the representation of the kind (46) and right-hand side $\hat{g}_m$ having the following representations:

$$\hat{Z}_m(v) = [\eta(v)]^{1/2} \quad \eta'(v) Z_m(\eta(v)); \tag{51b}$$

$$\hat{g}_m(u) = 2[\eta(u)]^{1/2} \quad u_m^i(\eta(u)). \tag{51c}$$

Thus, integral equation (29) has been equivalently reduced to the infinite set of non-inter-connected integral equation of the first kind of the type (51 - a). Each kernel of these equations is the sum of a canonic kernel with logarithmic singularity and some smooth function. Any solution of this equation has to have the representation of the form (46).

As we have proved before, the solution of equation (29) is unique, if it exists. Because every equation (51) was obtained by means of the above-described Fourier transform (see, first of all, formulas (33), (44)), it can easily be proved that the solution of each equation (51) is unique, if it exists.

4. Solving Procedure for Canonic Integral Equation with Logarithmic Singularity.

We will consider here the following canonic integral equation of the following form

$$\int_{-1}^{1} \left\{ -\frac{1}{\pi} \ln |u-v| + K(u,v) \right\} z(v) dv = b(u), \quad u \in [-1,1] \tag{52}$$

with unknown function $z(v)$; all other functions in (52) are supposed to be known. We additionally suppose that function $z(v)$ has the representation of the form

$$z(v) = (1-v^2)^{-1/2} m(v), \quad v \in [-1,1] \tag{53}$$

with function $m(v)$ which belongs to the Hölder class on $[-1,1]$, function $b(u)$ supposed to be smooth enough (see below) and function

$$\psi(\theta,\tau) = K(\cos\theta, \cos\tau), \quad \vartheta, \tau \in [0,\pi] \tag{54}$$

is supposed to be smooth: it is a continuous function with its first derivatives

$$\psi(\vartheta,\tau), \frac{\partial\psi(\vartheta,\tau)}{\partial\vartheta}, \quad \frac{\partial\psi(\vartheta,\tau)}{\partial\tau} \in C([0,\pi]\times[0,\pi]) \tag{55}$$

and its mixed derivative is a square-integrable function:

$$\frac{\partial^2\psi(\vartheta,\tau)}{\partial\vartheta\partial\tau} \in L_2([0,\pi]\times[0,\pi]) \tag{56}$$

Any of equations (51) is an equation of type (52). One of the purposes of the previous section was the reduction of equation (43) to this canonical form (52).

We use below the orthonormal Chebyshev's polynomials of the first kind $\hat{T}_n(x)$ which are connected with well-known standard Chebyshev polynomials of the first kind $T_n(x)$ by means of the formula:

$$\hat{T}_n(x) = d_n^{-1} T_n(x), \quad x \in [-1,1] \tag{57}$$

where $d_0 = \pi^{1/2}$ and $d_n = (\pi/2)^{1/2}$, $n \neq 0$ and, consequently,

$$\int_{-1}^{1} (1-x^2)^{-1/2} \hat{T}_n(x)\hat{T}_m(x)dx = d_{m,n}, \quad m,n = 0,1,2,3,\ldots, \tag{58}$$

δ_{mn} is the Kronecker delta: $\delta_{n,n} = 1$ and $\delta_{mn} = 0$ for $m \in n$.

From formula 5.4.2.9 of [23]:

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n} = -\ln\left|2\sin\frac{x}{2}\right|, \quad x \in [-2\pi, 2\pi] \tag{59}$$

one can obtain after elementary transformations that

$$-\frac{1}{\pi}\ln|u-v| = \sum_{n=0}^{\infty} \frac{\hat{T}_n(u)\hat{T}_n(v)}{\gamma_n^2}, \quad u,v \in [-1,1], \tag{60}$$

where

$$\gamma_0 = (\ln 2)^{-1/2}; \quad \gamma_n = |n|^{1/2}, \quad n \neq 0, \tag{61}$$

