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Global Stability Analysis of an End-to-End Congestion Control Scheme for General Topology Networks with Delay*

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Abstract

We analyze the stability properties of an end-to-end congestion control scheme under fixed heterogeneous delays, and for general network topologies. The scheme analyzed is based on the congestion control game of [1], with the starting point being the unique Nash equilibrium of that game. We prove global stability of this solution (and hence of the congestion control algorithm) under a mild symmetricity condition. We further demonstrate the stability of the algorithm numerically for various delays, user numbers, and topologies.

1. Introduction

In communication networks, delays between users and resources of the network are most of the time not negligible. In the context of the Internet, these delays vary from order of tens to hundreds of milliseconds, and affect the stability of end-to-end congestion control algorithms. The communication delays in the network are in general heterogeneous in the sense that forward delays between the users and the resources are different from the feedback delays. It is possible to consider end-to-end congestion control schemes in this setting as feedback systems with delay where users vary their flow rates in accordance with the delayed feedback they receive from the system resources. Depending on the specific network, this feedback signal may be in the form of packet losses as in Transfer Control Protocol (TCP), marked packets or variations in round trip time (RTT) the packets experience.

Several congestion control and pricing algorithms have been introduced in [2, 3, 4, 5], whose stability properties have recently been investigated in the presence of non-negligible delays [6, 7, 8]. Johari and Tan [8] have analyzed the local stability of a delayed system where the end user implements the 'primal algorithm' ([2, 3]) which is a TCP-like rate control algorithm. They have considered a single link accessed by a single user, as well as its multiple user extension under the assumption of symmetric delays. In both cases, they have provided sufficient conditions for local stability of the underlying system of equations.

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Massoulie [7] has extended these results to general network topologies and heterogeneous delays. In another study, Vinnicombe [6] has also provided sufficient conditions for local stability of a user source law which is a generalization of the same algorithm.

Within the distributed congestion control framework of [2, 3], Kunniyur and Srikant [9] have examined the question of how to provide the congestion feedback from the network to the user. They have proposed an explicit congestion notification (ECN) marking scheme combined with dynamic adaptive virtual queues, and have shown using a time-scale decomposition that the system is semi-globally stable in the no-delay case. Deb and Srikant [10], on the other hand, have focused on the case of single user and a single resource, and have investigated sufficient conditions for global stability of various congestion control schemes. Liu, Başar, and Srikant [11] have extended the framework of [2, 3] by introducing a primal-dual algorithm, which has dynamic adaptations at both ends (users and links), and have given a condition for its local stability using the generalized Nyquist criterion. Wen and Arcak [12] have used the passivity framework to unify some of the stability results on primal and dual algorithms without delay, have introduced and analyzed a larger class of such algorithms for stability, and have shown robustness to variations due to delay. In another recent study, Alpcan and Başar [13] have proposed a similar scheme based on game theory, where queueing delays in the network are used as congestion feedback. They have also provided sufficient conditions for global stability at a bottleneck link under non-negligible communication delays.

In this paper, we analyze in the presence of heterogeneous delays the global stability of the congestion control and pricing scheme introduced earlier in [1], which is based on noncooperative game theory. We provide sufficient conditions for the global stability under heterogeneous delays on a general topology network. Specifically, we make a mild symmetricity assumption on the variation of aggregate flows at links with respect to individual delays. We note that the condition obtained for the global stability shares a similar structure with the local stability results obtained in the earlier studies [6, 7, 8].

The rest of the paper is organized as follows: In Section 2 we introduce the network model as well as the game theory based congestion control framework, and state the existence and uniqueness of a Nash equilibrium (NE) solution. Global stability of this NE is established under a gradient algorithm in Section 3 for the ideal -no delay- case. Section 4 contains the stability analysis and sufficient conditions for global stability of the same algorithm in the presence of heterogeneous delays. In Section 5 we illustrate the theoretical results obtained numerically. The paper concludes with the remarks of Section 6.

2. A Congestion Control Game

The network model and the congestion control game in this section have been introduced in our earlier study [1]. They are presented here in a concise form for completeness. We consider a general network model based on fluid approximations, similar to the one in [13]. The topology of the network is characterized by a set of nodes $\mathcal{N} = \{1, \dots, N\}$ and a set of links connecting the nodes, $\mathcal{L} = \{1, \dots, L\}$, with each link $l \in \mathcal{L}$ having a fixed positive capacity $c_l > 0$, and an associated buffer size $b_l \geq 0$. There are M users sharing the network, with the set of users being $\mathcal{M} = \{1, \dots, M\}$. Each user is associated with a unique *connection*, or path R between a source and a destination node. The i^{th} user sends its nonnegative flow, $x_i \geq 0$, over its path R_i. A routing matrix, $\mathbf{A} := [(a_{l,i})]$ of ones and zeros, is defined as in [2], which describes the relation between the set of routes $\mathcal{R} = \{1, \dots, M\}$ associated with the users (connections) and links $l \in \mathcal{L}$. We assume without any loss of generality that $\mathbf{A}$ has no rows or columns that are identically zero.

Using the routing matrix $\mathbf{A}$, the capacity constraints of the links are given by $\mathbf{Ax} \leq \mathbf{c}$, where $\mathbf{x}$ is the $(M \times 1)$ flow rate vector of users and $\mathbf{c}$ is the $(L \times 1)$ link capacity vector. The flow rate vector, $\mathbf{x}$, is said to be feasible if it is nonnegative and satisfies this constraint. Let $\mathbf{x_{-i}}$ be the flow rate vector of all users except the i^{th} one. For a given fixed, feasible $\mathbf{x_{-i}}$, there exists a strict finite upper-bound $m_i(\mathbf{x_{-i}})$ on flow rate, x_i, of the i^{th} user based on the capacity constraints of the links:

$$m_i(\mathbf{x_{-i}}) = \min_{l \in R_i}(c_l - \sum_{j \neq i} A_{l,j}\ x_j)\ \geq\ 0.$$

For the given general topology network model, we propose a network game with M users. It is assumed that the routing problem has already been solved and individual routes, $R \in \mathcal{R}$, do not change during the connection. Our analysis is based on noncooperative game theory. Here, the users are noncooperative in the sense that they have no means of communicating with each other about their preferences, and each user tries to optimize his usage of the network independently. A specific cost function, J, is assigned to each user, which will be indexed by i for user i. This cost function not only models the user preferences but it also includes a feedback term capturing the current network state. The i^{th} user minimizes his cost, J_i, by adjusting his flow rate $0 \leq x_i \leq m_i(\mathbf{x_{-i}})$ given the fixed, feasible flow rates of all other users on its path, $\{x_j : j \in (R_j \cap R_i)\}$.

