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A Note on the Poisson Summation Formula and its Application to Electromagnetic Problems Involving Cylindrical Coordinates

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Abstract

A modified version of the Poisson Summation Formula is derived, involving the Fourier-Bessel Transform of the kernel function, as opposed to the conventional Fourier Transform engaged in the standard formula. This novel form can be applied to problems employing cylindrical coordinates, such as the fast evaluation of the dyadic Green's function in a waveguide or cavity of circular cross-section.

Key Words: *Poisson summation formula, electromagnetic problems, cylindrical coordinates, dyadic Green Functions*

1. Introduction

The Poisson Summation Formula (PSF) [1,2] is a powerful mathematical tool, utilized in several aspects of Electrical Engineering, such as Signal Processing, Communications, as well as in Electromagnetics. In the latter area, specifically, the PSF is very useful in the treatment of various Dyadic Green's Functions (DGFs) that cannot be represented in closed form, but only as a slowly converging series. Waveguiding structures constitute typical examples of such a behavior. The associated DGFs are given in terms of doubly infinite series involving the respective eigenfunctions of the boundary value problem. These expressions are not only very complicated, but exhibit very slow convergence characteristics as well, when the source and observation points lie in the vicinity of each other. This is not unexpected, since the series is actually bound to diverge for coinciding source and observation points, a property described by a well-known singularity. Nevertheless, this singularity is not easily extractable, unlike in the free space case, and the computational usage of the pertinent DGF in numerical techniques, such as the Moment Method, becomes impractical, if not impossible.

Convergence acceleration of the infinite series representing a DGF has been accomplished in special cases, such as general periodic structures [3,4], antenna arrays [5-14], periodically excited parallel plate waveguides [15], rectangular cavities [16], and rectangular waveguides [17]. In all papers associated with large antenna arrays [5-14], the PSF is applied directly to the series representing the composite radiation of the array elements, transforming the DGF from the spatial to the spectral domain. The truncation effect for finite arrays is also discussed in [7-14]. Appropriate combination of the resulting DGF spectral form

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with its original spatial form circumvents poor convergence problems [5]. On the other hand, treatment of the DGF in waveguiding structures [15-17] is far more complicated. In these cases, acceleration is usually achieved via the celebrated Ewalds' method [18,19], which expresses the free space Green's function as a combination of complementary error functions. Again, a prerequisite for the application of the technique is the transformation of the DGF, but this time in the opposite direction. As opposed to array analysis, the transformation is performed from the standard modal (spectral) representation towards the image (spatial) form, which is actually a superposition of appropriately weighted free space scalar Green's functions. The latter essential transformation can be performed through use of the PSF.

In all aforementioned papers, application of the PSF is almost straightforward, yielding closed form results, since the functions involved are merely algebraic or trigonometric, expressed in Cartesian coordinates. If the coordinate system ceases to be Cartesian, several mathematical complications occur, depleting the usefulness of the conventional PSF. The purpose of this short paper is to propose a modified version of the PSF, involving the Fourier-Bessel Transform [2,20], which is suitable for a similar analysis in cylindrical coordinates.

2. The Poisson Summation Formula and its Application in DGF Calculations

The Poisson Summation Formula (PSF) [1,2] implies that an infinite sum of periodic samples of a fairly generic function is equal to a similar sum in its Fourier Transform domain. The theorem can be extended to arbitrary dimensionality, but for engineering purposes it is normally invoked for functions of up to three independent variables. The basic formula for the one-dimensional case reads

$$\sum_{n=-\infty}^{+\infty} f(2\pi n) = \frac{1}{2\pi} \sum_{\nu=-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(\tau) e^{j\nu\tau} d\tau \tag{1}$$

provided that $f(x)$ is a continuous function of x and $\sum\limits_{n=-\infty}^{+\infty} f(2\pi n)$ converges absolutely. In a more compact form, (1) can be written as

$$\sum_{n=-\infty}^{+\infty} f(2\pi n) = \sum_{\nu=-\infty}^{+\infty} F(\nu) \tag{1a}$$

where $F(\cdot)$ is the inverse Fourier transform of $f(\cdot)$.