According to the above mentioned properties of functions $z(v), b(u)$ and $K(u,v)$, we can represent them as their Fourier-Chebyshev series:

$$z(v) = (1-v^2)^{-1/2} \sum_{n=0}^{\infty} z_n \hat{T}_n(v); \quad v \in (-1,1); \tag{62}$$

$$b(u) = \sum_{n=0}^{\infty} b_n \hat{T}_n(u), \quad u \in [-1,1]; \tag{63}$$

$$b(u) = \sum_{n=0}^{\infty} b_n \hat{T}_n(u), \quad u \in [-1,1]; \tag{64}$$

$$K(u,v) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} k_{mn} \hat{T}_m(u)\hat{T}_n(v), \quad u,v \in [-1,1], \tag{65}$$

where $\{z_n\}_{n=0}^{\infty}$ are unknown coefficients and $\{b_n\}_{n=0}^{\infty}$ and $\{k_{mn}\}_{m,n=0}^{\infty}$ are Fourier-Chebyshev coefficients of the functions $b(u)$ and $K(u,v)$ correspondently:

$$b_n = \int_{-1}^{1} (1-u^2)^{-1/2} b(u) \hat{T}_n(u) du = d_n^{-1} \int_{0}^{\pi} b(\cos\vartheta) \cos n\theta d\vartheta; \tag{66}$$

$$k_{mn} = \int_{-1}^{1}\int_{-1}^{1} \frac{K(u,v)\hat{T}_m(u)\hat{T}_n(v)}{(1-u^2)^{1/2}(1-v^2)^{1/2}} dudv = \tag{67}$$

$$= d_n^{-1} d_m^{-1} \int_{0}^{\pi}\int_{0}^{\pi} K(\cos\vartheta, \cos\tau) \cos m\vartheta \cos n\tau d\vartheta d\tau. \tag{68}$$

It can be proved that coefficients k_{mn} satisfy the following inequality:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (1+n^2)(1+m^2)|k_{mn}|^2 < \infty \tag{69}$$

which follows from formulas (54)-(56) and (67), and can be derived from formula (67) by means of integration by parts in integrals (67) and by Parsevale's equality.

Substitution of the right-hand sides of series (62)-(64) and (60) into equation (52), changing the order of integration and summation and using (59) one arrives at the following equation:

$$\sum_{n=0}^{\infty} \gamma_n^{-2} z_n \hat{T}_n(u) + \sum_{n=0}^{\infty} \left(\sum_{s=0}^{\infty} k_{ns} z_s \right) \hat{T}_n(u) = \sum_{n=0}^{\infty} b_n \hat{T}_n(u), \quad u \in [-1,1] \tag{70}$$

The orthogonal property (58) and completeness of the functions system $\{\hat{T}_n(u)\}_{n=0}^{\infty}$ give us the equality of the Fourier-Chebyshev coefficients of the left and right-hand sides of the last equation,

$$\gamma_n^{-2} z_n + \sum_{s=0}^{\infty} k_{ns} z_s = b_n, \quad n = 0,1,2,\ldots\ . \tag{71}$$

Equations (70) can be considered as an infinite algebraic system, which it is possible, in principle, to solve by means of a truncation procedure. Unfortunately, this system is evidently one of the first kind, and that is why it is necessary to transform it by means of a relevant regularization procedure.

For this purpose, let us define unknown coefficients:

$$y_n = z_n/\gamma_n, \quad n = 0,1,2,\ldots, \tag{72}$$

and multiply every equation (70) by γ_n. As a result, we obtain

$$y_n + \sum_{s=0}^{\infty} \gamma_n \gamma_s k_{ns} y_s = \gamma_n b_n, \quad 0,1,2,\ldots \tag{73}$$

Let us define coefficients

$$\hat{k}_{ns} = \gamma_n \gamma_s k_{ns}, \quad n = 0, 1, 2, \ldots, \tag{74}$$

matrix-operator

$$\hat{K} = \{k_{ns}\}_{n,s=0}^{\infty} \tag{75}$$

and vector-columns

$$y = \{y_n\}_{n=0}^{\infty}; \quad \hat{b} = \{\hat{b}_n\}_{n=0}^{\infty}; \quad \hat{b}_n = \gamma_n b_n. \tag{76}$$

After that, system (72) can be rewritten as

$$y_n + \sum_{s=0}^{\infty} \hat{k}_{ns} y_s = \hat{b}_n, \quad n = 0, 1, 2, \ldots \tag{77}$$

and as a functional equation of the form

$$(I + \hat{K})y = \hat{b}, \quad y, \hat{b} \in \ell_2. \tag{78}$$

From formulas (69) and (73), it follows that

$$\sum_{n=0}^{\infty} \sum_{s=0}^{\infty} (1+n)(1+s)|k_{ns}|^2 < \infty \tag{79}$$

and, consequently, operator $\hat{K}$ is a compact and even Hilbert-Schmidt operator.