The cost function of the i^{th} user, J_i, is the difference between a user-specific pricing function, P_i, and a utility function, U_i. The pricing function P_i depends on the current state of the network, and can be interpreted as the price a user pays for using the network resources. The utility function of the i^{th} user is defined to be increasing and concave in accordance with elastic traffic as well as with the economic principle, law of diminishing returns. We focus on the bandwidth as the main resource in the system. Therefore, the utility of the i^{th} user depends only on its own flow rate. Thus, the cost function is defined as

$$J_i(\mathbf{x}; \mathbf{c}, \mathbf{A}) = P_i(\mathbf{x}; \mathbf{c}, \mathbf{A}) - U_i(x_i). \tag{1}$$

We note that P_i does not necessarily depend on the flow rates of all other users; it can be structured to depend only on the flow rates of the users sharing the same links on the path of user i.

In the given context of the network game, the Nash equilibrium is defined as a set of flow rates, $\mathbf{x}^*$ (and corresponding costs J^*), with the property that no user can benefit by modifying its flow while the other players keep theirs fixed. Furthermore, if the Nash equilibrium, $\mathbf{x}^*$, meets the capacity constraints as well as the positivity constraint with strict inequality, then it is an *inner* solution. Mathematically speaking, $\mathbf{x}^*$ is in Nash Equilibrium, when x_i^* of any i^{th} user is the solution to the following optimization problem given that all users on its path have equilibrium flow rates, $\mathbf{x}_{-i}^*$:

$$\min_{0 \leq x_i \leq m_i(\mathbf{x_{-i}}^*)} J_i(x_i, x_{-i}^*, \mathbf{c}, \mathbf{A}) \ , \tag{2}$$

where $\mathbf{x_{-i}}$ denotes the collection $\{x_j : j \in R_j \cap R_i\}_{j=1,\dots,M}$. To proceed further, we make the following two assumptions.

A1. $P_i(\mathbf{x})$ is jointly continuous in all its arguments and twice continuously differentiable, non-decreasing and convex in x_i, i.e.

$$\partial P_i(\mathbf{x})/\partial x_i \geq 0\ ,\ \partial^2 P_i(\mathbf{x})/\partial x_i^2 \geq 0. \tag{3}$$

A2. $U(x_i)$ is jointly continuous in all its arguments and twice continuously differentiable, non-decreasing and strictly concave in x_i, i.e.

$$\partial U_i(x_i)/\partial x_i \geq 0\ \ ,\ \ \partial^2 U_i(x_i)/\partial x_i^2 < 0\ \ , \forall x_i$$

Moreover, the optimal solution is an *inner* one, $0 < \sum_j A_{l,j} x_j^* < c_l$, $\forall l$, under the additional assumption:

A3. The i^{th} user's cost function has the following properties at $x_i = 0$ $(x_i = m_i(\mathbf{x_{-i}}))$: $\partial J_i(\mathbf{x} : x_i = 0)/\partial x_i < 0 \; \forall \mathbf{x}$ $(\partial J_i(\mathbf{x} : x_i = m_i(\mathbf{x_{-i}}))/\partial x_i > 0 \; \forall \mathbf{x})$, respectively.

Theorem 2.1 below establishes that the congestion control game admits a unique NE under the following further assumption:

A4. The price function $P_i(\mathbf{x})$ of the i^{th} user is defined as the sum of link price functions on its path,

$$P_i = \sum_{l \in R_i} P_l(\sum_{j : l \in R_j} x_j),$$

where P_l is defined as a function of the aggregate flow on link l, and satisfies (3) with i replaced by l.

Theorem 2.1 *Under A1-A4, the network game admits a unique inner Nash equilibrium.*

Proof A slightly modified version of the proof in [1] is given here for completeness. Let $X := \{\mathbf{x} \in \mathbb{R}^M : \mathbf{A}\mathbf{x} \leq \mathbf{c} \, , \mathbf{x} \geq 0\}$ be the set of feasible flow rate vectors (or strategy space) of the users. The flow rate of a generic i^{th} user is nonnegative and bounded above by the minimum link capacity on its route, $0 \leq x_i < \min_{l \in R_i} c_l$. The set X is clearly closed and bounded, hence, compact. Next, we show that X has a nonempty interior and is convex. Define the following flow rate vector: $\mathbf{x}^{max} := \min_l c_l / M$. Clearly, $\mathbf{x}^{max} \in X$ is feasible and positive as $c_l > 0 \, \forall l$. Hence, there exists at least one positive and feasible flow rate vector in the set X, which is an interior point. Thus, the set X has a nonempty interior. Let $\mathbf{x}^1$, $\mathbf{x}^2 \in X$ be two feasible flow rate vectors, and $0 < \lambda < 1$ be a real number. We have, for any $\mathbf{x}^\lambda := \lambda \mathbf{x}^1 + (1 - \lambda)\mathbf{x}^2$,

$$\mathbf{A}\mathbf{x}^\lambda = \mathbf{A}(\lambda \mathbf{x}^1 + (1 - \lambda)\mathbf{x}^2) \leq \mathbf{c}$$

Furthermore, $\mathbf{x}^\lambda \geq 0$ by definition. Hence, $\mathbf{x}^\lambda$ is feasible and is in X for any $0 < \lambda < 1$. Thus, the set X is convex. By a standard theorem of game theory (Thm. 4.4 p.176 in [14]), the network game admits a NE.

We now prove uniqueness. Differentiating (1) with respect to x_i, and using assumptions *A1,A2*, we have

$$f_i(\mathbf{x}) := \frac{\partial J_i(\mathbf{x})}{\partial x_i} = \frac{\partial P_i(\mathbf{x})}{\partial x_i} - \frac{\partial U_i(x_i)}{\partial x_i} \; . \tag{4}$$

As a simplification of notation, $\mathbf{c}$ and $\mathbf{A}$ are suppressed as arguments of the functions for the rest of this proof.

