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An Efficient Preconditioner for Iterative Solvers

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Abstract

*The method of moments solution of the Maxwell's equations leads to a dense system of complex equations. Direct solution of these equations using LU factorization becomes unwieldy as the size of the scatterer increase in terms of wavelength. Iterative solvers, such as those based on Krylov projection methods, offer an alternative approach for solving large system of equations. Most often, the iterative methods are used in combination with some kind of preconditioning to improve the condition number of the system matrix **A** in order to acheive accelerated convergence [1-2]. This paper discusses the application of Multi-Frontal Preconditioners (MFPs) for the Krylov projection methods for an efficient solution of the dense system of linear equations. The MFP uses combined unifrontal/multi-frontal approach to handle arbitrary sparsity patterns and enables a general fill-in reduction[3]. The paper specifically focuses on the efficient solution of complex general systems, without making any assumptions regarding the positive definitness of the operators. Performances of several popular Krylov projection methods are presented to demostrate the computational efficiency of the present method, using the MFP.*

Key Words: *Method of Moments, iterative solution, LU factorization, Krylov projection, Conjugate Gradient Normal Solver*

1. Theory

In general, the conventional Method of Moments (MoM) formulation of the EM problem leads to a dense complex linear system of the form:

$$\mathbf{AX} = \mathbf{B} \tag{1}$$

where **A** is an N x N complex coefficient matrix, **B** and **X** are the known right hand side (RHS) and unknown vectors, respectively.

Most existing iterative techniques utilize a projection method in one form or another to solve (1). Such a process represents a canonical way of extracting an approximation to the solution of a linear system from a subspace. The solution vector is updated at each iteration step in a way such that the residual error ε, defined as

$$\varepsilon = \frac{\|\mathbf{AX} - \mathbf{B}\|_2}{\|\mathbf{B}\|_2} \tag{2}$$

is minimized. Typically, a residual error of 0.1% is adequate for most applications, although one may choose to reduce it if a higher level of accuracy is desired. The efficiency of an iterative solver depends strongly on the quality of the preconditioner used. It is difficult to find a good general *black box* type of preconditioner that works equally well for all the situations. To be effective, a preconditioner should be a good approximation of $\mathbf{A}$, and the solution $\mathbf{X}$ of the preconditioned system

$$\mathbf{M}^{-1}\mathbf{A}\mathbf{X} = \mathbf{M}^{-1}\mathbf{B} \tag{3}$$

where $\mathbf{M}$ is the preconditioner, should be easier to compute than that of the original system. Obviously, preconditioning an iterative solver involves a repeated solution of the above equation. Also, it is important to consider two attributes when evaluating a preconditioner, viz., the ease of preconditioner matrix formation and memory requirements for its storage. In the MoM solution of the integral equations, the diagonal elements of the system matrix $\mathbf{A}$ represents the self-term contributions, while the off-diagonal elements stands for the mutual coupling terms. The diagonal dominance of $\mathbf{A}$ is not always assured and this property of the matrix depends strongly on the scatterer geometry.

A sparse representation of the system matrix is generated by thresholding the matrix $\mathbf{A}$ row-wise. The individual elements of the resulting sparsified matrix $\mathbf{A}^s$ are generated as follows.

$$A^s_{i,j} = \begin{cases} A_{i,j} if |A_{i,j}| > \alpha |A^m_i| \\ 0 \end{cases} \tag{4}$$

where $|A^m_i|$ is the magnitude of the largest element in the row 'i', and α is the threshold factor which is taken to be 0.01 for all the calculations presented in this paper.

This form of sparsification results in a general nonzero pattern, which has to be preserved in order for the solution procedure to be efficient. The MFP uses a combined unifrontal/multi-frontal approach to handle arbitrary sparsity patterns to achieve a general fill-in reduction[3]. The frontal approach involves finding a permutation of $\mathbf{M}$, which when factorized into its LU factors (P$\mathbf{A}$Q = LU where P and Q are permutation matrices), preserves the sparsity and the numerical accuracy. The performance of the MFP when applied to several popular Krylov projection methods is presented in the next section.

2. Results

The performance of iterative solvers such as Conjugate Gradient Normal (CGN), BiCG-stabilized and Transpose Free QMR have been studied with MFP. A 10 x 10 planar hairpin array [4] illuminated by a plane wave at normal incidence has been selected as a representative test case (Figure 1). The array is 12λ x 12λ in size at 4000 MHz and requires a total number of 3700 unknowns. A detailed description of the structure is presented elsewhere [4] and is not repeated here. The above array is observed to have a resonance at 2125 MHz [4] and is ideal for verifying the robustness of the MFP. The preconditioner matrix $\mathbf{M}$ has been derived by sparcifying the original dense matrix $\mathbf{A}$ and its structure is presented in Figure 2. The figure shows an expanded view to highlight the finer details, although the entire matrix is found to exhibit a similar structure. The fill-in percentage of the matrix $\mathbf{M}$ is 0.55% of the original matrix $\mathbf{A}$ with 3700 unknowns. Using MF approach, this matrix is decomposed into its LU factors with a fill-in of just 0.74%. It should be noted that the normal LU factorisation of a sparse matrix with a general fill-in pattern neither ensures nor preserves the sparsity. However, the frontal approach adopted in this paper does presere the sparsity of the matrix M in the process of factorization. This is essential for reducing the solution time of (1) and also the storage required for the preconditioner.

