Turk J Elec Engin, VOL.10, NO.2 2002, © TÜBİTAK

Aspects of Radar Polarimetry

Ernst LÜNEBURG
EML Consultants, 82234 Wessling-GERMANY
e-mail: e.lueneburg@t-online.de

Abstract

This contribution is a tutorial introduction to the phenomenological theory of radar polarimetry for the coherent scatter case emphasizing monostatic backscattering and forward scattering (transmission). Characteristic similarities and differences between radar polarimetry and optical polarimetry and the role of linear and antilinear operators (time-reversal) are pointed out and typical polarimetric invariants are identified.

Key Words: *radar polarimetry, optical polarimetry, scattering, linear and antilinear operators*

1. Introduction

Polarimetry as part of classical optics started in the seventeenth century with the discovery of polarization by double reflection of Iceland spar by Erasmus Bartholinus. Since then polarization effects, their physical interpretation and technological applications in optical devices are by far too numerous to mention. The history of optical polarimetry is well documented in a variety of monographs, cf. for instance Shurcliff [1], Born and Wolf [2], Collet [3] and Brosseau [4]. For polarized light in nature see Können [5].

Radar polarimetry is the merging of the technological concept of radar (radio detection and ranging) and of the fundamental property of transversality of electromagnetic waves. It started about 40-50 years ago with the pioneering works of Sinclair [6], Graves [7], Kennaugh [8], Deschamps [9], Bickel [10], Booker et al. [11] and others. In particular the work of Kennaugh must be pointed out although most of his work at the Electro-Science Laboratory of the Ohio State University at Columbus, Ohio was classified and de-classified only in the late 1970's. The history of radar polarimetry is described by Boerner [12] where also an extended list of references can be found. The pioneering work in radar polarimetry culminated in the famous 1970 Ph.D. thesis of Richard Huynen [13].

In the last decades the importance and potential of radar polarimetry in problems of remote sensing with real and synthetic aperture radar, inverse scattering, radar meteorology, target recognition and classification, clutter suppression and target decomposition was fully recognized. Unfortunately only a limited number of monographs that cover fundamental aspects of radar polarimetry exist: Mott [14], the NATO ASI's [15, 16] edited by Boerner et al; some still untranslated Russian monographs [Kanareykin et al. [17], Bogorodsky et al. [18]].

During the last decade it became apparent that the full appreciation of polarimetric methods, especially in remote sensing using polarimetric relative phase measurements, is hampered by a lamentable lack of understanding and inconsistencies of basic polarimetric concepts obscuring the inherent polarimetric structures of radar responses. This is related to the problem of properly defining and measuring states of polarization for electromagnetic waves travelling in opposite directions (monostatic radar case), the definition of co- and cross-polarized components and the correct form of polarization basis transformations for coherent and incoherent scatter matrices.

There are several different but related topics in radar polarimetry that give rise to doubts concerning the present-day conventions, treatment, questionable conclusions and incorrect interpretations. These are essentially the following ones:

- Forward Scattering Alignment (FSA) convention versus Backscatter Alignment (BSA) convention;
- The correct voltage and power transfer equations for mono-, antimonostatic and bistatic scattering including its formulation in arbitrary orthonormal polarization bases;
- Change of polarization basis for Sinclair/Kennaugh, Jones/Mueller and bistatic scattering matrices;
- Polarimetric invariants including the Huynen fork for back scattering;
- Bistatic polarization characteristics (co-representations);
- The mathematical and physical interpretation of (relative) phases in mono- and bistatic scattering scenarios.

These problems need first be addressed in the coherent case. The generalization to the incoherent case generally does not involve additional fundamental difficulties. The unsatisfactory present state-of-the-art is reflected also in the fact that the relevant IEEE standards from 1979 and 1983 [19] have not been updated ever since.

Some if not all of these problems can be resolved by introducing the physico-mathematical concept of 'time-reversal' well-known in quantum mechanics as has been proposed by Lüneburg [20, 21]. Related concepts are due to Cloude [22] and Krogager [23] and in the framework of the 2-spinor theory to Bebbington [24] stressing geometrical aspects of radar polarimetry. The mathematical concept of time-reversal appears to be tailor-made for radar polarimetry and offers another example of the intimate interplay between mathematics and physico-engineering sciences.

Unfamiliarity within the engineering community with the antilinear time reversal operator concept enforced the extensive usage of the bilinear form of the voltage equation as a cornerstone of radar polarimetry giving the impression that radar polarimetry is a clever hodge-podge of electromagnetic wave theory and radar network performance. The radar voltage or power transfer equation obscures the implicit application of the time-reversal operation and its correct application and interpretation has become ever since a source of endless and fruitless debates in the literature up to the present time, cf. Kostinski and Boerner [25], Hubbert [26, 27] and Lüneburg [28]. The voltage equation can be applied for general bistatic scattering phenomena, in particular to forward and backscattering, using one and the same formal definition, but completely different conclusions can be derived from it. This indicates that the voltage equation written in its standard conventional form has to be modified or supplemented in order to accomodate different scattering scenarios.

2. The Polarization Ellipse

Plane harmonic monochromatic electromagnetic waves traveling in the direction of the wave vector $\mathbf{k} = k\hat{\mathbf{k}}$ are represented by the real analytical signal

$$\mathcal{E}(\mathbf{r}, t, \mathbf{k}) = \Re\{\mathbf{E}(\mathbf{k}) \exp\{i(\omega t - \mathbf{k} \cdot \mathbf{r})\}. \tag{1}$$

Here, $\mathbf{r}$ and $\mathbf{k}$ are 3–dimensional vectors in $\mathcal{R}^3$ whereas $\mathbf{E(k)}$ is a complex vector in the complexified 2-dimensional subspace $\mathcal{P}(\mathbf{k}) \in \mathcal{C}^2$, perpendicular to $\mathbf{k}$. The phase factor $\tau = \omega t - \mathbf{k} \cdot \mathbf{r}$ implies that the plane wave is propagating in space with increasing time in the direction of the wave vector $\mathbf{k}$. Plane waves are the simplest solutions of the free space wave equation $(\nabla^2 + k^2)\mathbf{E} = 0$ where $k = \omega\sqrt{\epsilon\mu} = 2\pi/\lambda$ with permeability μ and permittivity ϵ is the propagation constant or wave number; $Z = 1/Y = \sqrt{\mu/\epsilon}$ is the intrinsic impedance of the surrounding free space medium.

The time dependence $\exp\{i\omega t\}$ is assumed to be always the same for all directions $\mathbf{k}$ and, hence, can be omitted whenever convenient. This, however, is not true for the entire phase factor τ depending on $\mathbf{k}$. All vectors should be considered as coordinate-free vectors. The choice of a coordinate system is arbitrary but proper choices lead to simplification. The vector $\mathbf{E}(\mathbf{k}) \in \mathcal{P}(\mathbf{k})$ has only two components if an orthonormal right-handed Cartesian coordinate system $\mathcal{B} = \{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ is chosen in such a way that two basis vectors, $\mathbf{e_1}$ and $\mathbf{e_2}$, say, lie in the subspace $\mathcal{P}(\mathbf{k})$. In this case the third basis vector $\mathbf{e_3} = \mathbf{e_1} \times \mathbf{e_2}$ is either parallel or anti-parallel to the propagation vector $\mathbf{k}$: If $\mathbf{e_3}$ is parallel to $\mathbf{k}$ the ordered vectors $\mathbf{e_1}$, $\mathbf{e_2}$, and $\mathbf{k}$ form a right-handed coordinate system, if they are anti-parallel these vectors form a left-handed system. Both possibilities may and are actually used to describe the state of polarization associated with a plane wave. For a fixed value of the e_3-coordinate the endpoint of the electric vector $\mathbf{E}(t)$ describes the so-called polarization ellipse with increasing time. The form of this ellipse and the sense of rotation relative to the direction of propagation (right-hand rule) characterizes the state of polarization. The extension or size of the ellipse as well as the position of the electric vector at a certain reference time are in general not considered parts of the state of polarization.

Due to linear superposition an arbitrary complex vector $\in \mathcal{P}(\mathbf{k})$ assumes the form

$$\mathcal{E}(\mathbf{r}, t, \mathbf{k}) = [E_1(\mathbf{k})\mathbf{e}_1 + E_2(\mathbf{k})\mathbf{e}_2] \exp\{i(\omega t - \mathbf{k} \cdot \mathbf{r})\} \tag{2}$$

$$\doteq \begin{bmatrix} E_1(\mathbf{k}) \\ E_2(\mathbf{k}) \end{bmatrix} \exp\{i(\omega t - \mathbf{k} \cdot \mathbf{r})\} = \mathbf{E}(\mathbf{k}) \exp\{i(\omega t - \mathbf{k} \cdot \mathbf{r})\} \tag{3}$$

(where the component isomorphism is denoted by $\doteq$) with the complex vector components

$$E_1 = |E_1|e^{i\phi_1}, \quad E_2 = |E_2|e^{i\phi_2}. \tag{4}$$

The 2–component vector $\mathbf{E}(\mathbf{k}) \in \mathcal{P}(\mathbf{k})$ is called a Jones vector. This notion was introduced in the 40's by R. Jones [29] in optical polarimetry. Jones vectors are often assumed to be normalized $||\mathbf{E}(\mathbf{k})|| = 1$ and are determined only up to a multiplicative factor of modulus one, i.e. Jones vectors are actually not ordinary 'vectors' but 'rays', a concept introduced in quantum mechanics [30] to emphasize the fact that the absolute phase is not a physical observable.