Differentiating $J_i(\mathbf{x})$ twice with respect to x_i yields

$$\frac{\partial f_i(\mathbf{x})}{\partial x_i} = \frac{\partial^2 J_i(\mathbf{x})}{\partial x_i^2} = \frac{\partial^2 P_i(\mathbf{x})}{\partial x_i^2} - \frac{\partial^2 U_i(x_i)}{\partial x_i^2} > 0$$

Hence, J_i is unimodal and has a unique minimum. Based on $A3$, $f_i(\mathbf{x})$ attains the zero value at $m_i(\mathbf{x_{-i}}) > x_i > 0$ given a fixed feasible $\mathbf{x_{-i}}$. Thus, the optimization problem (2) admits a unique positive solution.

To preserve notation, let $\frac{\partial^2 J(\mathbf{x})}{\partial x_i^2}$ be denoted by B_i. Further introduce, for $i, j \in \mathcal{M}$, $j \neq i$,

$$\frac{\partial^2 J_i(\mathbf{x})}{\partial x_i \partial x_j} = \frac{\partial^2 P_i(\mathbf{x})}{\partial x_i \partial x_j} =: A_{i,j} \; ,$$

with both B_i and $A_{i,j}$ defined on the space where $\mathbf{x}$ is nonnegative, and bounded by the link capacities. Suppose that there are two Nash equilibria, represented by two flow vectors $\mathbf{x}^1$ and $\mathbf{x}^0$, with elements x_i^0 and x_i^1, respectively. Define the pseudo-gradient vector:

$$g(\mathbf{x}) := \left[\nabla_{x_1} J_1(\mathbf{x})^T \cdots \nabla_{x_M} J_M(\mathbf{x})^T\right]^T \tag{5}$$

As the Nash equilibrium is necessarily an inner solution, it follows from first-order optimality condition that $g(\mathbf{x}^0) = 0$ and $g(\mathbf{x}^1) = 0$. Define the flow vector $\mathbf{x}(\theta)$ as a convex combination of the two equilibrium points $\mathbf{x}^0$, $\mathbf{x}^1$:

$$\mathbf{x}(\theta) = \theta \mathbf{x}^0 + (1 - \theta)\mathbf{x}^1$$

where $0 < \theta < 1$. By differentiating $\mathbf{x}(\theta)$ with respect to θ,

$$\frac{dg(\mathbf{x}(\theta))}{d\theta} = G(\mathbf{x}(\theta))\frac{dx(\theta)}{d\theta} = G(\mathbf{x}(\theta))(\mathbf{x}^1 - \mathbf{x}^0) \ , \tag{6}$$

where $G(\mathbf{x})$ is the Jacobian of $g(\mathbf{x})$ with respect to $\mathbf{x}$:

$$G(\mathbf{x}) := \begin{pmatrix} B_1 & A_{12} & \cdots & A_{1M} \\ \vdots & & \ddots & \vdots \\ A_{M1} & A_{M2} & \cdots & B_M \end{pmatrix}_{M \times M} . \tag{7}$$

We also note that, by *A4*:

$$\sum_{l \in (R_i \cap R_j)} \frac{\partial^2 J_l(\mathbf{x})}{\partial x_i \partial x_j} = \sum_{l \in (R_i \cap R_j)} \frac{\partial^2 J_l(\mathbf{x})}{\partial x_i \partial x_j}$$

$$\Rightarrow A(i,j) = A(j,i) \ \ i,j \in \mathcal{M} \ .$$

Hence, $G(\mathbf{x})$ is symmetric. Integrating (6) over θ,

$$0 = g(\mathbf{x}^1) - g(\mathbf{x}^0) = \left[\int_0^1 G(\mathbf{x}(\theta))d\theta\right](\mathbf{x}^1 - \mathbf{x}^0) \ , \tag{8}$$

where $(\mathbf{x}^1 - \mathbf{x}^0)$ is a constant flow vector. Let $\overline{B_i(\mathbf{x})} = \int_0^1 B_i(\mathbf{x}(\theta))d\theta$ and $\overline{A_{ij}(\mathbf{x})} = \int_0^1 A_{ij}(\mathbf{x}(\theta))d\theta$. In view of *A2* and *A4*, $B_i(\mathbf{x}) > A_{ij}(\mathbf{x}) > 0$, $\forall i,j$. Thus, $\overline{B_i(\mathbf{x})} > \overline{A_{ij}(\mathbf{x})} > 0$, for any $\mathbf{x}(\theta)$. In order to simplify the notation, define the matrix $\mathcal{G}(\mathbf{x}^1, \mathbf{x}^0) := \int_0^1 G(\mathbf{x}(\theta))d\theta$, which can be shown to be full rank for any fixed $\mathbf{x}$. Rewriting (8) as, $0 = \mathcal{G} \cdot [\mathbf{x}^1 - \mathbf{x}^0]$, since $\mathcal{G}$ is full rank, it readily follows that $\mathbf{x}^1 - \mathbf{x}^0 = 0$. Therefore, the NE is unique.

Under *A3*, the NE has to be an inner solution, as the following argument shows. First, $\mathbf{x} \geq 0$, with $x_i = 0$ for at least one i, cannot be an equilibrium point since user i can decrease its cost by increasing its flow rate. Similarly, the boundary points $\{\mathbf{x} \in \mathbb{R}^M : \mathbf{A}\mathbf{x} \leq \mathbf{c} \ , \mathbf{x} \geq 0, \text{with } (Ax)_l = c_l \text{ for at least one link } l\}$ cannot constitute NE, as users whose flows pass through the link can decrease their flow rates under *A3*. Thus, under *A1-A4* the network game admits a unique inner NE. □

3. Global Stability in the Delay-Free Case

We consider a simple dynamic model of the network game where each user changes his flow rate in proportion with the gradient of his cost function with respect to his flow rate. Note that this corresponds to the well-known steepest descent algorithm in nonlinear programming [15]. Hence, the user update algorithm is:

$$\dot{x}_i(t) = \frac{dx_i(t)}{dt} = -\frac{\partial J_i(\mathbf{x}(t))}{\partial x_i} = \frac{dU_i(x_i)}{dx_i} - \sum_{l \in R_i} f_l\left(\sum_{j \in \mathcal{M}_l} x_j\right) := \Theta_i(\mathbf{x}), \tag{9}$$

for all $i = 1, \ldots, M$, where $\mathcal{M}_l(M_l)$ is the set (number) of users whose flows pass through the link, $l \in R_i$; t is the time variable, which we drop in the second line for a more compact notation; and f_l is defined as $f_l(\cdot) := \partial P_l(\cdot)/\partial x_i$.