Extension to two and three dimensions yields respectively

$$\sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} f(2\pi m, 2\pi n) = \frac{1}{(2\pi)^2} \sum_{\mu=-\infty}^{+\infty} \sum_{\nu=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(\tau_1, \tau_2) e^{j(\mu\tau_1 + \nu\tau_2)} d\tau_1 d\tau_2 \tag{2}$$

$$\begin{aligned} &\sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \sum_{p=-\infty}^{+\infty} f(2\pi m, 2\pi n, 2\pi p) \\ &\quad = \frac{1}{(2\pi)^3} \sum_{\mu=-\infty}^{+\infty} \sum_{\nu=-\infty}^{+\infty} \sum_{\xi=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(\tau_1, \tau_2, \tau_3) e^{j(\mu\tau_1 + \nu\tau_2 + \xi\tau_3)} d\tau_1 d\tau_2 d\tau_3 \end{aligned} \tag{3}$$

The usefulness of the theorem in practical problems may be easily overlooked, given the apparently complicated nature of (1)-(3). Nevertheless, the PSF can indeed facilitate the transformation between modal and image series expansions of DGFs occurring in Electromagnetics, as discussed in the Introduction.

A typical case of a successful DGF series acceleration is described in [17], where the rectangular waveguide problem is addressed. The standard (spectral/modal) form for the $\hat{x}\hat{x}$ component of the magnetic vector potential DGF is given [17] by

$$G_{xx}^{A} = \frac{\mu}{2ab} \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{\varepsilon_m \varepsilon_n}{\gamma_{mn}} exp\left(-\gamma_{mn}|z-z'|\right) cos\left(\frac{m\pi x}{a}\right) cos\left(\frac{m\pi x'}{a}\right) sin\left(\frac{n\pi y}{b}\right) sin\left(\frac{n\pi y'}{b}\right) \tag{4}$$

where

$$\varepsilon_i = \begin{cases} 1, & i = 0 \\ 2, & i \neq 0 \end{cases} \tag{4a}$$

and

$$\gamma_{mn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2 \quad . \tag{4b}$$

Furthermore, k, a, b are the medium wavenumber and the dimensions of the waveguide along the x and y axis respectively, whereas μ is the medium permeability.

It is well known that the series in (4) converges very slowly when the source and observation points are located close to each other, rendering the DGF numerical calculation extremely cumbersome. To overcome this difficulty, the two dimensional PSF given in (2) is invoked [21] to transform (4) into its equivalent image/spatial form, i.e.

$$G_{xx}^{A} = \frac{\mu}{4\pi} \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \sum_{i=0}^{3} A_i^{xx} \frac{\exp\left(-jkR_{i,mn}\right)}{R_{i,mn}} \tag{5}$$

where

$$A_i^{xx} = \begin{cases} +1, & i = 0, 2 \\ -1, & i = 1, 3 \end{cases} \tag{5a}$$

and

$$R_{i,mn} = \sqrt{\left(X_i + 2ma\right)^2 + \left(Y_i + 2nb\right)^2 + \left(z - z'\right)^2} \tag{5b}$$

with

$$X_i = \begin{cases} x - x', & i = 0, 1 \\ x + x', & i = 2, 3 \end{cases} \text{ and } Y_i = \begin{cases} y - y', & i = 0, 2 \\ y + y', & i = 1, 3 \end{cases} . \tag{5c}$$

The new expression (5) of the DGF obviously constitutes a superposition of weighted free space scalar Green's functions that are subject to Ewald's technique [18,19]. The series kernel is transformed into a sum of two definite integrals, given in terms of complementary error functions, resulting in a rapidly converging, easily computable series.

A similar procedure has been applied [16] to a rectangular cavity, where the associated DGF also exhibits slow convergence properties. The modal/spectral form for the $\hat{x}\hat{x}$ component of the magnetic vector potential DGF is given [16] by

$$G_{xx}^{A}=\frac{\mu}{abc}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\sum_{p=0}^{+\infty}\frac{\varepsilon_m\varepsilon_n\varepsilon_p}{\alpha_{mnp}^2}cos\left(\frac{m\pi x}{a}\right)cos\left(\frac{m\pi x'}{a}\right)sin\left(\frac{n\pi y}{b}\right)sin\left(\frac{n\pi y'}{b}\right)sin\left(\frac{p\pi z}{c}\right)sin\left(\frac{p\pi z'}{c}\right) \tag{6}$$

where

$$\varepsilon_i=\begin{cases}1, & i=0\\ 2, & i\neq 0\end{cases} \tag{6a}$$

and

$$\alpha_{mnp}^2=\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2+\left(\frac{p\pi}{c}\right)^2-k^2 \tag{6b}$$

Moreover, k, a, b, c are the medium wavenumber and the dimensions of the cavity along the x, y and z axis respectively, whereas μ is the medium permeability.

