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On the existence of common Lyapunov functions for consensus algorithms based on averaging

Mehmet AKAR[1,*], **Robert SHORTEN**[2]
[1] *Department of Electrical and Electronic Engineering, Bogazici University, Bebek, İstanbul-TURKEY*
e-mail: mehmet.akar@boun.edu.tr
[2] *Hamilton Institute, National University of Ireland, Maynooth, Co. Kildare-IRELAND*
e-mail: robert.shorten@nuim.ie

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Abstract

This paper addresses distributed deterministic consensus algorithms based on averaging. We relate the conditions for achieving consensus to the existence of a common norm for a set of row-stochastic matrices associated with the original set of averaging matrices. For a system to achieve consensus, it is shown that this associated set of matrices should have a Common Lyapunov Function, even if the original set might not have one.

Key Words: *Consensus, synchronization, scrambling matrix, common Lyapunov function, switched systems*

1. Introduction

There has been considerable recent research interest in developing scalable algorithms for the control of multi-agent systems, including autonomous teams of unmanned air/ground/sea vehicles (U/AGS/V) and sensor networks. The individual agents in such networks share a common physical quantity, e.g., heading angle for unmanned autonomous vehicles, and time value for sensor nodes. It is important that all nodes in the network acquire the same state value; for instance, in formation control it is desired that all UAVs fly in the same direction. In wireless sensor network applications, the sensor nodes are the energy constrained, not always dependable devices which are used to sense, compute, and communicate data. It is important for sensor nodes that they share a common clock in order for them to effectively compare their measured data (e.g., temperature, humidity, pollution level). While GPS is used for time stamping in many areas, it may not always be suitable for wireless sensor applications where the tiny sensors may not have a direct path to the GPS signal. There are many other types of problems where achieving a common value is the objective, e.g., in the study of bird and fish flocking behavior; in the study of social networks; in the study of disease propagation; and in the study biological oscillators.

*Corresponding author: Department of Electrical and Electronic Engineering, Boğazici University, Bebek, İstanbul-TURKEY

In this paper, we focus on one particular consensus algorithm based on averaging, where each node updates its value using information collected from its neighbors (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] and references therein). These and related problems arise in a number of applications across a wide variety of disciplines. While the control and systems theory community have studied these algorithms in the context of unmanned air vehicles, consensus related problems also arise in distributed sensing [11], signal processing [12], cloud computing [13], distributed load balancing [13], routing [14], and in congestion control applications [15]. In each of these applications, the problem under study can be reduced to establishing whether distributed algorithms converge to a desired fixed point. A major contribution of this paper is to develop new tools for establishing such facts using the infinity-norm Lyapunov function.

Mathematically, the consensus problem is related to determining classes of matrices whose infinite products converge to a rank one matrix. Wolfowitz's classical result requires each matrix in the product to be ergodic [16]. Convergence to a rank one matrix can also be established under different sets of assumptions that relax, in various ways, the ergodicity requirement on the individual matrices [1]–[10]. Convergence of consensus based strategies are usually studied using techniques from the Markov chain community, e.g., the properties of the Birkhoff coefficient [17, pp. 19]. A purported advantage of this strategy is that convergence, or stability, is derived without resorting to the use of common Lyapunov functions (which are typically hard to find); see for example, [1]. A similar line of justification was followed in [2] to motivate the contribution of [2]. Our motivation in this work is to show that an elementary common Lyapunov function (CLF) derived from the L_∞ norm always exists for such systems.

It is argued in [1] and [2] that the set of averaging matrices under consideration do not have *a common quadratic Lyapunov function* and alternative tools were employed in order to demonstrate the convergence of the consensus algorithm. Furthermore, the non-existence of a common quadratic Lyapunov function was demonstrated analytically in [7] for networks consisting of more than seven nodes. Although non-existence of a common Lyapunov function for the set of averaging matrices under consideration is an important finding, we demonstrate in this paper that this is not as crucial in proving the convergence of the consensus algorithm for the same set of conditions given in [1, 2]. The latter observation stems from the fact that we can prove that indeed such a common Lyapunov function exists for a related set of averaging matrices constructed from the original set of stochastic matrices. We show that this function is elementary to construct, and that many existing results and some new ones can be derived in an elementary manner using this framework of common Lyapunov functions.