By assumption *A4*, the partial derivative of f_l with respect to x_i, $\partial f_l(\cdot)/\partial x_i$, is non-negative. Furthermore, since $P_l(\mathbf{x})$ is convex and jointly continuous in x_i for all i whose flows pass through the link l, on the compact set of feasible flow rate vectors, $X := \{\mathbf{x} \in \mathbb{R}^M : \mathbf{Ax} \leq \mathbf{c}\ , \mathbf{x} \geq 0\}$, the derivative $\partial f_l(.)/\partial x_i$ can be bounded above by a constant $\alpha_l > 0$. Hence,

$$0 \leq \frac{\partial f_l(\bar{x}_l)}{\partial x_i} \leq \alpha_l, \tag{10}$$

where $\bar{x}_l = \sum_{i \in \mathcal{M}_l} x_i$.

Next, we establish the result that the system defined by (9) is asymptotically stable on the set X, which is invariant by assumption *A3* under the gradient update algorithm (9). In order to see the invariance of X , we investigate each boundary of X separately. When $x_i = 0$ for some $i \in \mathcal{M}$, we have $\dot{x}_i > 0$ from (9) under assumption *A3* due to the gradient descent algorithm of user i. Hence, the system trajectory moves toward inside of X. Likewise, in the case of $\bar{x}_l = c_l$ for some $l \in \mathcal{L}$, it follows from (9) and assumption *A3* that $\dot{x}_i < 0\ \forall i \in \mathcal{M}_l$, and hence, the trajectory remains inside the set X.

The equilibrium state of the system (9) in X is of course the unique NE, $\mathbf{x}^*$, referred to in Theorem 2.1. Let us define a candidate Lyapunov function $V : \mathbb{R}^M \rightarrow \mathbb{R}^+$ as

$$V(\mathbf{x}) := \frac{1}{2}\sum_{i=1}^{M} \Theta_i^2(\mathbf{x}),$$

which is in fact restricted to the domain X. Further let $\Theta := [\Theta_1, \ldots, \Theta_M]$. Taking the derivative of V with respect to t on the trajectories generated by (9), we obtain

$$\dot{V}(\mathbf{x}) = \sum_{i=1}^{M} \frac{d^2U_i(x_i)}{dx_i^2}\Theta_i^2(\mathbf{x}) - \Theta^T(\mathbf{x})A^T K A \Theta(\mathbf{x}),$$

where A is the routing matrix, and K is a diagonal matrix defined as $K := \mathrm{diag}\left[\frac{\partial f_1(\bar{x})}{\partial \bar{x}}, \frac{\partial f_2(\bar{x})}{\partial \bar{x}}, \ldots, \frac{\partial f_M(\bar{x})}{\partial \bar{x}}\right]$.

Since $A^T K A$ is non-negative definite and d^2U_i/dx_i^2 is uniformly negative definite, $V(\mathbf{x})$ is strictly decreasing, $\dot{V}(\mathbf{x}) < 0$, on the trajectory of (9). Thus, the system is asymptotically stable on the invariant set X by Lyapunov's stability theorem (see Theorem 3.1 in [16]).

Theorem 3.1 *Assume A1-A4 hold. Then, the unique Nash equilibrium of the network game is globally stable on the compact set of feasible flow rate vectors,* $X := \{\mathbf{x} \in \mathbb{R}^M : \mathbf{A}\mathbf{x} \leq \mathbf{c}\ ,\ \mathbf{x} \geq 0\}$ *under the gradient algorithm given by*

$$\dot{x_i} = -\frac{\partial J_i(\mathbf{x})}{\partial x_i}\ , i = 1, \dots, M.$$

4. Global Stability under Bounded and heterogeneous Communication Delays

We now investigate global stability of the gradient algorithm (9) under bounded and heterogeneous communication delays. The pricing function of the i^{th} user is defined in accordance with assumption *A4* as

$$P_i = \sum_{l \in R_i} P_l\big(\sum_{j \in \mathcal{M}_l} x_j\big),$$

where R_i is the path (route) of user i, and P_l is the pricing function at link $l \in \mathcal{L}$. The update algorithm with communication delays is then given by

$$\dot{x_i}(t) = \frac{dU_i(x_i(t))}{dx_i} - \sum_{l \in R_i} f_l\bigg(\sum_{j \in \mathcal{M}_l} x_j(t - r_{li} - r_{lj})\bigg) \tag{11}$$

where r_{li} and r_{lj} are fixed communication delays between the l^{th} link and the i^{th} and j^{th} users respectively. [1] To simplify the notation we define

$$\bar{x}_l^i(t - r) := \sum_{j \in \mathcal{M}_l} x_j(t - r_{li} - r_{lj}).$$

In addition, let q be an upper-bound on the maximum round-trip time (RTT) in the system:

$$q := 2 \max_i \sum_{l \in R_i} r_{li} - r_{(l-1)i},$$

where $r_{0i} = 0\ \forall i$. Finally, define $\mathbf{x}_t := \{\mathbf{x}(t+s),\ -q \leq s \leq 0\}$, and by a slight abuse of notation let $\Theta_i(\mathbf{x}_t)$ denote the right hand side of (11).

We next make use of the stability theory for autonomous systems of [17], and generalize the scalar analysis of [18] and also of Chapter 5.4 of [17] to the multidimensional (multi-user) case. Let $\phi_i \in C\big([-r_i, 0], \mathbb{R}\big)$ be a feasible flow rate function (initial condition) for the i^{th} user's dynamics (11) at time $t = 0$, where C is the set of continuous functions. In addition, let $\mathbf{x}(\phi)(t)$ be the solution of (11) through ϕ for $t \geq 0$, and $\dot{\mathbf{x}}(\phi_i)(t)$ be its derivative. In order to simplify the notation, we will use $\mathbf{x}(\phi)$ and $\mathbf{x}$ as well as $\Theta(\phi)$ and Θ and their respective derivatives interchangeably for the remainder of the paper.

A continuously differentiable and positive function $V : C^M \to \mathbb{R}^+$ is defined as

$$V(\mathbf{x}_t(\phi)) := \frac{1}{2} \sum_{i=1}^{M} \Theta_i^2(\mathbf{x}_t(\phi)) = \frac{1}{2} \Theta^T(\mathbf{x}_t(\phi)) \Theta(\mathbf{x}_t(\phi)).$$

[1] Here we implicitly make the assumption that queueing delays are negligible compared to the fixed propagation delays in the system.