It is clear now that any solution $z(v)$ of equation (51), if it exists, produces the solution y of equation (77) by means of formulas (62),(71) and (75); the same formulas, which are taken in the reverse order, produce the solution of equation (52) from a solution of equation (77), if the last solution exists. This means that equations (52) and (77) are equivalent.

Let us suppose now that equation (52) has the same property that we proved for equation (51): the solution $z(v)$ of equation (52) is unique, if it exists. From this and the above-mentioned equivalence of equations (52) and (77), it follows that equation (77) has the same property: the solution of equation (77) is unique, if it exists. From this fact and from the Fredholm alternative for an equation of the second kind with compact operator, as it is well known, it follows that equation (77) has the solution for arbitrary right-hand side vector $\hat{b}$, i.e., operator $(I + H)^{-1}$ exists.

In this situation it is possible to solve equation (77) numerically with any required accuracy (see Introduction).

Let us consider infinite set of integral equations (51) again. Every one of these equations has form (52) with

$$K(u, v) = K_m(u, v) \tag{80}$$

and consequently each of them can be reduced to the infinite algebraic equation of the kind (77) with matrix operator

$$\hat{K} = \hat{K}^m \equiv \{k_{ns}^m\}_{s,n=0}^{\infty} \tag{81}$$

which can be obtained by means of formulas (79),(66),(67),(73) and (74). Operators $\hat{K}^m$ have property (78) for their coefficients $k_{ns} = k_{ns}^m$, they are compact operators in ℓ_2, etc.

That is why an infinite set of integral equation (51) is reduced to the infinite set of the following equation in space ℓ_2

$$(I + \hat{K}^m)y^m = \hat{b}^m, \quad y^m, \hat{b}_m \in \ell_2, \quad m = 0 \pm 1 \pm 2, \dots \tag{82}$$

with compact in ℓ_2 operators $\hat{K}^m$.

As we proved before, the solution of every equation in (51) is unique, if it exists. This means, as we concluded above, that the same property has every one of equations (81), and from the Fredholm alternative it follows that every equation of set (81) has one and only one solution. It means, of course, that all equations (51),(35) and (29) have this property. The unique and existing solution $Z(p)$ of equation (29) produces by means of formula (19) the unique and existing solution of the Dirichlet diffraction boundary value problem (10)-(18). Function $Z(p)$ can be constructed from solutions $y^m, m = 0 \pm 1, \pm 2, \dots$ of equations (81) by means of formulas (33), (37), (44) and (51) and formulas of types (62) and (71), which are taken in reverse order. Thus, we have proved both:

- the equivalence of the Dirichlet diffraction problem (10)-(18) and set (81) of infinite algebraic systems of the second kind
- the existence and the uniqueness of the diffraction problem solution.

5. Numerical Results

Throughout this section the possibilities of the algorithm used will be illustrated with the typical results presented for verification and testing of the algorithm. Detailed investigation of the physical features of the problem lies outside the scope of this paper, and it will be the subject of our next publication.

In Fig. 2, the ring considered has $ka = 15$ and $kb = 30$, the wave is incident from a point source located at $(r, \phi, z) = (0, 0, 5)$ in cylindrical coordinates, and one can see the typical behavior of condition number v_N, depicted as a function of the truncated algebraic system size. It is clear from this figure that v_N tends to a constant value while the dimension of the system increases, which corresponds to our theoretical prediction (see Eq.(9)). That is why it is safe to solve system (81) for arbitrary big N without danger of "numerical catastrophe". Such behavior of v_N is qualitative only for algebraic systems of the second kind. For direct methods like the Moment Method and others the behavior of v_N is quite the opposite: it tends to infinity fast, destroying the truncated solutions completely.

Indeed, good convergence is observed in Table 1, where the typical behavior of the calculated current density at one point (i.e. kr =12.5) of the ring is presented, in the case when a point source at $(r, \phi, z) = (0, 0, 5)$ excites the ring with dimensions $ka = 10, kb = 15$ with different truncated systems of algebraic equations.

It is possible to estimate the efficiency of the solving procedure by looking at Table 2. The time of computation for three cases to obtain a 1% accurate value of the current density is shown.