The rest of the paper is organized as follows. In Section 2, the switched system formulation of the consensus problem with relevant mathematical preliminaries is discussed. The main contributions of the paper relating scrambling matrices and CLFs to the concepts of ergodicity, network connectivity and consensus are presented in Section 3. Finally, some concluding remarks are given in Section 4.

2. Mathematical preliminaries

Consider a group of n agents with $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \Re^n$ representing the vector of the values (that could be clocks, heading angle, etc) that are to be equalized. The objective is to achieve a common value for all nodes, ideally $x_1(t) = x_2(t) = \cdots = x_n(t)$, for all t. Clearly, this requires at least a single node transmitting to all others for all time; an assumption that usually fails for instance, in mobile ad hoc and sensor networks due to lack of connectivity at certain time intervals. Similarly, the flow of information among the agents in a multi-agent system is restricted due to mobility and variations in the communication neighborhoods. Therefore, we are interested in distributed algorithms that achieve consensus at least asymptotically, i.e., $x_i(t) \to c$ as $t \to \infty$. To this end, let each agent update its value using the averaging algorithm

$$x_i(t+1) = \sum_{j=1}^{n} w_{ij}(t) x_j(t), \tag{1}$$

where $w_{ij}(t)$ are non-negative averaging coefficients that satisfy the following assumption in this paper; see [1, 2, 3, 4, 5, 7].

Assumption 1 *(i) There exists a positive constant* α *such that* $w_{ii}(t) > \alpha$*, for all* i *and* t*.*

(ii) $w_{ij}(t) \in \{0\} \bigcup [\alpha, 1]$ *for all* i*,* j*,* t*.*

(iii) $\sum_{j=1}^{n} w_{ij}(t) = 1$ *for all* i *and* t*.*

The first part of the above assumption makes sure that the updates use individual data. The second part is a condition on whether the data from other agents are used in the update (i.e., with some non-vanishing weight w_{ij} such that $w_{ij} \in [\alpha, 1]$) or not (i.e., $w_{ij} = 0$). We are interested in deriving conditions under which synchronization can be achieved asymptotically, i.e., $x_i(t) \to c$ as $t \to \infty$.

The value update in (1) can also be described by the recursion

$$x(t+1) = W(t)x(t), \; W(t) \in \mathbb{W}, \tag{2}$$

where $W(t)$ in the set $\mathbb{W}$ is a non-negative row-stochastic matrix with positive diagonal elements. We are interested in the dynamic properties of (2) in the subspace

$$\Delta = \{\delta \in \Re^n : \delta^T e = 0\}, \tag{3}$$

where $e = [1, 1, \ldots, 1]^T \in \Re^n$. Let $\Delta^\perp = \text{span}\{e\} = \{ke : k \in \Re\}$. Note that any vector in $\Re^n$ is in the joint span of Δ and $\Delta^\perp$.

The entries of a row-stochastic matrix W are non-negative, and each row adds up to one, i.e., we have $We = e$. Another important property of such matrices is summarized in the following lemma.

Lemma 1 *(Theorem 1.1 in [18], pp. 3–4) Let* y *be a nonnegative vector and* W *a stochastic matrix. If* $z = Wy$ *then*

$$\max_i z_i - \min_i z_i \leq \tau(W)(\max_i y_i - \min_i y_i) \tag{4}$$

where

$$\tau(W) = \frac{1}{2} \max_{i,j} \sum_k |w_{ik} - w_{jk}|. \tag{5}$$

Definition 1 *The parameter* $\tau(W)$ *in (5) is referred to as the* coefficient of ergodicity *of* W*, and it satisfies* $0 \leq \tau(W) \leq 1$ *[18]. Furthermore, if* $\tau(W) < 1$*, the matrix* W *is called a* scrambling *matrix.*

Two important properties of the coefficient of ergodicity are

$$\|\delta^T W\|_1 \leq \tau(W)\|\delta\|_1, \tag{6}$$

for all $\delta \in \Delta$, and

$$S(Wz) \leq \tau(W)S(z), \tag{7}$$

for all vectors z, where $S(z)$ is the spread of a vector z, defined as $S(z) = \max_{i,j}|z_i - z_j|$ (see [18], page 5). From (6) it follows that a scrambling matrix W is contractive in Δ since $\tau(W) < 1$. Similarly from (7), it is seen that W is contractive on the difference of the entries in a vector if we are to consider left multiplication by vectors (see [18], page 5).