We introduce the candidate Lyapunov function $\overline{V} : \mathbb{R}^+ \times C^M \to \mathbb{R}^+$,

$$\overline{V}(t;\phi) := \sup_{t-2q \le s \le t} V(\mathbf{x}_s(\phi)).$$ [2]

Let $\dot{\overline{V}}(t;\phi)$ and $\dot{V}(t;\phi)$ be defined as the upper right-hand derivatives of $\overline{V}(t;\phi)$ and $V(t;\phi)$ respectively along $\mathbf{x}_t(\phi)$. In order for $\overline{V}(t;\phi)$ to be non-increasing, $\dot{\overline{V}}(t;\phi) \le 0$, the set

$$\Phi = \{\phi \in C : \overline{V}(t;\phi) = V(\mathbf{x}_t(\phi)); \dot{V}(\mathbf{x}_t(\phi)) > 0 \ \forall t \ge 0\} \tag{12}$$

has to be empty. To see this consider the case when the set Φ is not empty. Then, by definition, there exist a time t and an $h > 0$ such that $\dot{\overline{V}}(\mathbf{x}_{t+h}(\phi)) > \dot{\overline{V}}(\mathbf{x}_t(\phi))$, and hence, $\dot{V}(\mathbf{x}_t(\phi))$ cannot be non-increasing. We now show that the set Φ is indeed empty.

Assume otherwise. Then, for any given t, there exists an $\epsilon > 0$ such that

$$\overline{V}(t;\phi) = V(\mathbf{x}_t(\phi)) = \sum_{i=1}^{M} \Theta_i^2(\mathbf{x}_t(\phi)) = \epsilon \tag{13}$$

and

$$V(\mathbf{x}_s(\phi)) = \sum_{i=1}^{M} \Theta_i^2(\mathbf{x}_s) \le \epsilon\ ,\ s \in [t-2q\ ,\ t].$$

Thus, the following bound on Θ_i, and thus on $\dot{x}_i$, follows immediately:

$$|\Theta_i(\mathbf{x}_s)| = |\dot{x}_i(s)| \le \sqrt{\epsilon}\ ,\ s \in [t-2q\ ,\ t]. \tag{14}$$

Taking the derivative of $\dot{x}_i(t)$ with respect to t, we obtain

$$\ddot{x}_i(t) = \frac{\partial \dot{x}_i(t)}{\partial t} = \dot{\Theta}_i(\mathbf{x}_t) = \frac{d^2 U_i(x_i)}{dx_i^2}\dot{x}_i(t) - \sum_{l \in R_i} \frac{\partial f_l(\bar{x}_l^i(t-r))}{\partial \bar{x}_l^i} \sum_{j \in \mathcal{M}_l} \dot{x}_j(t - r_i - r_j). \tag{15}$$

Let $\delta_i := -\min_{x_i \in X} \dfrac{d^2 U_i(x_i)}{dx_i^2} > 0$. Using (14) and (15), it is possible to bound $\dot{\Theta}_i(\mathbf{x}_s)$ and $\ddot{x}_i(s)$ on $s \in [t-q, t]$ with

$$|\dot{\Theta}_i(\mathbf{x}_s)| = |\ddot{x}_i(s)| \le \delta_i |\dot{x}_i(s)| + \sum_{l \in R_i} \frac{\partial f_l(\bar{x}_l^i(s-r))}{\partial \bar{x}_l^i} |\bar{x}_l^i(s-r)| \le (\delta_i + \sum_{l \in R_i} M_l \alpha_l)\sqrt{\epsilon}. \tag{16}$$

To simplify the notation, define

$$y_i := \delta_i + \sum_{l \in R_i} M_l \alpha_l.$$

Hence, we have the following bound on $\Theta_i(\mathbf{x}_s)$, $s \in [t-q, t]$:

$$\Theta_i(\mathbf{x}_t) - q y_i \sqrt{\epsilon} \le \Theta_i(\mathbf{x}_s) \le \Theta_i(\mathbf{x}_t) + q y_i \sqrt{\epsilon}. \tag{17}$$

[2] Without any loss of generality, we define $V(\mathbf{x}_s) = 0,\ s \in [-2q, -q]$.

We next show that $V(\mathbf{x}_t(\phi))$ is non-increasing, and obtain a contradiction to the initial hypothesis that the set Φ is not empty. Assume that $\partial f_l(\bar{x}_l^i(t-r))/\partial \bar{x}_l^i = \partial f_l(\bar{x}_l^j(t-r))/\partial \bar{x}_l^j$, $\forall i, j \in \mathcal{M}_l$, $\forall t$ for each link l. This assumption holds for example when f_l is linear in its argument. Let B be defined in such a way that $B^T B := A^T K A$, where the positive diagonal matrix K is defined in Section 3. Also define the positive diagonal matrix

$$D(\mathbf{x}) := \text{diag}\left[|D_1(x_1)|, |D_2(x_2)|, \ldots, |D_M(x_M)|\right],$$

where $D_i(\mathbf{x}) := d^2 U_i(x_i)/dx_i^2$. Then, using (17), we obtain

$$\begin{aligned} \dot{V}(\mathbf{x}_t) = & -\sum_{i=1}^{M} D_i(x_i)\Theta_i^2(\mathbf{x}_t) - \sum_{i=1}^{M} \Theta_i(\mathbf{x}_t) \cdot \sum_{l \in R_i} \frac{\partial f_l(\bar{x}_l^i(t-r))}{\partial \bar{x}_l^i} \sum_{j \in \mathcal{M}_l} \Theta_j(\mathbf{x}_{t-r_{li}-r_{lj}}) \\ \leq & -\Theta^T D \Theta - \Theta^T B^T B \Theta + q\sqrt{\epsilon}|\Theta^T B^T B \mathbf{y}|, \end{aligned} \tag{18}$$

where everything is evaluated at t. Now, for any fixed trajectory generated by (11), and for a frozen time t, a sufficient condition for $\dot{V}(\mathbf{x}_t) \leq 0$ is

$$q\sqrt{\epsilon} \leq \frac{||B\Theta||^2 + ||\sqrt{D}\Theta||^2}{||B\Theta||\,||B\mathbf{y}||},$$

where $||\cdot||$ is the Euclidean norm.