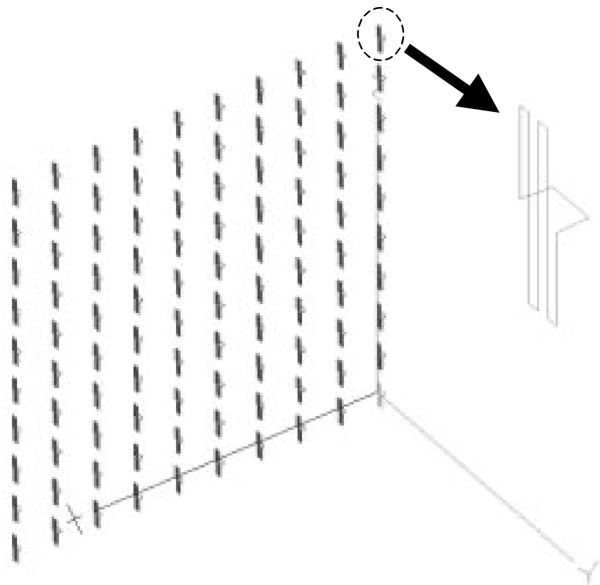

Figure 1. A 10 x 10 element planar hairpin array.

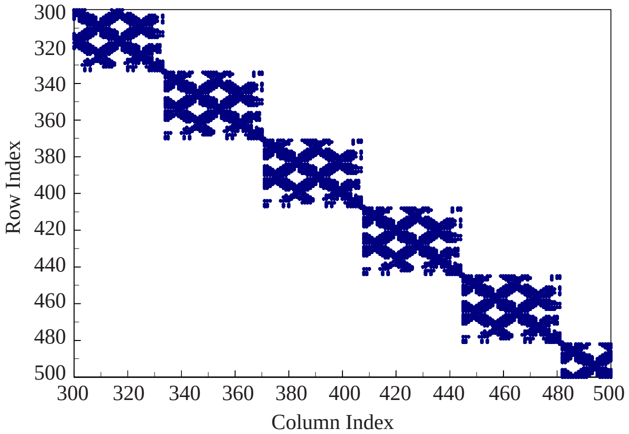

Figure 2. An expanded view of the MFP matrix derived from the original dense system matrix A, with 3700 unknowns. The scale is expanded to bringout the fine structure of the matrix.

The performances of various iterative solvers with MFP have been studied at 2125 MHz, which corresponds to the resonant frequency, as well as at the off-resonance frequency of 4500 MHz. The results are presented in Table 1 and Table 2, respectively. Also, the solution time for the LAPACK direct solver using complete LU decomposition is presented for the sake of comparision. All the computations have been carried out on Pentium III 662 MHz Xeon PC with 512Mb RAM., and a residual error of 0.001 has been chosen as the convergence criterion. The CPU time shown for the MFP includes the time taken to generate preconditioner matrix, its factorization, and the time taken by the iterative solver. We observe that, at 2125 MHz (resonant frequency), both the direct and iterative solvers take more time compared to that at off-resonance. The CGN solver was found to be reliable even without any preconditioning and showed a monotonic decrease in the residual, while the BiCGStab and TFQMR failed to converge. The performance of all the three iterative solvers improved by an order of magnitude when MFP is used. This is true not only at the resonant frequency where the matrix A is poorly conditioned, but also at off-resonance as shown in Table 2. It is observed that the iterative solvers with MFP are 10 times faster than the direct solver at off-resonance, while this factor comes down to 3 at resonance. The BiCGStab solver with MFP is found to exhibit better performance over the direct and other iterative solvers.

Table 1. Performance of various iterative solvers at 2125 MHz.

Solver	No Preconditioner		MFP	
	Iterations	CPU time(s)	Iterations	CPU time(s)
CGN	589	3261.00	39	218.40
BiCGStab	-	-	26	208.63
TFQMR	-	-	55	221.44
LAPACK	-	610.00	-	-

The eigen spectrum of the original system matrix and the preconditioned matrix with the MFP are shown in Figure 3a and Figure 3b, respectively. It can be seen that the MFP compressed the eigen spectrum of the original matrix successfully, leading to a significant improvement in the computaional time for solving the matrix equation. The residual error of the Conjugate Gradient method as a function of the iteration number is presented in Figure 3. It is observed that the use of MFP considerably enhanced the convergence rate of the conventional CG.