As $\tau(W) < 1$ for a scrambling matrix, it can be seen from (4) that successive iterations will lead to a strict decrease in the difference between the maximum and the minimum components of the vector. In the limit as time approaches infinity, this difference vanishes; hence the maximum and the minimum values become equal implying that all components have the same value, i.e., the nodes are synchronized.

A stochastic matrix W is said to be *ergodic* if $\lim_{t\to\infty} W^t = ed^{\mathbb{T}}$ for some d. A scrambling matrix is ergodic [18], but the converse is not true in general, e.g.,

$$W = \begin{bmatrix} 1/10 & 9/10 & 0 & 0 \\ 2/3 & 1/6 & 1/6 & 0 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}. \tag{8}$$

Using (5), the coefficient of ergodicity $\tau(W)$ can be computed to be equal to one, which implies by Definition 1 that this matrix is not scrambling. On the other hand, since we have

$$\lim_{t\to\infty} W^t = \begin{bmatrix} 0.2878 & 0.3885 & 0.1942 & 0.1295 \\ 0.2878 & 0.3885 & 0.1942 & 0.1295 \\ 0.2878 & 0.3885 & 0.1942 & 0.1295 \\ 0.2878 & 0.3885 & 0.1942 & 0.1295 \end{bmatrix} = ed^{\mathbb{T}},$$

where $d = [0.2878\,, 0.3885,\ 0.1942,\ 0.1295]^{\mathbb{T}}$, the matrix W in (8) is ergodic.

2.1. Common Lyapunov functions and pseudocontractivity

The main objective of this paper is to relate the notion of synchronization to the existence of common Lyapunov functions. We will say that $V(x(t))$ is a Lyapunov function for (2) if it satisfies (i) $V(x(t)_{|\Delta}) > 0$ for all $x(t)_{|\Delta} \neq 0$, and $V(0) = 0$, (ii) $V(x(t+1)_{|\Delta}) - V(x(t)_{|\Delta}) < 0$ for all $x(t)_{|\Delta} \neq 0$, where $x(t)_{|\Delta}$ denotes the projection of $x(t)$ onto the subspace Δ. Given a set of row-stochastic matrices, $\mathbb{W} = \{W_1, W_2, \ldots, W_N\}$, consider the N subsystems obtained by using the system matrix $W(t) = W_i$ in (2), i.e.,

$$\Sigma_i:\ x(t+1) = W_i x(t),\ i = 1, 2, \ldots, N. \tag{9}$$

If a Lyapunov function, $V(x(t))$, *that is common to all of the subsystems* Σ_i *exists*, then it is called a *common Lyapunov function* (CLF) for Σ_i, $i = 1, 2, \ldots, N$. Note that, when and if such a CLF exists, it is seen from the definition of the CLF that the trajectories in the subspace Δ will diminish and synchronization will be ensured. The notion of CLF as defined above is related to the pseudocontractivity defined in [8].

Definition 2 *Let T be an operator on $\Re^n$. The operator T is nonexpansive with respect to some norm $\|\cdot\|$ and a closed set $X^\star$ if*

$$\forall x \in \Re^n,\; x^\star \in X^\star,\; \|Tx - x^\star\| \leq \|x - x^\star\|. \tag{10}$$

T is pseudocontractive if it is nonexpansive with respect to $\|\cdot\|$ and $X^\star$ and

$$\forall x \notin \Re^n,\; d(Tx, X^\star) < d(x, X^\star), \tag{11}$$

where $d(x, X^\star)$ is the distance of x to $X^\star$ defined as $d(x, X^\star) = \|x - x_{|X^\star}\|$.

In [8], Su and Bhaya use the notion of pseudocontractivity in order to study the convergence of nonstationary iterative methods for linear systems in which the coefficient matrices are singular M-matrices. In [9], Fang and Antsaklis utilize these tools in the study of consensus algorithms. The following result in [10] relates pseudocontractivity of stochastic matrices to the scrambling condition.