Let $k := \frac{||B\Theta||}{||B\mathbf{y}||} > 0$. Rewriting the sufficient condition we obtain

$$q\sqrt{\epsilon} \leq k + \frac{1}{k}\mu,$$

where $\mu := \frac{||\sqrt{D}\Theta||^2}{||B\mathbf{y}||^2} > 0$. The following worst-case bound on q can be derived by a simple minimization:

$$q\sqrt{\epsilon} \leq 2\sqrt{\mu}. \tag{19}$$

We next find a lower bound on μ. From (13), it follows that $||\sqrt{D}\Theta(\mathbf{x}_t)||^2 \geq \bar{d}\epsilon$, where $\bar{d} := \min_i \min_{x_i \in X} \left|\frac{d^2 U_i(x_i)}{dx_i^2}\right|$, and $\sqrt{D}$ is the unique positive definite matrix whose square is D. Furthermore,

$$||B\mathbf{y}||^2 \leq \sum_{i=1}^{M} y_i \sum_{l \in R_i} \alpha_l \sum_{j \in \mathcal{M}_l} y_j.$$

Define also the following upper-bound on y_i:

$$b := \max_i \left(\delta_i + \sum_{l \in R_i} M_l \alpha_l\right)$$

Since $\delta_i > 0$, one obtains $||B\mathbf{y}||^2 \leq Mb^3$, and hence

$$\mu \geq \frac{\bar{d}\epsilon}{Mb^3}.$$

Thus, from (19) a sufficient condition for $V(\mathbf{x}_t)$ to be non-increasing is

$$q \leq \frac{2\sqrt{\bar{d}}}{\sqrt{M}b^{3/2}}, \tag{20}$$

which now holds for all $t \geq 0$.

Finally, we make use of Definition 3.1 and Theorem 3.1 of [17] to establish global asymptotic stability of the system (11). Let $S := \{\phi \in C : \dot{\overline{V}}(t;\phi) = \dot{V}(\mathbf{x}_t(\phi)) = 0\}$. From (11) and (18) it follows that

$$\begin{array}{l} S' = \{\phi \in C : \phi(\tau) = \mathbf{x}^*, \ -q \leq \tau \leq 0\} \subset S, \text{ as} \\ \Theta(\mathbf{x}_\tau) = \dot{\mathbf{x}}(\tau) = 0 \Leftrightarrow \mathbf{x}_\tau = \mathbf{x}^* \Rightarrow \dot{V}(\mathbf{x}_\tau) = 0. \end{array}$$

Hence, S' is the largest invariant set in S, and for any trajectory of the system that belongs identically to S, we have $\mathbf{x}_\tau = \mathbf{x}^*$. In other words, the only solution that can stay identically in S is the unique equilibrium of the system. This then leads to the following theorem:

Theorem 4.1 *Assume that*

$$\partial f_l(\bar{x}_l^i(s-r))/\partial \bar{x}_l^i = \partial f_l(\bar{x}_l^j(s-r))/\partial \bar{x}_l^j, \ \forall i,j \in \mathcal{M}_l \ \forall t.$$

Then, the unique Nash equilibrium of the network game is globally asymptotically stable on the compact set of feasible flow rate vectors, $X := \{\mathbf{x} \in \mathbb{R}^M : \mathbf{A}\mathbf{x} \leq \mathbf{c}\ ,\ \mathbf{x} \geq 0\}$ *under the gradient algorithm*

$$\dot{x}_i(t) = \frac{dU_i(x_i(t))}{dx_i} - \sum_{l \in R_i} f_l\Bigg(\sum_{j \in \mathcal{M}_l} x_j(t - r_{li} - r_{lj})\Bigg),$$

in the presence of fixed heterogeneous delays, $r_{li} \geq 0$*, for all users* $i = 1, \ldots, M$*, and links* $l \in \mathcal{L}$*, if the following condition is satisfied*

$$q \leq \frac{2\sqrt{\bar{d}}}{\sqrt{M}b^{3/2}},$$

where

$$b := \max_i \Bigg(- \min_{x_i \in X} \frac{d^2U_i(x_i)}{dx_i^2} + \sum_{l \in R_i} M_l \alpha_l \Bigg),$$

and

$$\bar{d} := \min_i \min_{x_i \in X} \left| \frac{d^2U_i(x_i)}{dx_i^2} \right|.$$

Remark 4.2 *If the user reaction function is scaled by a user-independent gain constant,* λ*, then the* i^{th} *user's response is given by*

$$\dot{x}_i = -\lambda \frac{\partial J_i(\mathbf{x}(t))}{\partial x_i},$$

and the sufficient condition for global stability turns out to be

$$q \leq \frac{2\sqrt{\bar{d}}}{\sqrt{M}\lambda^{3/2}b^{3/2}}.$$

Notice that, for any $\lambda < 1$*, the upper-bound on maximum RTT,* q*, is relaxed proportionally with* $\lambda^{3/2}$.

5. Numerical Evaluation

The results presented in Section 4 are evaluated numerically using MATLAB. The delay differential equations are solved using the delay differential solver, *dde23* [19]. The utility function for the i^{th} user is chosen as

$$U_i = u_i \log(x_i + 1),$$

where u_i is a user-specific positive preference parameter. The pricing function at the link l is defined as