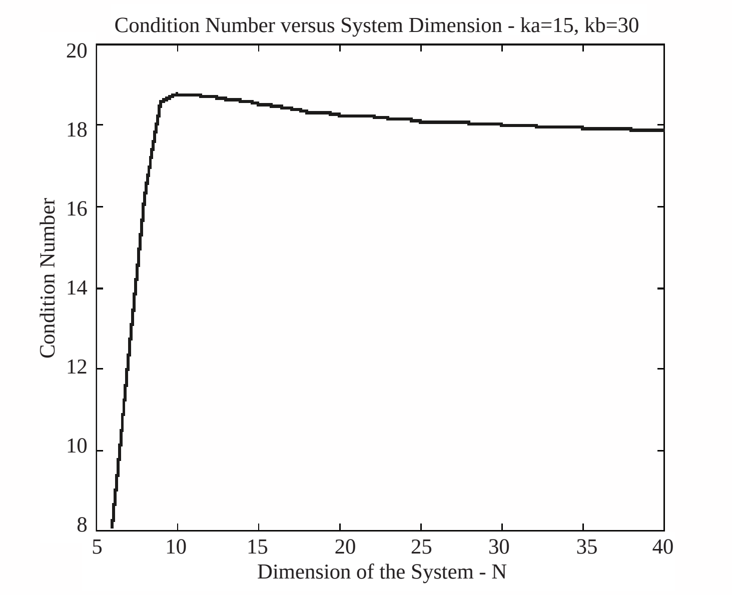

Figure 2. Condition number of the finite dimensional algebraic systems as a function of their sizes. Convergence to a constant guarantees stability of solutions

Table 1. Convergence of current density values (kJ) at $kr = 12.5$ when a point source at a point with cylindrical coordinates (0,0,5) excites the ring with $ka = 10, kb = 15$ for different truncation numbers (N) of the system.

(N)	(kJ)
2	0.00484911808976952
4	0.00533434200376563
10	0.00536650647080133
15	0.00536652842011560
20	0.00536648641943425
30	0.00536651528557421
40	0.00536650431820151
50	0.00536650925975500
60	0.00536650657671625
65	0.00536650674179517
68	0.00536650690114758

Table 2. Calculation time (T) of the solution of $N \times N$ dimensional algebraic system, made by a 450 MHz Pentium II processor PC with 64 MB RAM, with 1% accuracy for various ka and kb.

ka	kb	N	T (sec)
10	15	20	< 1
20	30	30	< 4
40	60	40	< 10

Fig. 3 plots the modulus of typical current density distribution on the ring surface in the case when a plane wave is incident normal to the surface of the ring (from top to bottom in Fig. 1) where $ka = 10$, $kb = 30$. Fig. 4 stands for the same property of another ring having $ka = 15, kb = 30$ in the case of point source excitation, which is located at $(r, \phi, z) = (0, 0, 5)$. It should be noted that in both cases, a good agreement can be observed clearly between the method proposed and Kirchhoff approximation [1-4, 24], which does not take into account the current density behavior near thin edges, but is known to be reliable enough at the points rather far from the edges[1]. And the evident difference, which one should expect, is that the singular behavior near both edges is obtained with the exact solution.

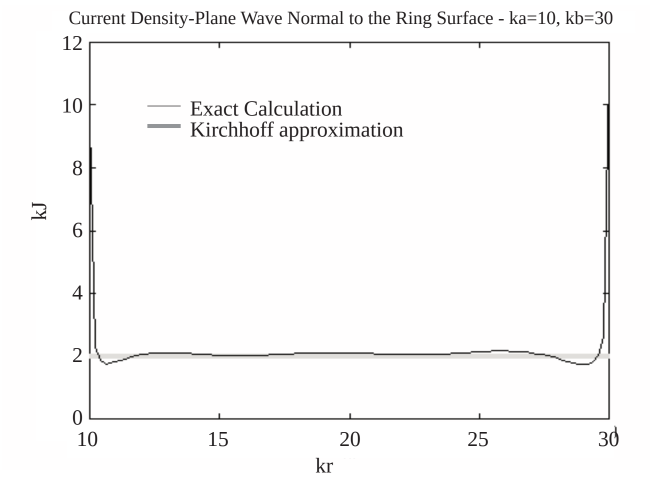

Figure 3. Current density induced by normally incident plane wave to the ring.