In our notation we do not distinguish between abstract vectors and vectors of components over the real or complex field whence a basis has been introduced due to the existing natural isomorphism. The vectors of components depend, however, on the chosen basis. In particular any vector $\mathbf{E(k)}$ in the subspace $\mathcal{P}(\mathbf{k})$ is represented by a complex 2–column vector $\in \mathcal{C}^2$, a complex 2-dimensional unitary vector space. This means that inner product and norm are defined as usual: For $\mathbf{x}, \mathbf{y} \in \mathcal{C}^{\mathbf{2}}$ the inner product reads $\mathbf{x} \cdot \mathbf{y} \equiv \mathbf{y}^\dagger \mathbf{x}$ and the norm is $||\mathbf{x}||^{\mathbf{2}} = \mathbf{x}^\dagger \mathbf{x}$ where the dagger symbol † means transposition and complex conjugation. It should be obvious that for different values of the propagation vector $\mathbf{k}$ the vector spaces $\mathcal{P}(\mathbf{k})$ are independent, i.e., Jones vectors from $\mathcal{P}$-spaces with different propagation vectors $\mathbf{k}$ cannot be added nor be used to form inner products. The familiar voltage equation of radar polarimetry involving Jones vectors for opposite directions of propagation is no exception from this rule: it will be shown that it is actually a disguised standard inner product assuming, however, the form of a bilinear Euclidean form in a linear polarization basis.

The state of polarization (including the direction of rotation) is completely determined by the Jones vector representation except the sense (right- or left-handedness) of polarization, i.e., the direction of rotation of the electric vector with respect to the direction of propagation of the plane wave. Often a right-handed triplet $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ with $\hat{\mathbf{k}} = \mathbf{e_3}$, the accompanying tripod, is taken as standard convention. This is called the wave-oriented coordinate system or the 'Forward Scattering Alignment' (FSA) convention. In this case the explicit designation of its dependence on $\mathbf{E}$ on $\mathbf{k}$ is generally omitted or neglected.

In order to derive the shape (locus) of the polarization ellipse we introduce the time varying real fields and use a right-handed coordinate system with $\mathbf{e_1} = \hat{\mathbf{x}}$, $\mathbf{e_2} = \hat{\mathbf{y}}$, $\mathbf{e_3} = \hat{\mathbf{z}} = \hat{\mathbf{x}} \times \hat{\mathbf{y}}$, and $\mathbf{k} \cdot \mathbf{r} = kz$

$$\mathcal{E}(\mathbf{r}, t, \mathbf{k}) = \mathcal{E}_x(t)\hat{\mathbf{x}} + \mathcal{E}_y(t)\hat{\mathbf{y}} \tag{5}$$

with

$$\mathcal{E}_x(t) = |E_x| \cos(\omega t - kz + \phi_x), \qquad \mathcal{E}_y(t) = |E_y| \cos(\omega t - kz + \phi_y). \tag{6}$$

Expansion of the cosine terms and elimination of the term $\omega t - \mathbf{k} \cdot \mathbf{r}$ leads to the parametric equation of an ellipse

$$\frac{\mathcal{E}_x^2(t)}{|E_x|^2} - 2\frac{\mathcal{E}_x(t)\mathcal{E}_y(t)}{|E_x||E_y|} \cos(\phi) + \frac{\mathcal{E}_y^2(t)}{|E_y|^2} = \sin^2(\phi) \tag{7}$$

with the phase difference

$$\phi = \phi_y - \phi_x \equiv \phi_2 - \phi_1, \quad \phi \in \{-\pi < \phi \leq \pi\}. \tag{8}$$

The ellipse equation can be written in the form

$$\mathcal{E}^{\,T}(t) M \mathcal{E}(t) = 1 \tag{9}$$

with the vector $\mathcal{E}(t) = [\mathcal{E}_x(t), \mathcal{E}_y(t)]^T$ and the time-independent matrix

$$M = \begin{bmatrix} a & -b \\ -b & c \end{bmatrix} \tag{10}$$

with the matrix elements

$$a = \frac{1}{\sin^2(\phi)} \frac{1}{|E_x|^2}, \quad b = \frac{1}{\sin^2(\phi)} \frac{\cos(\phi)}{|E_x||E_y|}, \quad c = \frac{1}{\sin^2(\phi)} \frac{1}{|E_y|^2}. \tag{11}$$

Since M is real symmetric and hence normal there is a real orthogonal matrix Q such that

$$QMQ^{\,T} = \mathrm{diag}[\lambda_1, \lambda_2] \tag{12}$$

is a diagonal matrix with real eigenvalues λ_1 and λ_2. Equation (9) implies that these eigenvalues are positive cf. Horn and Johnson [31, 32].

The eigenvalues of the matrix (10) read

$$\lambda_{1,2} = \frac{1}{2}\left\{\operatorname{trace} M \pm \sqrt{\operatorname{trace}^2 M - 4\det M}\right\} \tag{13}$$

where

$$\operatorname{trace} M = \lambda_1 + \lambda_2 = a + c = \frac{|E_x|^2 + |E_y|^2}{\sin^2(\phi)|E_x|^2|E_y|^2} \tag{14}$$

$$\det M = \lambda_1\lambda_2 = ac - b^2 = \frac{1}{\sin^2(\phi)}\frac{1}{|E_x|^2|E_y|^2}. \tag{15}$$

This implies the relations

$$\begin{aligned} b^2 &= ac - \lambda_1\lambda_2 = a(a+c) - a^2 - \lambda_1\lambda_2 = a(\lambda_1+\lambda_2) - a^2 - \lambda_1\lambda_2 = (\lambda_1 - a)(a - \lambda_2) \\ &= c(a+c) - c^2 - \lambda_1\lambda_2 = c(\lambda_1+\lambda_2) - c^2 - \lambda_1\lambda_2 = (\lambda_1 - c)(c - \lambda_2) \end{aligned} \tag{16}$$

The normalized orthogonal eigenvectors

$$M\mathbf{x}_i = \lambda_i \mathbf{x}_i \qquad (i = 1, 2) \tag{17}$$

are given by

$$\mathbf{x}_1 = \frac{1}{N}\begin{bmatrix} -b \\ \lambda_1 - a \end{bmatrix} = \begin{bmatrix} -\sin(\tau) \\ \cos(\tau) \end{bmatrix}, \quad \mathbf{x}_2 = \frac{1}{N}\begin{bmatrix} \lambda_1 - a \\ b \end{bmatrix} = \begin{bmatrix} \cos(\tau) \\ \sin(\tau) \end{bmatrix} \tag{18}$$

where

$$\sin(\tau) = \frac{\sqrt{|\sin^2(\alpha) - \sin^2(\epsilon)|}}{\sqrt{\cos(2\epsilon)}}, \quad \cos(\tau) = \frac{\sqrt{|\cos^2(\epsilon) - \sin^2(\alpha)|}}{\sqrt{\cos(2\epsilon)}} \tag{19}$$

Here the following standard definitions have been used

$$\sin(2\epsilon) = \sin(2\alpha)\sin(\phi), \quad \tan(2\tau) = \tan(2\alpha)\cos(\phi) \tag{20}$$

with

$$\sin(\alpha) = \frac{|E_y|}{||\mathbf{E}||}, \quad \cos(\alpha) = \frac{|E_x|}{||\mathbf{E}||} \qquad \left(||\mathbf{E}|| = \sqrt{|E_x|^2 + |E_y|^2}\right) \tag{21}$$

and, hence,

$$\sin(2\alpha) = \frac{2|E_x||E_y|}{||\mathbf{E}||^2}, \ \cos(2\alpha) = \frac{|E_x|^2 - |E_y|^2}{||\mathbf{E}||^2}, \ \tan(2\alpha) = \frac{2|E_x||E_y|}{|E_x|^2 - |E_y|^2}, \tag{22}$$

where ϵ is the ellipticity angle and τ the tilt or orientation angle of the polarization ellipse.

It can easily be shown that

$$\lambda_1 - \lambda_2 = (\lambda_1 + \lambda_2)\cos(2\epsilon) \quad \text{and} \quad \lambda_1 + \lambda_2 = \frac{1}{||\mathbf{E}||^2}\frac{1}{\sin^2(\epsilon)\cos^2(\epsilon)} \tag{23}$$

and, hence,

$$\lambda_1 = \frac{1}{||\mathbf{E}||^2 \sin^2(\epsilon)}, \quad \lambda_2 = \frac{1}{||\mathbf{E}||^2 \cos^2(\epsilon)} \tag{24}$$

and

$$r_{\min} = \frac{1}{\sqrt{\lambda_1}} = ||\mathbf{E}|| \, |\sin(\epsilon)|, \quad r_{\max} = \frac{1}{\sqrt{\lambda_2}} = ||\mathbf{E}|| \cos(\epsilon). \tag{25}$$

Here $r_{\min}$ denotes the minor half-axis and $r_{\max}$ the major half-axis of the polarization ellipse. This implies that the ellipticity angle ϵ is determined by the axial ratio of the polarization ellipse

$$\tan(\epsilon) = \frac{r_{\min}}{r_{\max}} \tag{26}$$

The orthogonal rotation matrix

$$Q = \begin{bmatrix} \cos(\tau) & -\sin(\tau) \\ \sin(\tau) & \cos(\tau) \end{bmatrix} \quad \text{with} \quad QQ^T = Q^TQ = I \tag{27}$$

transforms the matrix M into diagonal form Σ

$$Q^T M Q = \Sigma = \mathrm{diag}[\lambda_2, \lambda_1] \tag{28}$$

and the vector $\vec{\mathcal{E}}$ into the new variables

$$\vec{\mathcal{E}}\,' = Q^T \vec{\mathcal{E}} \quad \text{with} \quad ||\vec{\mathcal{E}}|| = ||\vec{\mathcal{E}}\,'||, \tag{29}$$

i.e.