$$P_l = \frac{\alpha}{2}\Big(\sum_{j: l \in R_j} x_j\Big)^2,$$

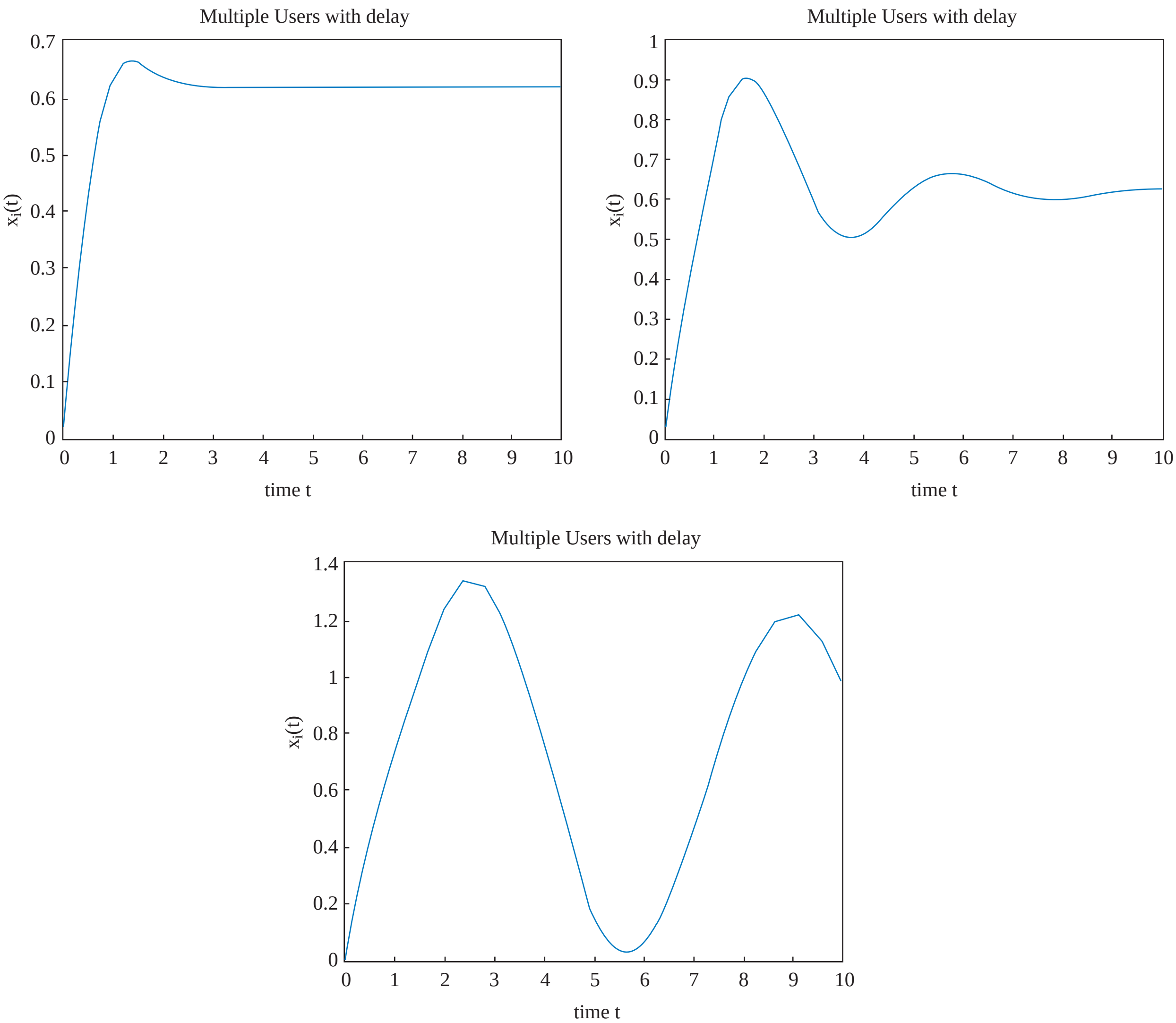

Figure 1. Flow rate versus time of a single-user on a single-link with utility and pricing functions $U = \log(x+1)$, $P = \frac{1}{2}x^2(t-2r)$, and communication delays (from top to bottom) $r = 0.5,\ 1,\ 2$.

where α is the pricing constant. We choose here the cost function, cost parameters, and link capacities in accordance with assumptions *A1-A4*. [3] In the case of a single link shared by M users, the user update algorithm follows directly from (11), and is given by

$$\dot{x_i}(t) = \frac{u_i}{x_i(t)+1} - \alpha \sum_{j:l \in R_j} x_j(t - r_j - r_i), \quad \forall i = 1, \dots, M.$$

We first investigate single-user on a single-link case for illustrative purposes. The parameters in user's cost function are chosen as $u = 1$ and $\alpha = 1$. We simulate the system under communication delays of $r = 0.5$, 1, and 2 as observed in Figure 1. From Theorem 4.1, the condition for stability is calculated as $r < 0.42$. Since this condition is only sufficient the system could remain stable for $r > 0.42$, which is indeed what happens. However, the rate of convergence decreases significantly for increasing r, and for delays $r > 2$ the system becomes unstable.

The effect of the number of users on system stability is demonstrated in the next simulation. The delay in the system is symmetric and chosen as $r_i = 0.5, \forall i$. Figure 2 shows that increasing the number of users has a similar effect as increasing the delay as captured in Theorem 4.1.

In the next set of simulations, the number of users sharing a single link is $M = 10$. The user utility parameters, u_i and delay r_i are randomly chosen with uniform distribution in the ranges $u_i \in [10, 20]$ and $r_i \in [0.1, 0.5]$, respectively. Figure 3a shows flow rates of individual users for the initial condition being the origin for all users, and Figure 3b shows the same for a randomly picked initial condition. Although the delays are larger than the theoretical bound $r_{max} < 0.05$, the system can be seen to be stable.

We next explore extensions to general topologies. Due to computational limitations of the *dde23* solver, we choose a simple network topology with two links as shown in Figure 4. The propagation delay between the two links is 0.2 and users' delays to their corresponding links are chosen randomly with a uniform distribution in the range $[0.1, 0.3]$. Cost parameters are $u = 5$ and $\alpha = 0.5$ and are symmetric for all three users. The results observed in Figure 4 are also in accordance with the results of Theorem 4.1.

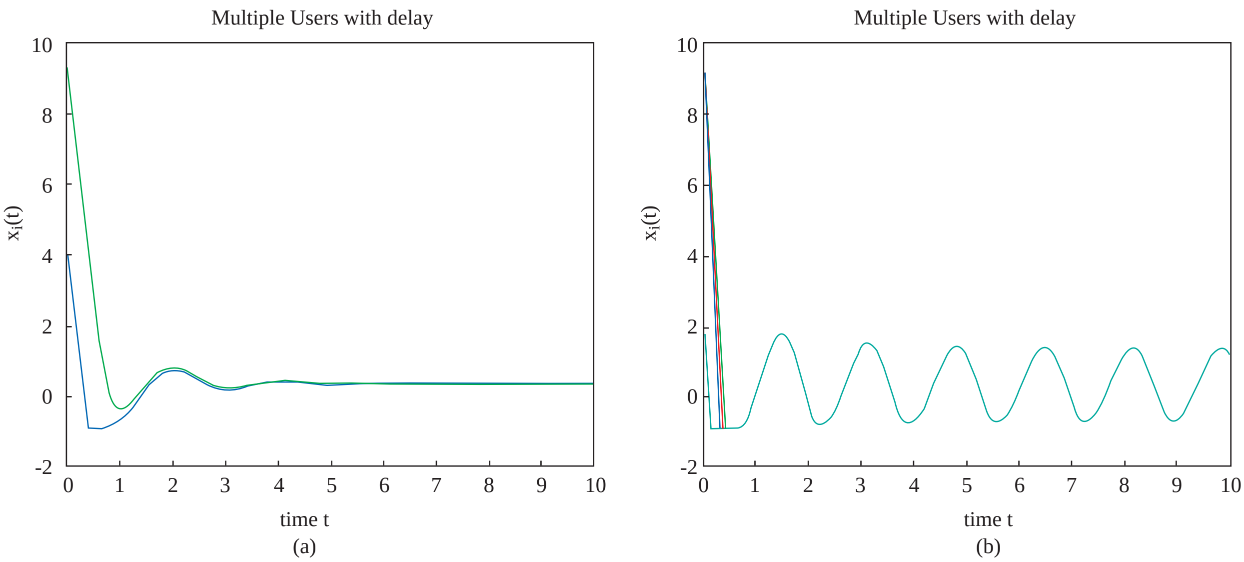

Figure 2. Flow rates of two and four users versus time under the delay $r_i = 0.5, \forall i$ shown in (a) and (b), respectively.