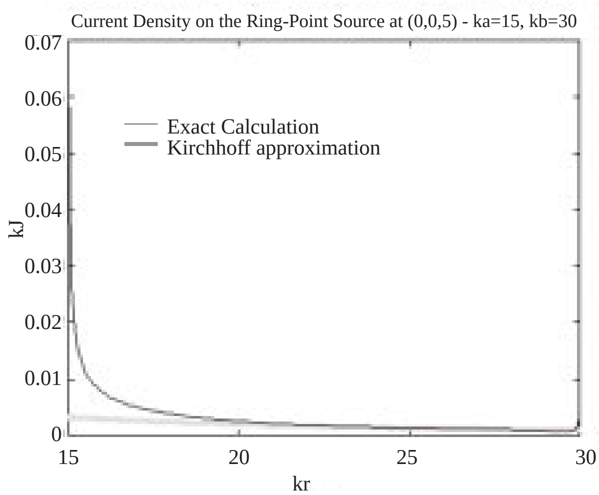

Figure 4. Current density on the ring induced by a point source.

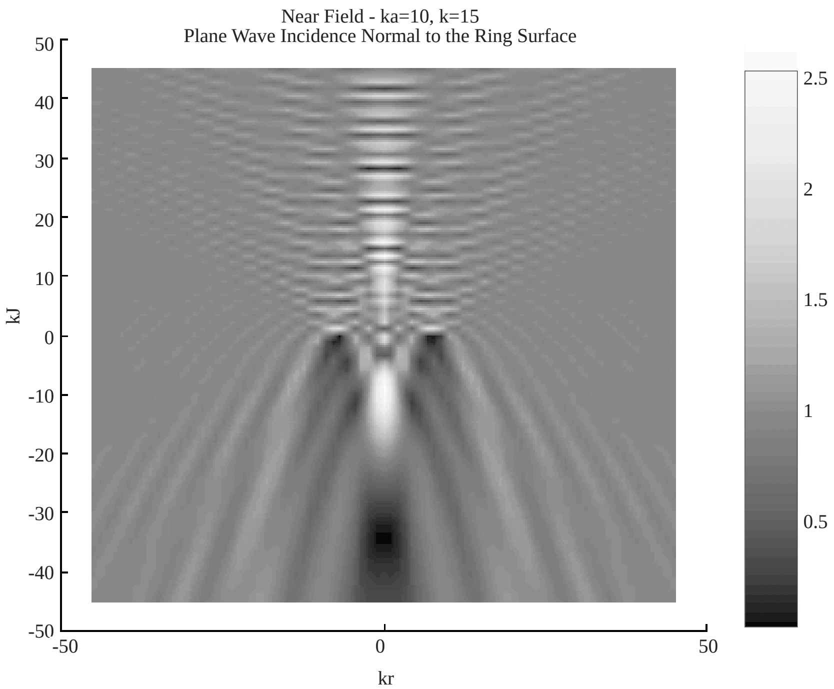

Figure 5. Total near field. Plane wave propagates from up to down in the picture.

[1]Double-sided current density of Kirchhoff approximation is the one of the infinite plane tangential to the obstacle point with the same illumination, and hence can be calculated easily.

Fig. 5 and Fig. 6 are for the near field of a ring which is shown as a cross-sectional picture $ka = 10, kb = 15$ in both cases, and Fig. 6 is a magnified view of the center of Fig. 5. It can be observed clearly that a steady state wave in the top half of the picture exists as a result of the interference of incident and reflected waves, the shadow zone took place in some bounded vicinity of the ring, and a spherical wave reradiated somewhere from the central disc hole of the ring.

Fig. 7 shows a far field pattern of a ring having $ka = 10, kb = 15$ in the case of a normally incident plane wave, as in the previous case. The difference between far field patterns of our method and Kirchhoff approximation is difficult to distinguish in Fig. 8 for a ring of $ka = 15, kb = 30$, even if it is plotted in logarithmic scale. But the difference exists and can be seen from the figure.