$$\vec{\mathcal{E}}^{\,T} M \vec{\mathcal{E}} = \vec{\mathcal{E}}'^{T} \Sigma \, \vec{\mathcal{E}}' = \lambda_1 |\mathcal{E}_x\,'|^2 + \lambda_2 |\mathcal{E}_y\,'|^2 = \left\{ \frac{|\mathcal{E}_x\,'|}{r_{\min}} \right\}^2 + \left\{ \frac{|\mathcal{E}_y\,'|}{r_{\max}} \right\}^2 = 1. \tag{30}$$

The sense of rotation can be derived by forming the vector product

$$\frac{\partial}{\partial t}\mathcal{E}(t) \times \mathcal{E}(t) = \omega \, |E_x||E_y| \sin(\phi) \hat{\mathbf{z}}. \tag{31}$$

Looking into the direction of $\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2$, the rotation is counter-clockwise, i.e. from $\hat{\mathbf{e}}_2$ to $\hat{\mathbf{e}}_1$ if and only if $\sin(\phi) > 0$, i.e. $0 < \phi < \pi$. The sense of rotation is clockwise if and only if $\sin(\phi) < 0$, i.e. $-\pi < \phi < 0$. Hence, the sense of rotation is independent from the direction of propagation which can still be either $\mathbf{k} = \hat{\mathbf{e}}_3$ or $\mathbf{k} = -\hat{\mathbf{e}}_3$ and follows directly from the Jones vector and the general assumed time dependence (as related to the phases $\phi_{1,2}$). This ambiguity is resolved by introducing the concept of handedness. The polarization is said to be right-handed if the rotation as time increases is clockwise looking into the direction of wave propagation (such a wave is said to have positive helicity) and is said to be left-handed if the rotation is counter-clockwise (negative helicity). Summarizing

- $\mathbf{k} = \hat{\mathbf{e}}_3 = \hat{\mathbf{z}} = \hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{x}} \times \hat{\mathbf{y}}$

$$\text{Polarization is} \left\{ \begin{array}{c} \text{left-handed} \\ \text{right-handed} \end{array} \right\} \text{ if and only if } \left\{ \begin{array}{c} 0 < \phi < \pi \\ -\pi < \phi < 0 \end{array} \right\} = \left\{ \begin{array}{c} \epsilon > 0 \\ \epsilon < 0 \end{array} \right\} \tag{32}$$

- $\mathbf{k} = -\hat{\mathbf{e}}_3 = -\hat{\mathbf{z}} = -\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = -\hat{\mathbf{x}} \times \hat{\mathbf{y}}$

$$\text{Polarization is} \left\{ \begin{array}{c} \text{left-handed} \\ \text{right-handed} \end{array} \right\} \text{ if and only if } \left\{ \begin{array}{c} -\pi < \phi < 0 \\ 0 < \phi < \pi \end{array} \right\} = \left\{ \begin{array}{c} \epsilon < 0 \\ \epsilon > 0 \end{array} \right\} \tag{33}$$

The inclusion of the sign of ϕ as sign of the ellipticity angle ϵ is common usage and is most convenient. It should be noted that in optics the sense of rotation is defined differently increasing the often prevailing confusion.

Linear polarizations are characterized by $\epsilon = 0$ or $\phi = 0 \mod(\pi)$. In this case $\tau = \alpha$. If $\cos\phi = 0$ or $\phi = \pm\pi/2$ the matrix M is already in diagonal form and the orientation angle $\tau = 0$. If additionally also $|\epsilon| = 1$ we have circular polarization. In this case the matrix M is scalar (a multiple of the unit matrix) and τ becomes undetermined: any rotation leaves a scalar matrix invariant. For the right and left orthogonal circular polarizations the normalized Jones vectors assume the form

- $\mathbf{k} = \hat{\mathbf{e}}_3 = \hat{\mathbf{z}} = \hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{x}} \times \hat{\mathbf{y}}$

$$\mathbf{e}_{rc} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} \text{ (right-circular);} \quad \mathbf{e}_{lc} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} \text{ (left-circular)} \tag{34}$$

- $\mathbf{k} = -\hat{\mathbf{e}}_3 = -\hat{\mathbf{z}} = -\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = -\hat{\mathbf{x}} \times \hat{\mathbf{y}}$

$$\mathbf{e}_{rc} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} \text{ (right-circular);} \quad \mathbf{e}_{lc} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} \text{ (left-circular)} \tag{35}$$

Nearly all of the recommendations expressed in the IEEE 1983 Standard Definitions of Terms for Antennas [19] are still highly valuable but some of them caused profound misunderstandings and hampered theoretical radar polarimetry over the last decades. These definitions also may have contributed to the poor acceptance of Graves's visionary concept of directional Jones vectors formulated already in 1956 [7]. In more general terms it is the profound difference between the Forward Scattering Alignment (FSA) convention and the generalized Backscatter Alignment (BSA) convention which are important also for the bistatic scattering case.

Essentially there is one concept that requires reconsideration due to its far-reaching consequences. This is the connection of the tilt angle with the direction of propagation, i.e., the orientation of the major axis of the polarization ellipse with respect to a line of reference. According to IEEE Standard 1983 [19] the tilt angle (of a polarization ellipse) is defined in the following way:

> When the plane of polarization is viewed from a specified side, the angle measured clockwise from a reference line to the major axis of the ellipse. Notes:
> (1) For a plane wave the plane of polarization shall be viewed looking in the direction of propagation;
> (2) The tilt angle is only defined up to a multiple of π radians and is usually taken in the range $(-\pi/2, +\pi/2)$ or $(0, \pi)$.

This definition makes the mathematical formulation of the relation between the representation of the polarization properties of a transmitting and a receiving antenna difficult and confusing since according to the IEEE 1983 Standard **the tilt angle changes into its negative value when the direction of propagation is reversed**.

It should also be mentioned that the expression for the voltage (power transfer) equation and the proper definition of co-polar polarization (radiation pattern) is not given in the IEEE 1983 Standards [19]. The cross-polar polarization (radiation) pattern) on the other hand always corresponds to the polarization orthogonal to the co-polarization. Anticipating results to be introduced later on in the context of time-reversal nowadays the term co-polarized polarization (radiation) of a plane wave refers to the polarizations of two plane waves propagating in opposite direction. Their polarizations are said to be co-polar if they have the same state of polarization, i.e. the same polarization ellipse (locus) but opposite sense of rotation. In particular this refers to the monostatic back scatter case. This terminology coincides with the IEEE Standard definition [19] for an antenna for transmission and reception:

> Polarization (receiving (of an antenna)). That polarization of a plane wave, incident from a given direction and having a given power flux density, which results in maximum available power at the antenna terminals.
>
> - Notes: (1) The receiving polarization of an antenna is related to the antenna's polarization on transmit (see definition above) in the following way: In the same plane of polarization, the polarization ellipses have the same axial ratios, the same sense of polarization and the same spatial orientation. Since their senses of polarization and spatial orientation are specified by viewing their polarization ellipses in the respective directions into which they are propagating, one should note that (a) although their senses of polarization are the same, they would appear to be opposite if both waves were viewed in the same direction; (b) their tilt angles are such that they are the negative of one another with respect to a common reference.

Part (b) of these Notes (1) applied to the backscatter case is based upon the crucial assumption that the direction of propagation $\mathbf{k}$ is changed continuously by a rotation around one of the axes perpendicular to it (the y-axis, say) from the original position to $-\mathbf{k}$ carrying with it its accompanying tripod of wave-oriented coordinate vectors. These continuous rotations correspond to linear operators, i.e. matrix representations of the S0(3) Lie group. This implies the introduction two right-handed cartesian coordinate systems for the wave transmitted by the antenna and the incident plane wave that is best received, both propagating in opposite directions, as detailed in Section 3.

Conceptually there is only one polarization for the transmit and the receive case for a reciprocal antenna. What changes, however, is their representations. Even if one and the same coordinate system is used there will be a difference due to the different directions of propagation to which they refer. This is different from ordinary three-dimensional vectors. Their polarization ellipse are the same including the sense of polarization. Since both plane waves are propagating in opposite directions the rotation of the electric vector is opposite when both ellipses are contemplated from one and the same point of observation. It is always possible to go over from two different cartesian coordinate system to one common cartesian system. This is requested for an application of the so-called voltage equation, see Mott [14]; further details are presented in Section 6.

The difficulty with the IEEE 83 Standard Definition is the extension of the definition of transmit/receive antennas from Cartesian coordinates to general (orthonormal) elliptical polarization bases. Cartesian or more generally linear polarization bases are independent from the direction of propagation of the plane waves they describe apart from the subtle question of right- or left-handedness (a different although related topic) whereas a general elliptical polarization basis involves explicitly the direction of propagation of the wave. These difficulties with the IEEE convention have also clearly been pointed out by Feinstein [33] although we do not agree with his conclusions.

All this leads to the mathematically very unsatisfactory effect that (i) the domain and the range of the scattering matrix do not belong to the same propagation space, $\mathcal{P}(\mathbf{k})$ versus $\mathcal{P}(-\mathbf{k})$, and (ii) they also refer to different coordinate systems. Both these drawbacks prohibit the straightforward application of eigenvector analysis for finding optimal states of polarizations. The solution of this difficult conundrum is the 'time inversion (reversion)' concept and the associated complex conjugation operation. In this respect the report may also be of interest for a requested forthcoming revision of the 1983 IEEE Standard Definitions of Terms for Antennas [19].