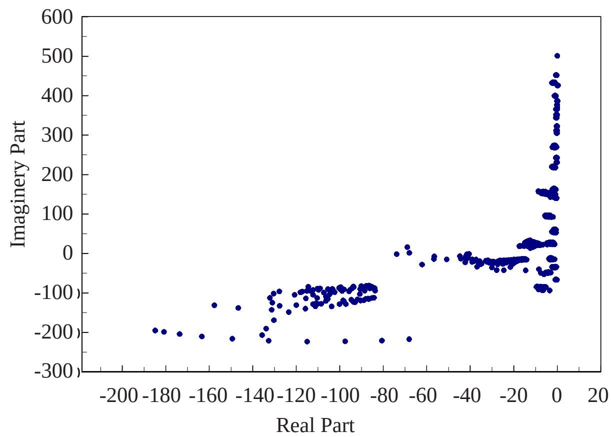

Figure 3a. Eigen spectrum of the original system matrix at 4500 MHz.

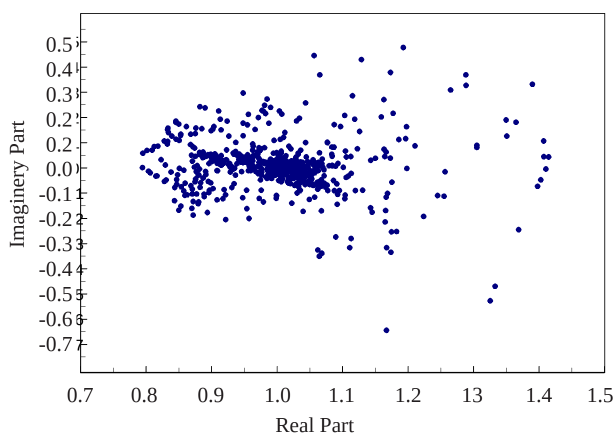

Figure 3b. Eigen spectrum of the preconditiod matrix at 4500 MHz.

Table 2. Performance of varius iterative solvers at 4500 MHz.

Solver	No Preconditioner		MFP	
	Iterations	CPU time(s)	Iterations	CPU time(s)
CGN	138	766.36	11	73.82
BiCGStab	-	-	4	42.53
TFQMR	-	-	10	48.58
LAPACK	-	570.74	-	-

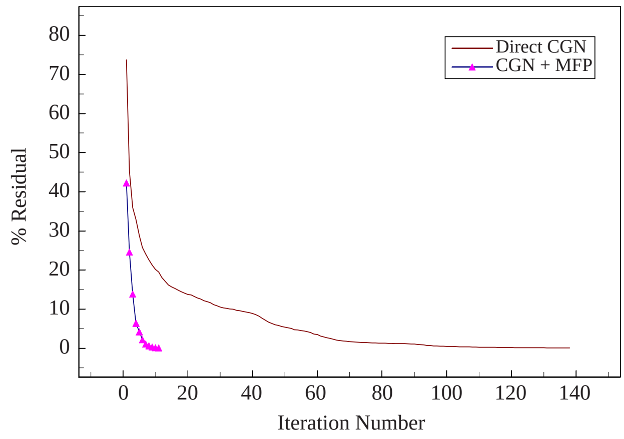

Figure 4a. Residual error *vs* iteration number of CG at 4500 MHz.

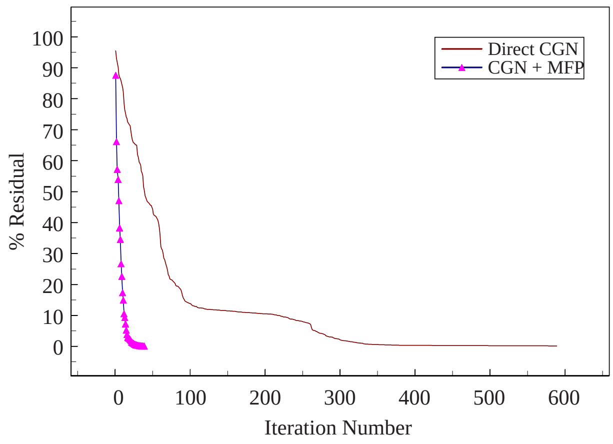

Figure 4b. Residual error *vs* iteration number of CG at 2125 MHz.

3. Conclusion

A Multi-Frontal Preconditioner for a class of Krylov projection methods useful for an efficient solution of the dense system of linear equations arising from integral equation formulations, has been presented in this work. The MFP uses a combined unifrontal/multi-frontal approach to handle arbitrary sparsity patterns, reduces general fill-in reduction, and is independent of the special properties of the system matrix. Performances of several popular iterative methods have been studied using the MFP for the specific problem of scattering from a hair-pin array, and it is shown that the approach significantly improves the computational speed as much as by an order of magnitude. It is expected that the method would find useful applications for a large class of array type problems for which the matrix thresholding is expected to yield a fairly sparsified matrix.

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