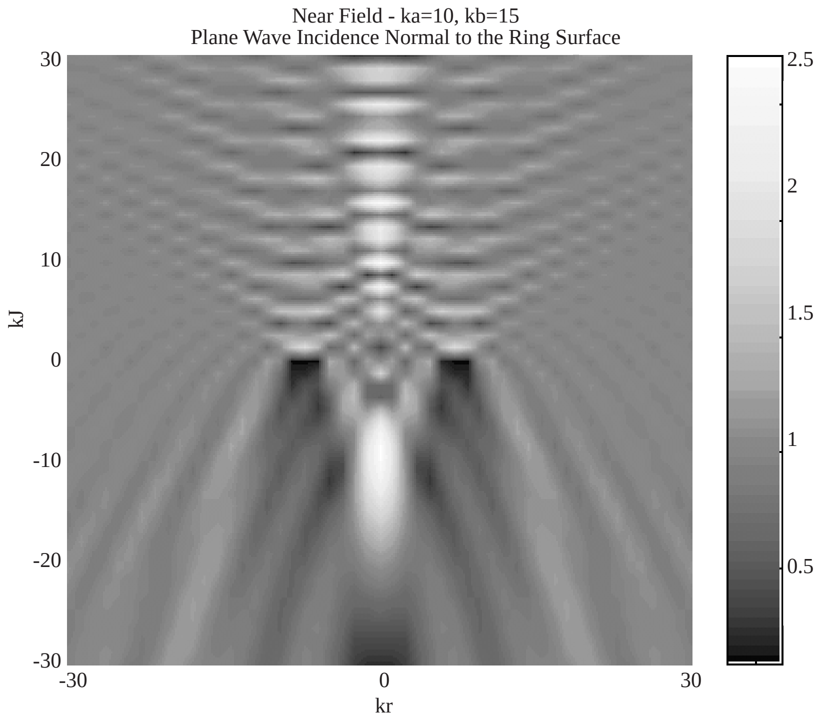

Figure 6. Total near field. Magnified view of Figure 5.

Fig 9. helps to make a comparison of far fields calculated by the exact method and Kirchhoff approximation. Now, in the case of excitation by point source, the difference between far fields is essentially greater, which is correlated with the great differences in current densities of the two methods. One can conclude that having less difference of the current densities for the case Fig. 8 leads to less distinguishable difference between calculated far fields of the mentioned two methods.

Fig. 10 illustrates frequency dependence of radar cross section,

$$RCS/b^2 = \lim_{r\to\infty}\left[4\pi r^2 \frac{|u^s|^2}{|u^i|^2}\right]/b^2, \tag{83}$$

normalized by b^2 where it is clearly seen that for the high frequency limit of large kb values the exact solution tends to be the RCS given by Kirchhoff approximation or geometrical optics.

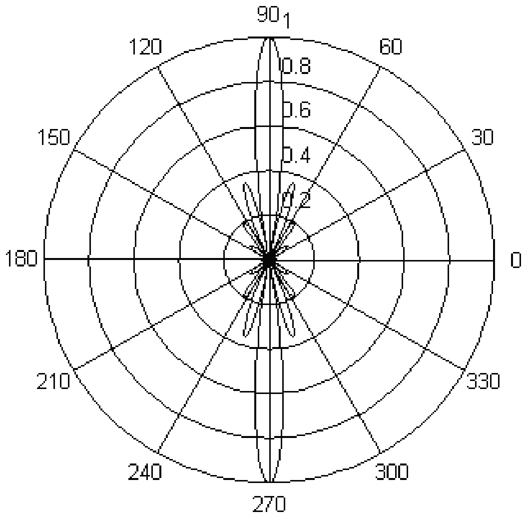

Figure 7. Far field pattern for normally incident plane wave.

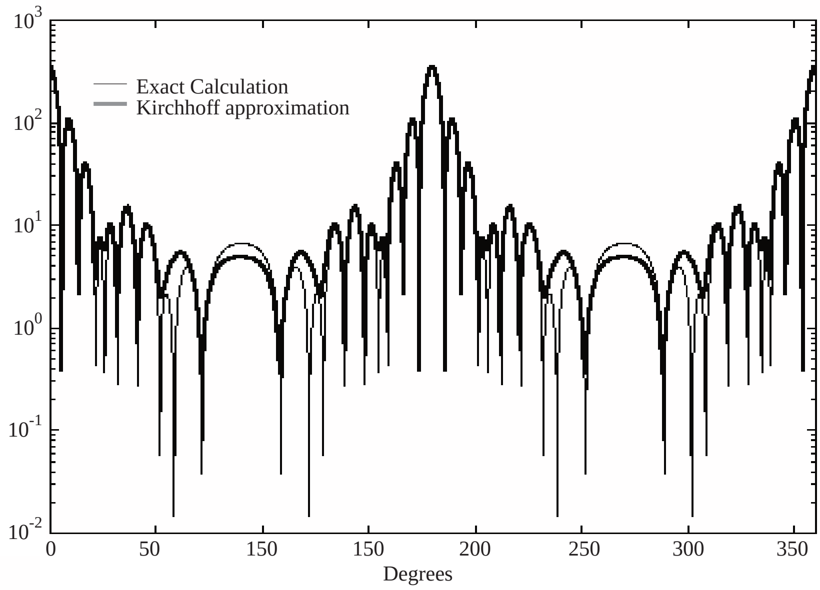

Figure 8. Far field pattern for normally incident plane wave; **ka=15, kb=30**. Comparison with Kirchhoff approximation.