Lemma 2 *(Theorem 6.35 in [10]) Let W be a stochastic matrix. The matrix W is pseudocontractive with respect to $\|\cdot\|_\infty$ and $\Delta^\perp$ if and only if W is a scrambling matrix.*

Remark 1 *Lemma 2 is easily extended to the case of sets of row stochastic matrices. Joint pseudocontractivity of these matrices imply convergence of trajectories to the ray spanned by the vector e (under certain scrambling type assumptions). Note in this case it is not necessary to assume any invariant properties of the subspace Δ.*

By taking $X^\star = \Delta^\perp$ and using the above lemma, we note that $V(x(t)) = \|x(t)_{|\Delta}\|_\infty$ is a Lyapunov function for a system with a scrambling system matrix. Similarly, the same function is also a CLF for a set of systems with corresponding set of scrambling matrices $\mathbb{W}$. Given that such a CLF exists, it is straightforward to note that any trajectory in the subspace Δ is contracting, hence consensus is achieved under arbitrary switching. In the case that the existence of a CLF is not so obvious, it is important to relate the structural properties of the weighting matrices to synchronization.

Example 1 *Consider the averaging matrices*

$$W_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/3 & 2/3 \end{bmatrix}, \tag{12}$$

$$W_2 = \begin{bmatrix} 1/4 & 3/4 & 0 & 0 \\ 0 & 2/3 & 1/3 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}. \tag{13}$$

that correspond to the network topologies in Figures 1, 2, respectively. Similar to the matrix in (8), it can easily be shown that both matrices are ergodic, although neither is scrambling.

Consider the situation in which the network topology is changing arbitrarily between the two shown in Figures 1, 2. The natural question that arises is whether synchronization is achieved under this scenario. Since neither matrix is scrambling, $V(x(t)) = \|x(t)_{|\Delta}\|_\infty$ is not a CLF. On the other hand, we will show that $V(x(t)) = \|x(t)_{|\Delta}\|_\infty$ is indeed a CLF for an associated set of matrices; hence synchronization is achieved under arbitrary switching of these system matrices.

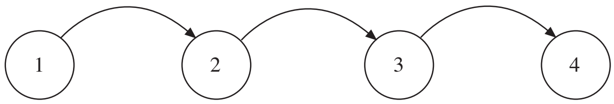

Figure 1. The topology corresponding to W_1 in Example 1.

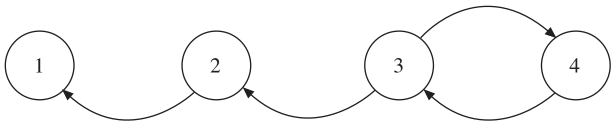

Figure 2. The topology corresponding to W_2 in Example 1.

3. Consensus and CLFs

Given a deterministic and finite set of matrices $\mathbb{W} = \{W_1, W_2, \ldots, W_N\}$ that satisfies Assumption 1, it is important to know whether infinite products of such matrices is ergodic. In this section, we relate the existence conditions of consensus to the existence of CLFs and to scrambling matrices. To this end, define the sets π_k, $k \geq 1$, which contain all products of matrices of length k from the set $\mathbb{W}$, e.g., $\pi_1 = \mathbb{W} = \{W_1, W_2, \ldots, W_N\}$, $\pi_2 = \mathbb{W} \times \pi_1 = \{W_1^2, W_1W_2, \ldots, W_1W_N, W_2W_1, W_2^2, \ldots, W_2W_N, \ldots, W_NW_1, W_NW_2, \ldots, W_N^2\}$, $\pi_k = \mathbb{W} \times \pi_{k-1}$, $k \geq 2$. Associated with π_k, consider the class of switched systems:

$$S_k: \; x(t+1) = W(t)x(t), \; W(t) \in \pi_k. \tag{14}$$

The following result is useful to deduce stability properties of (2).

Theorem 1 *For the systems in (2) and (14), the following statements are equivalent.*

(i) The system in (2) achieves consensus.

(ii) For some positive constant K, all matrices in π_K are scrambling, and $V(x(t)) = \|x(t)_{|\Delta}\|_\infty$ is a CLF.

(iii) For some positive constant K, the system S_K in (14) achieves consensus.