In a way similar to the rectangular waveguide, the three dimensional PSF (3) is applied according to the methodology described in [21], and (6) is again transformed into its equivalent image/spatial form

$$G_{xx}^{A}=\frac{\mu}{4\pi}\sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty}\sum_{p=-\infty}^{+\infty}\sum_{i=0}^{7}A_i^{xx}\frac{e^{-jkR_{i,mnp}}}{R_{i,mnp}} \tag{7}$$

where

$$A_i^{xx}=\begin{cases}+1, & i=0,3,4,7\\ -1, & i=1,2,5,6\end{cases} \tag{7a}$$

and

$$R_{i,mnp}=\sqrt{(X_i+2ma)^2+(Y_i+2nb)^2+(Z_i+2pc)^2} \tag{7b}$$

with

$$X_i=\begin{cases}x-x', & i=0,1,2,3\\ x+x', & i=4,5,6,7\end{cases}, \tag{7c}$$

$$Y_i = \begin{cases} y - y', & i = 0,1,4,5 \\ y + y', & i = 2,3,6,7 \end{cases}, \tag{7d}$$

and

$$Z_i = \begin{cases} z - z', & i = 0,2,4,6 \\ z + z', & i = 1,3,5,7 \end{cases}. \tag{7e}$$

Again, Ewald's method can be applied to (7), yielding a rapidly convergent equivalent series.

Moreover, Lampe, Klock and Mayes [3] have also utilized the PSF together with Kummer's transformation to rapidly calculate periodic scalar Green's functions associated with structures involving infinite arrays of charge or current sources. Suppose that the periodic Green's function to be computed is of the form $\sum\limits_{n=-\infty}^{+\infty} f(n)$, where $f(n)$ is the respective non-periodic free space Green's function associated with the Helmholtz or the Laplace equation, representing the basic element of the infinite current or charge distribution. Assume that there exists another function, equal to $\sum\limits_{n=-\infty}^{+\infty} g(n)$, which is smooth, periodic and *asymptotically* equal to the original one. If $\sum\limits_{n=-\infty}^{+\infty} G(2\pi n)$ is its equivalent, one-dimensional Poisson Sum, then the original troublesome periodic Green's function is well approximated by

$$\sum_{n=-\infty}^{+\infty} f(n) \approx \sum_{n=-K}^{K} [f(n) - g(n)] + \sum_{n=-\infty}^{+\infty} G(2\pi n) \tag{8}$$

which converges much more rapidly than the original series.

Further applications of the PSF in phased antenna arrays are described in [5-14], where final acceleration is achieved through the suitable combination of both the spectral and the spatial DGF forms. A finite version of the PSF, applied to truncated arrays, is thoroughly discussed in [14].

3. The Poisson Summation Formula in Cylindrical Coordinates

In all above cases the functions involved are expressed in Cartesian coordinates, where the application of the PSF of any dimensionality is straightforward. It would be interesting to investigate the applicability of the PSF to similar problems involving other useful coordinate systems, such as the cylindrical one. If the PSF could be invoked, waveguides and cavities with circular cross-sections could be analyzed in a manner similar to their rectangular counterparts. Unfortunately, such a procedure is not as easy as expected. For example, consider the $\hat{z}\hat{z}$ component of the electric DGF for the circular waveguide [22]

$$G_{zz}(\mathbf{R}, \mathbf{R}') = \frac{-j}{4\pi k^2} \sum_{n=0}^{+\infty} \sum_{m=1}^{+\infty} \frac{(2-\delta_{0n})2\lambda^2}{a^2 J_{n+1}^2(\lambda a) k_z} J_n(\lambda\rho) J_n(\lambda\rho') cos[n(\phi-\phi')] e^{-jk_z|z-z'|} \tag{9}$$

where

$$k_z = \sqrt{k^2 - \lambda^2}, \qquad \lambda = \frac{p_{nm}}{a}, \qquad \delta_{0n} = \begin{cases} 1, & n = 0 \\ 0 & n \neq 0 \end{cases} \tag{9a}$$

and p_{nm} is the m^{th} root of the n^{th} order Bessel function $J_n(\cdot)$, whereas a is the waveguide radius and k is the wavenumber. The major problem in (9), with respect to the standard PSF applicability, is that contributions from the two indices m and n are not clearly separated. Moreover, the Fourier integrals emerging in the procedure cannot be calculated in closed form, mostly due to the elaborate denominator occurring in (9). The main reason of these mathematical complications is the incompatibility of the standard PSF, which possesses an inherent Cartesian character, with the cylindrical coordinate system.