3. Time Reversal and Antilinear Transformations

In this section we are primarily concerned with the time reversal operation and the associated antilinear transformations. The time evolution of a physical system is described by its time-dependent trajectory or path (coordinates and velocities or momenta). Time reversal, denoted by $\mathcal{T}$, is described by considering a set of trajectories from the point of view of two different space-time coordinate systems, related by a reflection of the time coordinate: $t \rightarrow \bar{t} = -t$ (and $\mathbf{x} \rightarrow \bar{\mathbf{x}} = \mathbf{x}$). Given a trajectory described by coordinates $\mathbf{x}(t)$ and velocities $\dot{\mathbf{x}}(t)$ as functions of time t in the first frame, the time reversed trajectory will be obtained by the same coordinate values $\mathcal{T}\mathbf{x}(t) = \mathbf{x}(-t)$ traced in the opposite time direction with reversed velocity values $\mathcal{T}\dot{\mathbf{x}}(t) = -\dot{\mathbf{x}}(-t)$. The time-reversed trajectory is traced out forwardly in the new time $\bar{t}$ or towards the 'coordinate past' in the t-system. This latter result is the origin for the term *time reversal*. This does not imply that physical systems evolves towards the past, see Doughty [34].

Classical electrodynamics are time-reversal covariant, i.e. the form of the equations remain invariant under $\mathcal{T}$. The new trajectories will be physical trajectories of the original system. The time-reversal operation in quantum systems was first formulated by Wigner [35] in 1932, see also Wigner [36], Gottfried [30], Herbut and Vujičić [37] and Ohnuki [38].

The following reasoning using a linear polarization basis is convenient for the application of time reversal operation. We consider a plane electromagnetic wave propagating in the positive z-direction, cf. Equation (1)

$$\Re\mathbf{E}^{+}(t,z) = \Re\begin{bmatrix} E_x^{+}(t,z) \\ E_y^{+}(t,z) \end{bmatrix} = \Re\begin{bmatrix} E_x^{+}e^{i(\omega t - kz)} \\ E_y^{+}e^{i(\omega t - kz)} \end{bmatrix} = \begin{bmatrix} |E_x^{+}|\cos(\omega t - kz + \phi_x) \\ |E_y^{+}|\cos(\omega t - kz + \phi_y) \end{bmatrix} \tag{36}$$

Time reversal $\mathcal{T} : t \rightarrow -t$ implies

$$\Re\mathbf{E}^{+}(t,z) \rightarrow \Re\mathbf{E}^{+}(-t,z) =: \Re\mathbf{E}^{-}(t,z) = \Re\begin{bmatrix} E_x^{-}e^{i(\omega t + kz)} \\ E_y^{-}e^{i(\omega t + kz)} \end{bmatrix} = \tag{37}$$

$$= \begin{bmatrix} |E_x^{+}|\cos(\omega t + kz - \phi_x) \\ |E_y^{+}|\cos(\omega t + kz - \phi_y) \end{bmatrix} = \Re\begin{bmatrix} E_x^{+*}e^{i(\omega t + kz)} \\ E_y^{+*}e^{i(\omega t + kz)} \end{bmatrix} = \Re\begin{bmatrix} E_x^{-}(t,z) \\ E_y^{-}(t,z) \end{bmatrix} \tag{38}$$

The Hilbert transform $\mathcal{H}$ (cf. Brosseau [4]) of the cosine term (real part) yields the corresponding sine term (imaginary part) which together lead to the exponential or analytic signal representation according to $\cos(x) + i\mathcal{H}(\sin(x)) = e^{ix}$. A comparison yields the following relation between the complex amplitudes of plane waves propagating in opposite directions which are said to posses the same state of polarization

$$\mathbf{E}^{-} = \begin{bmatrix} E_x^- \\ E_y^- \end{bmatrix} = \begin{bmatrix} E_x^{+*} \\ E_y^{+*} \end{bmatrix} = \mathbf{E}^{+*}, \tag{39}$$

where a common linear $\{x, y\}$ polarization basis is used.

Let us consider the general orthonormal elliptic polarization basis $\mathcal{B}_+ = \{\mathbf{b}_1^+, \mathbf{b}_2^+\}$. The basis vectors $\mathbf{b}_1^+$ and $\mathbf{b}_2^+$ assume the following form in the linear orthonormal polarization basis $\mathcal{E} = \{\mathbf{e}_x, \mathbf{e}_y\}$

$$\mathbf{b}_1^+ = a\mathbf{e}_x + b\mathbf{e}_y, \quad , \mathbf{b}_2^+ = -b^*\mathbf{e}_x + a^*\mathbf{e}_y \tag{40}$$

or in components

$$[\mathbf{b}_1^+]_{\mathcal{E}} = \begin{bmatrix} a \\ b \end{bmatrix}, [\mathbf{b}_2^+]_{\mathcal{E}} = \begin{bmatrix} -b^* \\ a^* \end{bmatrix}, \quad \mathbf{b}_1^{+\dagger}\mathbf{b}_2^+ \equiv \mathbf{b}_1^{+*} \cdot \mathbf{b}_2^+ = 0, \quad |a|^2 + |b|^2 = 1. \tag{41}$$

The column vector of the basis vectors $\mathbf{b}_1^+$ and $\mathbf{b}_2^+$ reads

$$\begin{bmatrix} \mathbf{b}_1^+ \\ \mathbf{b}_2^+ \end{bmatrix} = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} = U \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \end{bmatrix} \tag{42}$$

with the unitary matrix U $U^\dagger U = I$.

Any Jones vector $\mathbf{x}^+$ pertaining to propagation in the $+z$-direction may be written in the linear basis $\mathcal{E}$ and in the elliptic basis $\mathcal{B}^+$ as

$$\mathbf{p}^+ = p_x^+\mathbf{e}_x + p_y^+\mathbf{e}_y = p_1^+\mathbf{b}_1^+ + p_2^+\mathbf{b}_2^+ \tag{43}$$

or

$$[\mathbf{p}^+]_{\mathcal{E}} = \begin{bmatrix} p_x^+ \\ p_y^+ \end{bmatrix}, \quad [\mathbf{p}^+]_{\mathcal{B}^+} = \begin{bmatrix} p_1^+ \\ p_2^+ \end{bmatrix}. \tag{44}$$

Inserting the expressions for $\mathbf{b}_1$ and $\mathbf{b}_2$ and a comparison yields

$$\mathbf{p}^+|_{\mathcal{E}} = \begin{bmatrix} p_x^+ \\ p_y^+ \end{bmatrix} = \begin{bmatrix} a & -b^* \\ b & a^* \end{bmatrix} \begin{bmatrix} p_1^+ \\ p_2^+ \end{bmatrix} = U^T\mathbf{p}^+|_{\mathcal{B}^+}, \tag{45}$$

i.e. the contragredient transformation with respect to Equation (42).

The vector $\mathbf{p}^-$ propagating in the negative z-direction and having the same polarization as $\mathbf{p}^+$, $\mathrm{Pol}(\mathbf{p}^-) = \mathrm{Pol}(\mathbf{p}^+)$, is given as

$$\mathbf{p}^- = p_x^-\mathbf{e}_x + p_y^-\mathbf{e}_y = p_1^-\mathbf{b}_1^- + p_2^-\mathbf{b}_2^- \tag{46}$$

where

$$\mathbf{p}^-|_{\mathcal{E}} = \begin{bmatrix} p_x^- \\ p_y^- \end{bmatrix} = \begin{bmatrix} p_x^{+*} \\ p_y^{+*} \end{bmatrix} = \mathbf{p}_{\mathcal{E}}^{+*}. \tag{47}$$

On the other hand in the linear polarization basis

$$\mathbf{b}_1^-|_{\mathcal{E}} = \mathbf{b}_1^{+*}|_{\mathcal{E}} = \begin{bmatrix} a^* \\ b^* \end{bmatrix}, \quad \mathbf{b}_2^-|_{\mathcal{E}} = \mathbf{b}_2^{+*}|_{\mathcal{E}} = \begin{bmatrix} -b \\ a \end{bmatrix}. \tag{48}$$

A comparison yields

$$\mathbf{p}^-|_{\mathcal{E}} = \begin{bmatrix} p_x^- \\ p_y^- \end{bmatrix} = \begin{bmatrix} a^* & -b \\ b^* & a \end{bmatrix} \begin{bmatrix} p_1^- \\ p_2^- \end{bmatrix} = U^\dagger \mathbf{p}^-|_{\mathcal{B}^-}. \tag{49}$$

Hence, starting from the abstract directional Jones vector in the $\{\mathbf{b}_1^+, \mathbf{b}_2^+\}$-basis pertaining to propagation in the $+z$-direction

$$\mathbf{p}^+ = p_1^+ \mathbf{b}_1^+ + p_2^+ \mathbf{b}_2^+ \tag{50}$$

the application of the time-reversal operation $\mathcal{T}$ yields a vector with the same polarization as $\mathbf{p}^+$

$$\mathbf{p}^- = p_1^- \mathbf{b}_1^- + p_2^- \mathbf{b}_2^- = \mathcal{T}\mathbf{p}^+ = \mathcal{T}\left[p_1^+ \mathbf{b}_1^+ + p_2^+ \mathbf{b}_2^+\right] = p_1^{+*} V \mathbf{b}_1^+ + p_2^+ V \mathbf{b}_2^+ \tag{51}$$

where $V\mathbf{p_i}^+ = \mathbf{p}_i^-$ $(i = 1, 2)$. The time-reversal operator $\mathcal{T}$ maps any polarization state into its motion-reversed counterpart that by definition has the same polarization. This transition can be accomplished by means of an unitary operator V:

$$V\mathbf{p}_i^+ = \mathbf{p}_i^- \qquad (i = 1, 2) \tag{52}$$

with the basis-dependent fundamental transition matrix

$$V = \sum_{i=1}^{2} \mathbf{p}_i^- \mathbf{p}_i^{+\dagger}. \tag{53}$$

With the given expressions for $\mathbf{p}_i^\pm$ the matrix V expressed in the linear polarization basis $\mathcal{E}$ reads

$$V = \begin{bmatrix} a^* \\ b^* \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}^\dagger + \begin{bmatrix} -b \\ a \end{bmatrix} \begin{bmatrix} -b^* \\ a^* \end{bmatrix}^\dagger = \begin{bmatrix} a^{*2} + b^2 & a^*b^* - ab \\ a^*b^* - ab & a^2 + b^{*2} \end{bmatrix} \quad (VV^\dagger = I). \tag{54}$$

The matrix V is unimodular $(\det V = 1)$, unitary and symmetric, hence coninvolutory $VV^* = i$.