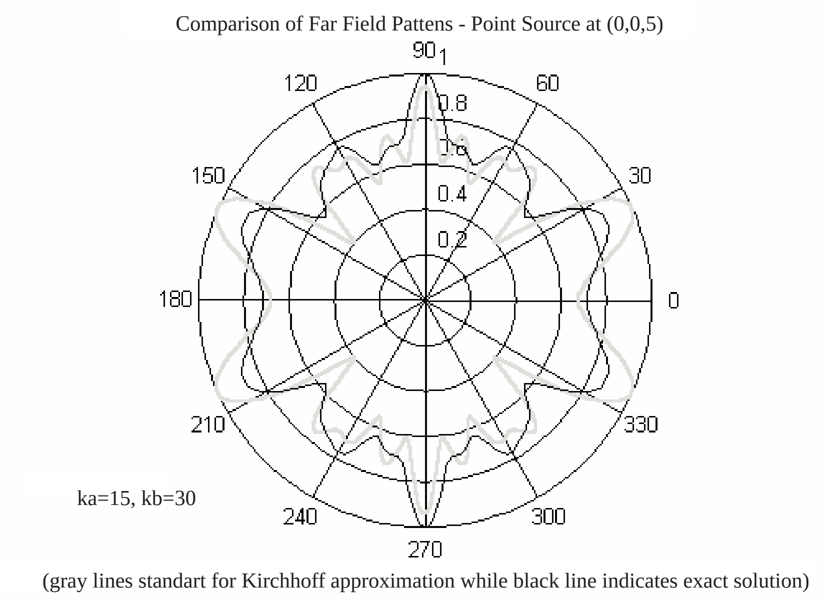

Figure 9. Far field pattern for a point source excitation. Comparison with Kirchhoff approximation.

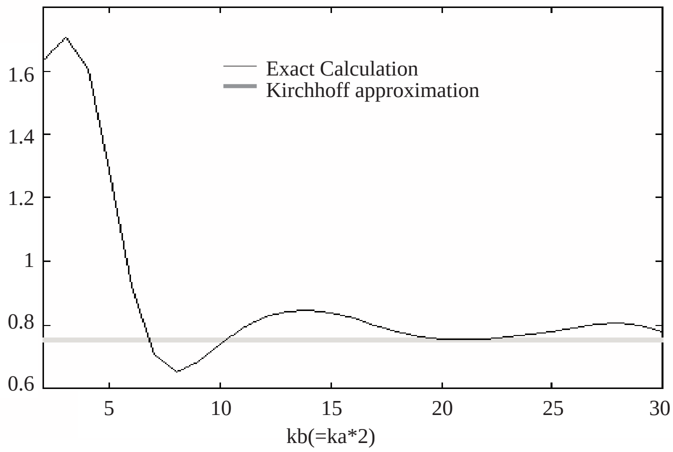

Figure 10. Radar cross section. Comparison with Kirchhoff approximation.

6. Conclusion

The Dirichlet boundary value problem of scalar wave diffraction by infinitely thin circular ring has been solved, i.e., it has been reduced equivalently in the mathematical sense to the infinite set of functional equations of the second kind in space ℓ_2 with compact operators in ℓ_2. Each of these equations, which belong to an infinite system of algebraic equations, has one and only one solution. This solution can be obtained with any necessary accuracy by means of the truncation procedure for the algebraic system. The set of solutions of these functional equations produces the unique solution of the boundary value problem.

These results have been obtained by means of the suggested new method, which is the combination of the conventional Orthogonal Polynomials Method and the original version of the Analytical Regularization Method.

Numerical experiments demonstrate fast converging results of truncated system dimension $N \to \infty$, and the results and algorithm are numerically stable. Thus, our approach is a powerful tool for theoretical study and numerical solving of the diffraction problems considered, and the approach can be generalized for a rather wide class of axially symmetrical diffraction problems.

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