To circumvent this difficulty, a modified version of the PSF can be derived, which is indeed compatible with cylindrical coordinates. The derivation is based on the two-dimensional standard PSF given in (2). Let f be a function of the quantity $r = \sqrt{x^2 + y^2}$ only. Now consider a polar coordinate system (ξ, η) associated with the Cartesian coordinate system (τ_1, τ_2), such that $\xi \equiv \sqrt{\tau_1^2 + \tau_2^2}$ and $\eta \equiv arctan\left(\frac{\tau_2}{\tau_1}\right)$. Moreover, consider a discrete cylindrical coordinate system (ψ, ω) such that $\psi = \sqrt{\mu^2 + \nu^2}$ and $\omega \equiv \arctan\left(\frac{\nu}{\mu}\right)$ (see Figure). It follows that the exponent of the term $e^{j(\mu\tau_1 + \nu\tau_2)}$ in (2) can be written as $j(\mu\tau_1 + \nu\tau_2) = j\xi\eta cos(\eta - \omega)$. Hence, (2) becomes

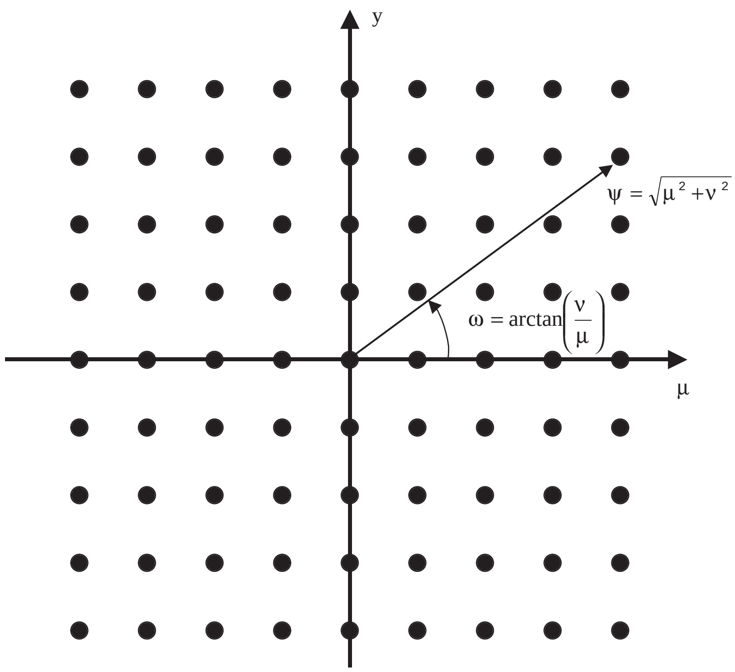

Figure 1. A discrete cylindrical coordinate system (ψ, ω) associated with a discrete Cartesian system (μ, ν).

$$\begin{aligned}
\sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} f\left(2\pi\sqrt{m^2+n^2}\right) &= \frac{1}{(2\pi)^2}\sum_{\mu=-\infty}^{+\infty}\sum_{\nu=-\infty}^{+\infty}\int_0^{+\infty}\int_0^{2\pi} f(\xi)\, e^{j\xi\psi\cos(\eta-\omega)}\xi d\xi d\eta \\
&= \frac{1}{(2\pi)^2}\sum_{\mu=-\infty}^{+\infty}\sum_{\nu=-\infty}^{+\infty}\int_0^{+\infty}\int_0^{2\pi} f(\xi)\, e^{j\xi\psi\cos\eta}\xi d\xi d\eta \\
&= \frac{1}{2\pi}\sum_{\mu=-\infty}^{+\infty}\sum_{\nu=-\infty}^{+\infty}\int_0^{+\infty} \xi f(\xi)\left(\frac{1}{2\pi}\int_0^{2\pi} e^{j\xi\psi cos\eta} d\eta\right) d\xi \\
&= \frac{1}{2\pi}\sum_{\mu=-\infty}^{+\infty}\sum_{\nu=-\infty}^{+\infty}\int_0^{+\infty} \xi f(\xi)\, J_0(\xi\psi)\, d\xi
\end{aligned} \tag{10}$$

where $J_0(\cdot)$ is the Bessel function of order 0. Thus, we conclude that

$$\sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} f\left(2\pi\sqrt{m^2+n^2}\right) = \frac{1}{2\pi}\sum_{\mu=-\infty}^{+\infty}\sum_{\nu=-\infty}^{+\infty} F^B\left(\sqrt{\mu^2+\nu^2}\right) \tag{11}$$

where $F^B(\psi) \equiv \int_0^{+\infty} \xi f(\xi)\, J_0(\xi\psi)\, d\xi$ is the inverse Fourier-Bessel transform of $f(\xi)$ [2,20]. Equation (11) has been written in a compact form to directly compare with (1a).

4. Conclusions

A novel analytical version of the Poisson Summation Formula (PSF) has been derived. This, newly developed, analytical expression which utilizes the Fourier-Bessel transform, involves functions represented in cylindrical coordinates. Therefore, the incompatibility of the conventional PSF with the cylindrical coordinates system is lifted, facilitating the application of the PSF in electromagnetic problems engaging cylindrical geometries, such as the acceleration of the Dyadic Green's Function (DGF) in waveguides and cavities of circular cross-section.

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