The time-reversal operator thus assumes the representation

$$\mathcal{T} = VK \tag{55}$$

which means that K is to take the complex conjugate of all the expansion coefficients of the arbitrary Jones vector in terms of the particular polarization basis with which one is working, and which must be known if V is to be specified explicitly. Therefore, the time-reversal operator $\mathcal{T}$ is *anti-unitary.* Application of $\mathcal{T}$ to the expansion

$$\mathbf{p}^+ = p_1^+ \mathbf{b}_1^+ + p_2^+ \mathbf{b}_2^+ \tag{56}$$

yields

$$\mathcal{T}\mathbf{p}^+ = \mathbf{p}^- = p_1^{+*} V \mathbf{b}_1^+ + p_2^{+*} V \mathbf{b}_2^+ = p_1^- \mathbf{b}_1^- + p_2^- \mathbf{b}_2^- \tag{57}$$

or in components

$$\mathcal{T}\mathbf{p}^+|_{\mathcal{B}^-} = \mathbf{p}^-|_{\mathcal{B}^-} = \begin{bmatrix} p_1^- \\ p_2^- \end{bmatrix}. \tag{58}$$

An arbitrary change of orthonormal polarization basis with the unimodular unitary matrix W reads

$$\begin{bmatrix} a \\ b \end{bmatrix} = W \begin{bmatrix} a' \\ b' \end{bmatrix} \quad W^{\dagger} = I,\ \det W = 1. \tag{59}$$

Using the relation

$$ZWZ^T = \det W\, W^* = \det W\, W^{-1T} \tag{60}$$

it follows from Equation (54) that

$$V' = W^T V W \qquad \text{(unitary consimilarity).} \tag{61}$$

This transformation conserves symmetry, unimodularity, unitarity as well as coninvolutority of the new transition matrix $V' : V'^T = V',\ V'V'^* = I$. This emphasizes the previous statement that the explicit form of the unitary matrix in the time reversal operator can be stated only after a set of basis vectors has been chosen.

The fact that the matrix V indeed depends upon the polarization basis actually used will be demonstrated by the following examples:

-
$$V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \equiv I \quad \text{for the linear } \{\mathbf{h}, \mathbf{v}\} \text{ polarization basis;} \tag{62}$$

-
$$V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} I \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \equiv -iB \tag{63}$$

  for a right/left circular polarization basis where $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is the backward identity matrix;

-
$$V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} I \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \equiv iB \tag{64}$$

  for a left/right circular polarization basis.

The first relation is actually true for any linear polarization basis. The simple transformation $V = I$ for basis transformations under the time-reversal points out the advantage for using a linear polarization basis for this operation. Often the alternatives are not even mentioned nor discussed notwithstanding the fact that circularly polarized radars are in use with success.

The general Jones vector has the form

$$\mathbf{E} = \begin{bmatrix} E_h \\ E_v \end{bmatrix} = \begin{bmatrix} |E_h|^{i\phi_h} \\ |E_v|e^{i\phi_v} \end{bmatrix}$$

where by the direction of propagation of the plane wave is $\hat{\mathbf{k}} = \hat{\mathbf{h}} \times \hat{\mathbf{v}}$ (right-handed coordinate system). If the amplitude ratio E_v/E_h is real the wave is linearly polarized with orientation angle $\psi = \arctan\{E_V/E_H\}$. If for instance $E_v = E_h$ then $\phi = \pi/4$ or $\phi = 5^o$ and if $E_v = -E_h$ then $\phi = -\pi/4$ or $\phi = -45^o$.

Thus the Jones vector

$$\mathbf{E}_{45} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{65}$$

represents a signal whose electric vector oscillates in the $\{\hat{\mathbf{h}},]hat\mathbf{v}\}$ plane along a line at 45 degrees to the positive $\hat{\mathbf{h}}$- and $\hat{\mathbf{v}}$-axes for forward propagation $\hat{\mathbf{k}} = \hat{\mathbf{h}} \times \hat{\mathbf{v}}$.

For backward propagation the 2-dimensional $\{\hat{\mathbf{h}}, \hat{\mathbf{v}}\}$-system and the propagation vector $\mathbf{k}$ are no longer right-handed but left-handed. A right-handed system can be obtained by rotating the coordinate system 180^o around the vector $\hat{\mathbf{v}}$, then the rotation axis remains invariant whereas $\hat{\mathbf{h}} \to -\hat{\mathbf{v}}$ and of course $\hat{\mathbf{k}} \to -\hat{\mathbf{k}}$. Other choices are of course possible. We denote the new coordinate system with a dash

$$\hat{\mathbf{v}}' = -\hat{\mathbf{v}}, \quad \hat{\mathbf{h}}' = \hat{\mathbf{h}}. \tag{66}$$

From

$$\mathbf{E} = E_h\hat{\mathbf{h}} + E_v\hat{\mathbf{v}} = E_h{}'\hat{\mathbf{h}}' + E_v{}'\hat{\mathbf{v}}' = -E_h{}'\hat{\mathbf{h}} + E_v{}'\hat{\mathbf{v}} \tag{67}$$

follows

$$\mathbf{E}_{45}{}' = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \tag{68}$$

This is still the same linear polarization in physical space but now $\psi' = -45$ degree by definition. The representation for an orientation angle of $+45^0$ now would correspond to the expression for -45^o in the old system. This change of coordinates is a pure rotation of the coordinate axes $\in SO(3)$. This is the FSA convention. In the BSA convention one and the same expression is used for both directions of propagation. No change and choice of change of coordinate system is used (i.e. the question of right- or left-handed coordinate system in $\mathcal{R}^3$ is obsolete) but the direction of propagation is denoted by a tag. Time reversal $t \to -t$ has no effect in this case.

For circular polarizations we have for $\hat{\mathbf{k}} = \hat{\mathbf{h}} \times \hat{\mathbf{v}}$

$$\mathbf{E}_{rc} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} \text{ (right-circular);} \quad \mathbf{E}_{lc} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} \text{ (left-circular)} \tag{69}$$

and for propagation in the opposite direction (but keeping the same coordinate system) $\hat{\mathbf{k}} = -\hat{\mathbf{h}} \times \hat{\mathbf{v}}$

$$\mathbf{E}_{rc} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} \text{ (right-circular);} \quad \mathbf{E}_{lc} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} \text{ (left-circular)} \tag{70}$$

If we now make the same change of coordinate systems as before we arrive at the correct representations for forward and back scattering (apart from an unimportant common sign factor). But the same result is obtained by using the complex conjugation operation of the time reversal operation.

A general elliptical polarization lies between the two extremes of linear and circular polarizations. Neither the BSA convention (using always a right-handed accompanying tripod of coordinates) nor the BSA convention (based upon the time-reversal operation) keep the representations of one and the same state of polarization for plane waves propagating in opposite directions 'form invariant'. The latter one accomplishes the reduction to only one propagation space simply by complex conjugation and allows the direct application of the powerful tools of matrix analysis for the calculation of optimal states of polarization.

4. Basic Equations of Radar and Optical Polarimetry

Choosing a **common** cartesian coordinate system $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$ for the domain and range of the scattering matrix S the forward and backward scattering processes can both be described by the following equations for radar **and** optical polarimetry

$$\mathbf{E}^{s} = S\mathbf{E}^{i} \qquad \text{(field equation)} \tag{71}$$

$$V = \mathbf{E}^{s} \cdot \mathbf{h} \qquad \text{(voltage equation).} \tag{72}$$

where 's' and 'i' denotes scattering and incidence. These relations are considered as **cornerstones** of the Theory of Polarimetry. The rather unusual role and importance of the voltage equation should be noted.

We note that for radar backscatter the generally complex 2×2 matrix S, the so-called Sinclair (scattering) matrix, is symmetric: $S = S^T$. For forward scattering or transmission the matrix S is often denoted by J and is called the Jones matrix. The Jones matrix J is in general not symmetric. Many important elementary optical devices have Jones matrices that are normal: $JJ^\dagger = J^\dagger J$ where $^\dagger \equiv {}^{T*}$.

The voltage equation refers to measurements of target characteristics and seems to indicate a relation between basic electromagnetic theory and field descriptors on one side and receiver network performance on the other side. This point of view only exists in radar polarimetry and has hampered research for many years distracting attention of the radar community from the profound differences existing between optical transmission polarimetry and radar backscatter polarimetry.