[3] We note, however, that in a network implementation cost parameters α and u can be adjusted "online" through an adaptive algorithm, which takes capacity constraints on the network into account, in order to satisfy *A3* and make the NE an inner solution.

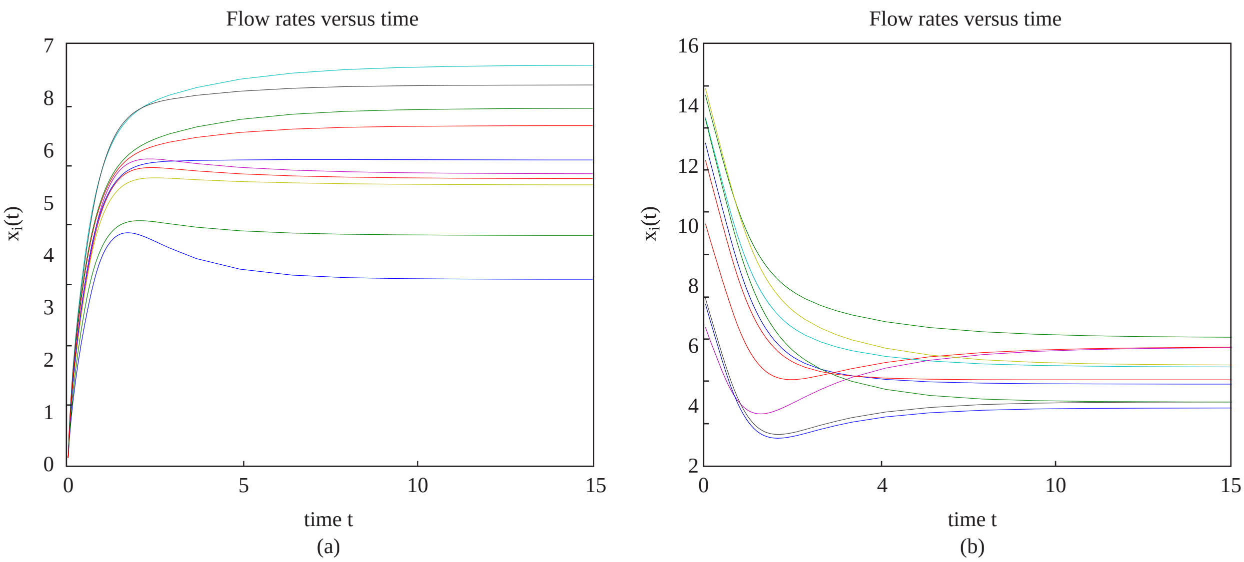

Figure 3. Flow rates of 10 users versus time with origin as the starting point (a), and random initial conditions (b).

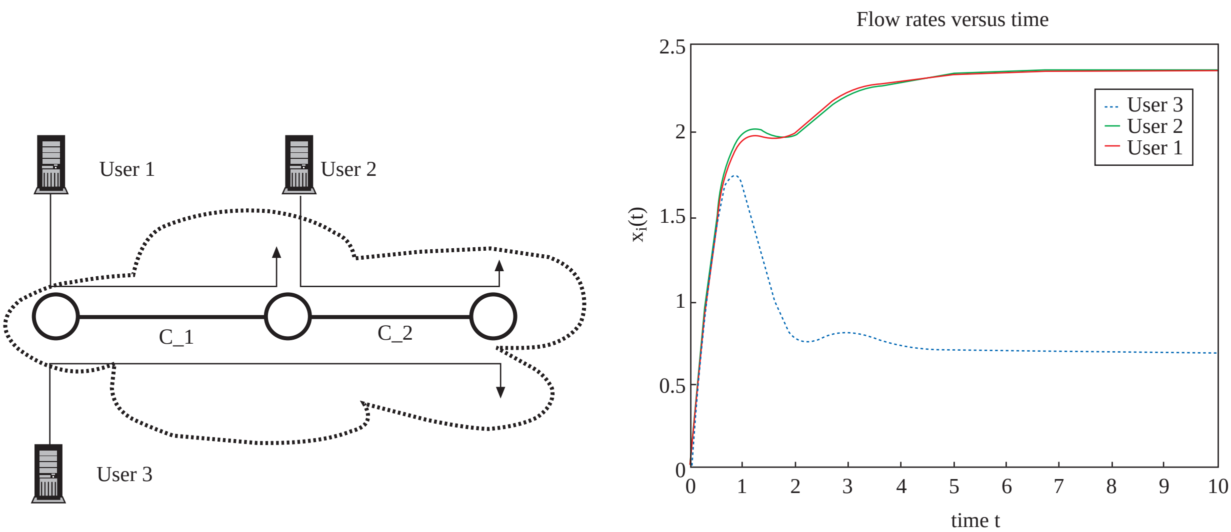

Figure 4. Flow rates of three users versus time on the basic topology shown.

6. Conclusion

In this paper we have analyzed the stability properties of an end-to-end congestion control scheme under fixed heterogeneous delays, and general network topologies. The scheme analyzed is based on a congestion control game within the framework of noncooperative game theory and captures a fairly general class of cost functions. The users under this scheme use a standard gradient algorithm to update their flow rates. We present a sufficient condition for global stability of the unique Nash equilibrium of this game for general network topologies and under a mild symmetricity assumption. The upper-bound on communication delays given in the sufficient condition is inversely proportional to the square root of the number of users multiplied by the cube of a gain constant. We note that this structure is similar to those of local stability results reported in other studies [6, 7, 8].

Finally, we evaluate stability of the congestion control scheme numerically for various delays, user numbers, and simple topologies for a specific cost structure. The theoretical and numerical analyzes indicate a fundamental tradeoff between the responsiveness of the user update algorithm and the stability properties of the system under communication delays.

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