Let us first explain the role of polarimetric unitary basis or coordinate transformation given by

$$\mathbf{E}^{i,s} \to U\mathbf{E}'^{\,i,s}, \quad \mathbf{h} \to U\mathbf{h}' \quad \text{(U unitary).} \tag{73}$$

Unitarity follows from the requirement of norm invariance: $||\mathbf{E}^{i,s}|| = ||\mathbf{E}'^{\,i,s}||$. Application of these transformations in the field equation apparently yields

$$S \to S' = U^{-1}SU \quad \text{(unitary similarity)} \tag{74}$$

and in the voltage equation

$$S \to S' = U^{T}SU \quad \text{(unitary consimilarity).} \tag{75}$$

This discrepancy has been the source of endless and fruitless debates up to the present time, cf. Huynen [13], Kostinski and Boerner [25], Hubbert [26, 27] and Lüneburg [28]. Obviously these equations require a more detailed notion to resolve these inconsistencies. Considering propagation along the positive and negative z-axis we write for Radar Polarimetry (RP) and Optical transmission or Polarimetry (OP)

Field equations

$$\mathbf{E}^{s\,-} = S\mathbf{E}^{i\,+} \qquad \text{(RP with Sinclair matrix S),} \tag{76}$$

$$\mathbf{E}^{s\,+} = J\mathbf{E}^{i\,+} \qquad \text{(OP with Jones matrix J).} \tag{77}$$

The generic symbols $+$ and $-$ indicate opposite directions of propagation and imply that domain and range of the scattering matrices S and J belong to the different propagation vector spaces $\mathcal{P}(\mathbf{z})$ and $\mathcal{P}(-\mathbf{z})$. They denote directional Jones vectors as introduced by Graves [7] as early as 1956 but nearly never employed afterwards until their relation with the time reversal operation was emphasized by Lüneburg [20, 21]. The antilinear **Time Reversal** $\mathcal{T}$ operation (inverse orbits) introduced in the preceding chapter maps any state of polarization into its motion-reversed counterpart and takes the complex conjugate of any number that may happen to multiply that state: $\mathcal{T} = UK$ by which is meant that K takes the complex conjugate of all the expansion coefficients of the arbitrary state of polarization in terms of the particular basis that is used and which must be known if U is to be specified (Gottfried [30]). For linear polarization bases we note $U = I$, the identity matrix. The time reversal operation has the effect that domain and range of the scattering matrices and of the antennas involved, respectively, belong to one and the same propagation space. Thus for the vector components this implies

$$\mathbf{E}^{+} = \mathbf{E}^{*-} \text{ or } \mathbf{E}^{-} = \mathbf{E}^{*+} \quad \text{(linear bases).} \tag{78}$$

The radar equation now reads

$$\mathbf{E}^{s+*} = S\mathbf{E}^{i+} \qquad \text{(RP)} \tag{79}$$

where the Sinclair matrix S on the right-hand side has domain and range in the common linear bases of the propagation space $\mathcal{P}^{+}$.

In contrast the equation for forward scattering of Optical Polarimetry remains unchanged

$$\mathbf{E}^{s+} = J\mathbf{E}^{i+} \qquad \text{(OP).} \tag{80}$$

The voltage equations assume the form

$$V = \mathbf{E}^{s-} \cdot \mathbf{h}^{+} = \mathbf{E}^{s+*} \cdot \mathbf{h}^{+} \qquad \text{(RP)} \tag{81}$$

and for transmission or forward scattering

$$V = \mathbf{E}^{s+} \cdot \mathbf{h}^{-} = \mathbf{E}^{s+} \cdot \mathbf{h}^{+*} \qquad \text{(OP)} \tag{82}$$

where in this case the antenna vector $\mathbf{h} \equiv \mathbf{h}^{-}$ is facing the incident wave $\mathbf{E}^{s} \equiv \mathbf{E}^{s+}$.

Having reduced all quantities involved to one and the same propagation space, $\mathcal{P}(\mathbf{z})$ in the present discussion, the propagation mark $+$ can be omitted.

The coordinate transformations

$$\mathbf{E}^{i,s+} \rightarrow U\mathbf{E}'^{\,i,s+}, \quad \mathbf{E}^{i,s-} \rightarrow U^{*}\mathbf{E}'^{\,i,s-} \tag{83}$$

imply for the field equations for radar polarimetry

$$\mathbf{E}^{s+*} = S\mathbf{E}^{i+} \Rightarrow U^{*}\mathbf{E}'^{\,s+} = SU\mathbf{E}'^{\,i+} \Rightarrow \mathbf{E}'^{\,s+} = S'\mathbf{E}'^{\,i+} \tag{84}$$

with the characteristic basis transformation

$$\mathbf{S'} = \mathbf{U^{T}SU} \qquad \text{(Radar Polarimetry)} \tag{85}$$

and for optical polarimetry

$$\mathbf{E}^{s+} = J\mathbf{E}^{i+} \Rightarrow U\mathbf{E}'^{\,s+} = JU\mathbf{E}'^{\,i+} \Rightarrow \mathbf{E}'^{\,s+} = J'\mathbf{E}'^{\,i+} \tag{86}$$

with the characteristic basis transformation

$$\mathbf{J}' = \mathbf{U}^{-1}\mathbf{J}\mathbf{U} \qquad \text{(Optical Polarimetry)}. \tag{87}$$

It is very interesting to realize that the same conclusions can be drawn from the voltage equations for radar and optical polarimetry. Equation (85) is a unitary consimilarity transformation and will be considered in detail in the following Section 5. Equation (87) on the other hand is an ordinary unitary similarity transformation, cf. for instance Horn and Johnson [31]. Both transformations are fundamentally different characterizing changes of bases for antilinear and for linear operators, respectively.

The extension to Stokes vectors for partially polarized waves is straightforward with direct consequences for the 4×4 Sinclair/Kennaugh and the Jones/Mueller scattering matrices for incoherent scattering phenomena.

5. Consimilarity Transformation and Optimal Polarizations

Introducing a new orthonormal polarization basis the Sinclair backscatter matrix S transforms according to

$$S \to S' = U^T S U \quad (UU^\dagger = I). \tag{88}$$

This is nowadays called a unitary consimilarity transformation, see [31, 39]. A general consimilarity transformation with an arbitrary nonsingular A is given by $S \to A^{-1*}SA$. This kind of transformation is characteristic for antilinear operators. It may also be called a unitary ^{T}congruence transformation.

It should be pointed out that symmetry of the Sinclair matrix is conserved under unitary consimilarity: $(S')^T = (U^T SU)^T = U^T S^T U = U^T SU = S'$. All the matrices S' obtained in this way form an equivalence class

$$C(S) = \{S' = U^T SU | UU^\dagger = I\} \tag{89}$$

and lead to a partition of all Sinclair matrices into disjoint equivalence classes. All matrices belonging to one and the same equivalence class are different matrix representations of a single (abstract) scattering operator.

We cite a remarkable theorem that can be considered as the basic theorem of radar polarimetry which allows to select from one particular equivalence class one member that has the very convenient form of a diagonal matrix. S is said to be condiagonalizable.

Takagi's Theorem [31, 39]:

A matrix $S \in M_n$ is unitarily condiagonalizable if and only if it is symmetric.

$$U^T SU = \Sigma = \text{diag}[\lambda_1, \lambda_2]. \tag{90}$$

This theorem has been rediscovered several times. Historical priority must be given to Autonne [40] for det(S) $\neq 0$ as early as 1915. It is often cited by the name of Takagi [41]. The diagonal elements λ_1 and λ_2 where we assume that $|\lambda_1| \geq |\lambda_2|$ are called *coneigenvalues*. If U is a unitary matrix that condiagonalizes the symmetric matrix S then with $U = [\mathbf{x}_1, \mathbf{x}_2]$ follows

$$S[\mathbf{x}_1, \mathbf{x}_2] = [\mathbf{x}_1^*, \mathbf{x}_2^*]\Sigma = [\lambda_1 x_1^*, \lambda_2 x_2^*] \tag{91}$$

or

$$S\mathbf{x}_i = \lambda_i \mathbf{x}_i^* \quad \text{for} \quad i = 1, 2. \tag{92}$$

Since U is unitary the columns of U, i.e. the vectors x_i $(i = 1, 2)$ are linearly independent and orthogonal.

In the radar community the general equation

$$S\mathbf{x} = \lambda \mathbf{x}^* \tag{93}$$

is known as Kennaugh's pseudo-eigenvalue equation [8, 13]. In mathematics it is said to be a coneigenvector equation. The prefix 'con' means 'conjugation' [31, 39].

In the context of the backscattering the coneigenvectors are those states of polarization that remain invariant under the mapping of the Sinclair scattering matrix S from the space $\mathcal{P}^+$, the domain of S, to the conjugate space $\mathcal{P}^-$, the range of S.

If λ is a coneigenvalue of S then $e^{2j\phi}\lambda$ is also a coneigenvalue of S for arbitrary phase factors ϕ since

$$S(e^{i\phi}\mathbf{x}) = e^{2i\phi}\lambda(e^{i\phi}\mathbf{x})^* \tag{94}$$

This is in contrast to the standard eigenvalue problem with unique eigenvalues. The phase indeterminacy of the coneigenvalue is an essential feature of the antilinear time-reversal operation in backscattering. Its interpretation and significance for target characterization and classification purposes (Huynen's skip angle) is at present not fully understood. In any case the diagonalizing matrix U in equation (91) may be chosen such that the coneigenvalues of S are real and nonnegative. On the other hand the matrix U is often chosen as unimodular $(\det(U) = 1)$, i.e. $U \in SU(2)$.

The symmetric Sinclair scattering matrix be given as

$$S = S^T = \begin{bmatrix} s_{11} & s_x \\ s_x & s_{22} \end{bmatrix} \qquad s_x = s_{12} = s_{21}. \tag{95}$$

The voltage equation and the power transfer equation assume the form

$$V_{\mathbf{y},\mathbf{x}} = \mathbf{y} \cdot S\mathbf{x} \equiv \mathbf{y}^T S\mathbf{x}, \quad P_{\mathbf{y},\mathbf{x}} = |V_{\mathbf{y},\mathbf{x}}|^2 \tag{96}$$

with

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\ ||\mathbf{x}|| = 1; \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},\ ||\mathbf{y}|| = 1. \tag{97}$$

Here, the normalized Jones vectors $\mathbf{x}$ and $\mathbf{y}$ are the transmitting and receiving antenna length or antenna height, respectively. Reference is given to Boerner et al. [42]. We distinguish the cases

- Co-polar power with $\mathbf{y} = \mathbf{x}$:

$$P_{co}(\mathbf{x}) = |\mathbf{x}^T S\mathbf{x}|^2, \quad P_{co}(\mathbf{x}_\perp) = |\mathbf{x}_\perp{}^T S\mathbf{x}_\perp|^2, \tag{98}$$

- Cross-polar power with $\mathbf{y} = \mathbf{x}_\perp$ or $\mathbf{x} = \mathbf{y}_\perp$:

$$P_x(\mathbf{x}) = |\mathbf{x}_\perp{}^T S\mathbf{x}|^2 = P_x(\mathbf{x}_\perp) = |\mathbf{x}^T S\mathbf{x}_\perp|^2. \tag{99}$$

The voltage V is in general complex-valued. Since the phase is of no concern here we consider only the co- and cross-polar power terms. Those vectors $\mathbf{x}$ and $\mathbf{x}_\perp$ (considered simultaneously) for which $|V_{co}|$ assumes the maximum value are denoted as $\mathbf{x}_{co,max}$ and $\mathbf{x}^\perp_{co,max}$ and those vectors $\mathbf{x}$ for which they assume the minimum value are denoted as $\mathbf{x}_{x,min}$ and $\mathbf{x}^\perp_{x,min}$; similarly for the cross-polar power terms.

5.1. The co-polar maxima

The co-polar power maximum is given by the first coneigenvector $\mathbf{x}_1$

$$\max_{||\mathbf{x}||=1} P_{co}(\mathbf{x}) = P_{co}(\mathbf{x}_1) = |\lambda_1|^2. \tag{100}$$

The solution $\mathbf{x}_2$ with the second coneigenvalue λ_2 ($|\lambda_2| \leq |\lambda_1|$) is orthogonal to $\mathbf{x}_1$ and denotes a local minimum of the co-polar power. Referring to Hong and Horn [39] It can be shown in general that the coneigenvectors $\mathbf{x}_i$ and the squared absolute values of the coneigenvalues $|\lambda_i|^2$ (i = 1, 2) of S are the ordinary eigenvectors and eigenvalues of Graves positive semidefinite linear power density matrix $G = S^\dagger S$ [7] (actually introduced by Kennaugh in his M.Sc. Thesis in 1952 [8])

$$G\mathbf{x}_i \equiv S^\dagger S\mathbf{x}_i = \mu_i \mathbf{x}_i = |\lambda_i|^2 \mathbf{x}_i \quad (i = 1, 2). \tag{101}$$

This is also the easiest way to calculate the coneigenvectors. The method fails however for coinciding eigenvalues of G and must be replaced by a more detailed reasoning based upon the coneigenvalue equation, see Horn and Johnson [31].

5.2. The co-polar nulls

The so-called Kennaugh-Huynen co-polar nulls can be derived in different ways.

- In the coneigenvector basis $\mathcal{E}$ the symmetric Sinclair matrix S assumes the form of a diagonal matrix

$$S|_{\mathcal{E}} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{102}$$

  where λ_1 and λ_2 are the coneigenvalues taken here as real nonnegative. Hence

$$V = \mathbf{x}^T S|_{\mathcal{E}}\mathbf{x} = [x_1\ x_2] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda_1 x_1^2 + \lambda_2 x_2^2 = 0 \tag{103}$$

  with the solutions

$$\mathbf{x}_{co,ni} = \frac{1}{\sqrt{\lambda_1 + \lambda_2}} \begin{bmatrix} \sqrt{\lambda_2} \\ \pm i\sqrt{\lambda_1} \end{bmatrix} \qquad i = 1, 2. \tag{104}$$

  These vectors are orthogonal if and only if $\lambda_1 = \lambda_2$. This implies that the matrix $A = [\mathbf{x}_{co,n1}, [_{co,n2}]$ constructed from the co-polar nulls is in general non unitary. Unitary matrices, however, can be constructed in the form $U_1 = [\mathbf{x}_{co,n1}, \mathbf{x}_{co,n1}^\perp]$ and $U_1 = [\mathbf{x}_{co,n2}, \mathbf{x}_{co,n2}^\perp]$ that produce zero in the 1-1 and 2-2 elements of S/ The resulting matrices are said to be off-triangular.

- The antisymmetric matrix (the standard metric spinor or the Levi-Civita symbol)

$$\beta = -\beta^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{105}$$

  yields by substituting β rather than V in the co-polar voltage equation

$$\mathbf{x}^T \beta \mathbf{x} = 0 \qquad \forall \mathbf{x}. \tag{106}$$

This implies that in the voltage equation one can replace

$$S \Rightarrow S_\lambda \doteq S + \lambda\epsilon = \begin{bmatrix} s_{11} & s_x + \lambda \\ s_x - \lambda & s_{22} \end{bmatrix} \tag{107}$$

without changing the co-polar voltage. The matrix S_λ becomes singular, $\det(S_\lambda) = 0$, for $\lambda_{1,2} = \pm\sqrt{s_x^2 - s_{11}s_{22}} = \pm\sqrt{-\det S}$. For these values of λ the matrix S_λ has rank 1 and can be written in the form of an outer product

$$S_{\lambda_1} = \frac{1}{s_{11}} \begin{bmatrix} s_{11} \\ s_x - \lambda_1 \end{bmatrix} \begin{bmatrix} s_{11} & s_x + \lambda_1 \end{bmatrix} = \frac{1}{s_{11}} ab^T, \tag{108}$$

$$S_{\lambda_2} = \frac{1}{s_{11}} \begin{bmatrix} s_{11} \\ s_x - \lambda_2 \end{bmatrix} \begin{bmatrix} s_{11} & s_x + \lambda_2 \end{bmatrix} = \frac{1}{s_{11}} \begin{bmatrix} s_{11} \\ s_x + \lambda_1 \end{bmatrix} \begin{bmatrix} s_{11} & s_x - \lambda_1 \end{bmatrix} = \frac{1}{s_{11}} ba^T. \tag{109}$$

Corresponding forms can be found if $s_{11} = 0$. The co-polar nulls are then given as orthogonal to the components a and b of $S_{\lambda_{1,2}}$ in the Euclidean sense:

$$x_{co,n1} = \begin{bmatrix} s_x - \lambda_1 \\ -s_{11} \end{bmatrix} \quad \text{and} \quad x_{co,n2} = \begin{bmatrix} s_x - \lambda_2 \\ -s_{11} \end{bmatrix} = \begin{bmatrix} s_x + \lambda_1 \\ -s_{11} \end{bmatrix}. \tag{110}$$

These vectors are not normalized.

- We obtain $V = 0$ in the co-polar voltage equation if and only if

$$S\mathbf{p} = \lambda\, \mathbf{p}_\perp^* \tag{111}$$

where $\mathbf{p}_\perp$ is perpendicular (orthogonal) to x in the unitary sense: $\mathbf{p}^\dagger \mathbf{p}_\perp = 0$. Now

$$\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \Longleftrightarrow \mathbf{p}_\perp = \begin{bmatrix} -p_2^* \\ p_1^* \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_1^* \\ p_2^* \end{bmatrix} = -\beta \mathbf{p}^*. \tag{112}$$

Hence, we obtain the following eigenvalue/eigenvector equation

$$\beta S\mathbf{p} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} S\mathbf{p} = \begin{bmatrix} s_x & s_{22} \\ -s_{11} & -s_x \end{bmatrix} \mathbf{p} \equiv \widetilde{S}\mathbf{p} = \lambda\mathbf{p} \tag{113}$$

for the matrix $\widetilde{S}$ with the eigenvalues $\lambda_{1,2} = \pm\sqrt{s_x^2 - s_{11}s_{22}}$ that agree with the previously found values and with the same eigenvalues $x_{co,n1}$ and $x_{co,n2}$. This is reflected in the relation

$$S_\lambda = \beta(\lambda I - \widetilde{S}). \tag{114}$$

- Any symmetric matrix S can be brought into any other symmetric matrix B of the same rank by means of ^{T}congruence, i.e., there exist a nonsingular matrix A such that $S = A^T BA$, cf. Horn and Johnson [31]. In particular S can be transformed into the backward identity matrix

$$A^T SA = \nu \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad A = [\mathbf{p}_1\, \mathbf{p}_2]. \tag{115}$$

- The last equation can be written in the form

$$S\mathbf{x}_i = \nu \mathbf{x}^*_{\perp,i} \qquad i = 1, 2. \tag{116}$$

  In the coneigenvector basis these normalized vectors assume the previous form.

- The 1-1 and 2-2 elements read $\mathbf{x}_i^T S\mathbf{x}_i = 0$ $(i = 1, 2)$ and indicates that the $\mathbf{x}_i$ are co-polar null vectors.

The co-polar maxima (cross-polar nulls) and the co-polar nulls form the famous Huynen fork on the Poincaré sphere with the Huynen-Boerner extension [13, 42, 14, 21] that completely characterize coherent radar targets. For forward propagation (transmission) in general no Huynen fork exists.

The analysis of the Jones matrices as representations of linear operators has been deleted in this paper since excellent treatments can be found in the classical literature, cf Naylor and Sell [43].

6. Voltage and Power Transfer Equations

The effective complex length $\mathbf{h}$ of an antenna (antenna vector) is defined in terms of the produced electric radiation field in the far-field zone. Using Lorentz's reciprocity theorem the received open-circuit voltage across the open terminals of the antenna is given by the so-called voltage equation:

$$V = V(\mathbf{h}, \mathbf{E}) = \mathbf{h} \cdot \mathbf{E} \equiv \mathbf{h}^T\mathbf{E} \tag{117}$$

where $\mathbf{E}$ is the incident externally produced electric field vector. It is important to realize that both vectors, $\mathbf{h}$ and $\mathbf{E}$ in equation (117), refer to one and the same cartesian coordinate system $\{\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y\}$ or of a spherical coordinate system. The voltage equation is bilinear in the $\{\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y\}$-basis. It is however not a scalar or inner product since the vectors $\mathbf{h}$ and $\mathbf{E}$ are in general complex. If $\mathbf{E} = \mathbf{h}$ then $V = \vec{h}^T\mathbf{h}$ is indefinite for complex $\mathbf{h}$, i.e. positive ($\mathbf{h}$ real), negative ($\mathbf{h}$ pure imaginary), zero for ($\mathbf{h} = [1, \pm i]^T$) or generally complex.

We introduce the vector components

$$\mathbf{h} = h_x e_x + h_y e_y \quad \text{or} \quad \mathbf{h} = \begin{bmatrix} h_x \\ h_y \end{bmatrix}, \qquad \mathbf{E} = E_x e_x + E_y e_y \quad \text{or} \quad \mathbf{E} = \begin{bmatrix} E_x \\ E_y \end{bmatrix}. \tag{118}$$

Here we denote abstract vectors and its representations as column vectors in the two-dimensional complex space $\mathcal{C}^2$ by the same symbol. The basis vectors $\hat{\mathbf{e}}_x = [1\ 0]^T$ and $\hat{\mathbf{e}}_y = [0\ 1]^T$ are real and orthonormal in their own basis $\{\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y\}$. Hence, the induced voltage also reads

$$V = h_x E_x + h_y E_y \quad \text{and} \quad P = V^*V = |(\mathbf{h} \cdot \mathbf{E})|^2. \tag{119}$$

The squared absolute value of V is said to be the power transfer P (from the incident wave to the antenna system).

Under a unitary change of basis with

$$\mathbf{h} \to \mathbf{h}' = U\mathbf{h}, \quad \mathbf{E} \to \mathbf{E}' = U\mathbf{E} \tag{120}$$

the voltage equation takes on the form

$$V = \mathbf{h} \cdot \mathbf{E} \to \mathbf{h}' \cdot U^T U\mathbf{E}' \neq V' = \mathbf{h}' \cdot \mathbf{E}'. \tag{121}$$

The conclusion is that the bilinear form of the voltage equation is not invariant with respect to general unitary basis transformations $UU^\dagger = U^\dagger U = I$ but only if U is restricted to real orthogonal transformations $U = O$ with $OO^T = O^T O = I$.

This conclusions is true of course whether the incident field is the field of another antenna or if $\mathbf{E}$ is the field back scattered from a target characterized by a symmetric Sinclair radar backscatter matrix S.

An often seen reasoning for the use of the voltage equation in fundamental polarimetry is the following. We found that the Sinclair back scatter matrix S under a change of polarizations basis with the unitary matrix U performs like

$$S \to S' = U^T S U \quad \text{(unitary consimilarity).} \tag{122}$$

Now using the radar equation for backscattering in the form $\mathbf{E}^s = S\mathbf{E}^i$ where $\mathbf{E}^i$ is the incident and $\mathbf{E}^s$ the back scattered wave the voltage equation assumes the form

$$V = \mathbf{h} \cdot S\mathbf{E}^{\,i} \tag{123}$$

and under a unitary change of polarization basis

$$V \to U\mathbf{h}' \cdot SU\mathbf{E}^{\,i\,\prime} = \mathbf{h}' \cdot U^T SU\mathbf{E}^{\,i\,\prime}. \tag{124}$$

Since the voltage equation must be invariant with respect to coordinate transformations one can set $V = V'$ and obtains

$$V = \mathbf{h}' \cdot U^T SU\mathbf{E}^{\,i\,\prime} = V' = \mathbf{h}' \cdot S'\mathbf{E}^{\,i\,\prime} \Longrightarrow S' = U^T SU. \tag{125}$$

The result agrees with that obtained with the help of the time reversal operation but the present derivation from the voltage equation (123) is not correct since the original voltage equation (117) is not invariant under general unitary transformation but only for (real) orthogonal transformations. This is one of the examples of the many intriguing pitfalls for which radar polarimetry is known.

We are getting one step closer to the solution of this riddle if we consider the question for which antenna polarization the received power is maximal for a given incident electric field vector $\mathbf{E}$. For sake of simplicity we take $\mathbf{h}$ and $\mathbf{E}$ as normalized: $||\mathbf{h}|| = 1$, $||\mathbf{E}|| = 1$. From the Cauchy-Schwarz inequality follows

$$\max P = \max |V|^2 = \mathbf{E}^\dagger \mathbf{E} = 1 \quad \text{for} \quad \mathbf{h} = \mathbf{E}^*. \tag{126}$$

In this case of complete polarization matching the polarization ellipses of the antenna polarization $\mathbf{h}$ and the incident plane electric wave $\mathbf{E}$ have the same locus but opposite sense of rotation. Going over to another polarization basis for $\mathbf{E}$ according to $\mathbf{E}' = U\mathbf{E}$ we must require that the optimal antenna polarization now reads

$$\mathbf{h}' = \{\mathbf{E}'\}^* = U^*\mathbf{E}* = U^*\mathbf{h}. \tag{127}$$

This implies that the antenna polarization $\mathbf{h}$ and the incident plane electric field vector $\mathbf{E}$ transform differently (conjugate) under a change of polarization basis for matched polarizations. But what is different for $\mathbf{h}$ and for $\mathbf{E}$? The answer is simple and obvious: $\mathbf{h}$ and $\mathbf{E}$ are Jones vectors belonging to plane waves with opposite directions of propagation.

Polarimetric antennas are characterized by only one state of polarization whether they are used for transmission or for reception. The antenna polarization is defined as the polarization of the plane wave field in the far zone that is produced by the antenna, the polarization of the antenna on reception is defined as that polarization of a plane incident electric wave that is best received by the antenna. If the polarization of the incident wave coincides with the polarization $\mathbf{h}$ of the antenna then

$$\text{polarization}\,\{\mathbf{h}\}|_{\text{receive}} = \text{polarization}\,\{\mathbf{h}\}|^{*}_{\text{transmit}}. \tag{128}$$

These antenna characteristics implies the use of directional Jones and Stokes vectors as suggested 1956 by Graves [7]. We denote these vectors as $\mathbf{E}^{+}$ and $\mathbf{E}^{-}$ or by some other symbols indicating their opposite direction of propagation.

We come to the bewildering conclusion that the bilinear voltage equation leads to correct result for the backscatter matrix but not for the back scattered field $\mathbf{E}^{s}$ or generally the field $\mathbf{E}$ of another source like an antenna as far as arbitrary bases of polarization are concerned and not only linear orthogonal bases. What is needed is a generalization of the voltage equation expressed in general polarization bases that reduces to the standard equation for the special cases of linear orthonormal bases.

The solution is surprisingly simply. We only have to realize that the voltage equation (117) always involves the Jones vectors of two plane waves travelling in opposite directions. Hence, we can write explicitly for the backscatter case

$$V = V(\mathbf{h}^{+}, \mathbf{E}^{-}) = \mathbf{h}^{+} \cdot \mathbf{E}^{-} \equiv \mathbf{h}^{+T}\mathbf{E} \tag{129}$$

and

$$V = V(\mathbf{h}^{-}, \mathbf{E}^{+}) = \mathbf{h}^{-} \cdot \mathbf{E}^{+} \equiv \mathbf{h}^{-T}\mathbf{E}^{+} \tag{130}$$

for the forward scatter (transmission) case. Replacing the term propagating in the negative (positive) direction by the complex conjugate of the corresponding term propagating in the opposite direction using the time-reversal operation of equation (39) we obtain a voltage equation that involves terms for propagation in only one direction. For the monostatic backscatter case we obtain

$$V = V(\mathbf{h}^{+}, \mathbf{E}^{+*}) = \mathbf{h}^{+} \cdot \mathbf{E}^{+*} \equiv \mathbf{h}^{+T}\mathbf{E}^{+*} \equiv \mathbf{E}^{+\dagger}\mathbf{h}^{+} \equiv [\mathbf{E}^{+}, \mathbf{h}^{+}] \tag{131}$$

where the brackets $[\mathbf{x}, \mathbf{y}] := \mathbf{x}^{\dagger}\mathbf{y}$ indicate the common scalar or inner product in an unitary space. In the expressions (131) the propagation index $^{+}$ can safely be omitted. A similar expression hold for the voltage equation for forward scattering. Now it is without any doubt possible to go over also to other (orthonormal) polarization bases.

The use of the bilinear form of the voltage equation (117) has become standard in the radar engineering community. It should be considered with reservation, however, if conclusions are drawn from it for which it was not designed. For instance for passage to nonlinear polarization bases it is best to use the form of the unitary scalar product.

7. Conclusions

The careful distinction of incoming and outgoing wave polarization spaces and the introduction of the time reversal operator leads quite natural to the concept of antilinearity in radar backscattering description. This procedure restores logical consistency and removes misconceptions related to the radar and the voltage (power transfer) equations. Backscatter and forward scatter (transmission) matrices have fundamentally different representations under changes of orthonormal polarization bases. The concepts of (unitary) consimilarity for backscattering and ordinary (unitary) similarity for forward scattering (transmission) are properly addressed and identified.

The definition of the state of polarization for the back scattered wave relies on the time reversal operation which is a discrete operation. Time reversal is described by an anti-unitary operation (Wigner 1931). This choice is perfectly adapted to the fact that an antenna has one and only one polarization for transmission and reception.

Optimal polarizations in the coherent scatter case either starting from the Sinclair and the Jones matrices are considered in some detail. The extension to the incoherent scatter case is straightforward without any new fundamental aspects but requires consideration of stochastic aspects combined with computational techniques. Other fundamental aspects of polarimetry like the 2×2 complex and the 4×4 real Minkowski spinor representations, the Poincaré sphere description and the roles of the Lorentz group and of the SU(2) universal covering group have not been addressed in this article which is devoted to illuminate the importance of the time-reversal concept for radar polarimetry.

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