Proof The equivalence of (i) and (iii) follow from the construction of the matrices. The assertion (ii) $\Rightarrow$ (iii) is straightforward. We now prove (i) $\Rightarrow$ (ii). Suppose that there is no positive constant K such that all matrices in π_K are scrambling. Then for the matrix product sequence which is not scrambling, $\|x(t)_{|\Delta}\|_\infty$ cannot decrease in the subspace Δ (by definition of a scrambling matrix and by Lemma 2), and hence consensus cannot be achieved for (2). □

Remark 2 *Theorem 1 relates the synchronization properties of a system to that of an associated system which contains a larger number of system matrices. Although the induced one-norm may not be decreasing in the subspace Δ along the trajectories of each system at each time step, it has to be so for all of the system matrices of the associated system. A similar principle was used in [19] to show that there exists a common quadratic Lyapunov function for the associated system, and hence one could conclude that the switched system is exponentially stable although there exists no such function for the original subsystems.*

Remark 3 *The proof of Theorem 1 notes the existence of the induced one-norm as a CLF for the associated system* S_K *for some positive* K*. Due to the equivalence of norms on finite dimensional spaces, it might be worth reminding the reader that a quadratic CLF also exists for some associated system* $S_{\bar{K}}$*, although* K *and* $\bar{K}$ *are not necessarily equal to each other.*

3.1. Graph representation and relating topology to consensus

In the study of distributed consensus algorithms, it is important to relate the convergence properties of the update rules to the underlying topology of the network [1]–[9]. To this end, we can associate a graph $(V, E(t))$ with (1), where $V = \{1, 2, \ldots, n\}$ is the set of vertices, $E(t)$ is the set of directed edges, and $(j, i) \in E(t)$ holds if and only if $w_{ij}(t) > 0$ (i.e., there is communication from node j to node i). The graph is said to be *symmetric* if $(j, i) \in E(t)$ implies $(i, j) \in E(t)$. A graph is said to be connected if it is symmetric and if there is a path between any two vertices. A connected graph is complete if there is a direct connection between all vertices. An adjacency matrix for (V, E) is defined as an $n \times n$ matrix $G = [g_{ij}]$, where $g_{ij} = 1$ if $w_{ij} > 0$, and $g_{ij} = 0$ if $w_{ij} = 0$. A complete graph has a positive adjacency matrix.

Given the set $\mathbb{W}$, let $\mathbb{G} = \{G_1, G_2, \ldots, G_N\}$ be the set of adjacency matrices for the graphs associated with the matrices W_1, W_2, $\ldots$, W_N. Let $\mathbb{G}_k$ be the similar set for π_k. Under Assumption 1, we have the following results which will subsequently be used to relate ergodicity to CLFs:

Lemma 3 *(i) If every* $G \in \mathbb{G}$ *is connected, then there exists an integer* $K \leq n - 1$ *for which every element of* $\mathbb{G}_K$ *is complete.*
(ii) Assume that every graph G *of* $\mathbb{G}$ *has a node from which there exists a path to all other nodes of the graph. Then there exists an integer* $K \leq n - 1$ *for which every graph of* $\mathbb{G}_K$ *has a node (not necessarily the same) from which there is a direct link to every other node.*

Proof Suppose that every graph $G \in \mathbb{G}$ is connected. Consider an arbitrary product of the adjacency matrices $G = G_{i_1} G_{i_2} \ldots G_{i_K} = [g_{ij}]$ for an arbitrary index set $i_j \in \{1, 2, \ldots, N\}$. Then g_{ij} is given by

$$g_{ij} = \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \cdots \sum_{k_K=1}^{n} g_{i,k_1 k_2} \, \cdots \, g_{i_K, k_K j} \tag{15}$$

where $g_{k,lm}$ is the (l, m)-th component of the adjacency matrix G_k. Recall that $g_{k,lm}$ is non-zero if there is a path of length one from node m to l. Hence g_{ij} in (15) will be non-zero for all i, j and for $K = n - 1$, as the graphs are connected and every node is accessible from every other in at most $n - 1$ steps (This is guaranteed by Assumption 1). This concludes the proof for part (i).

For the second part assume that every graph G of $\mathbb{G}$ has a node j_0 from which there exists a path to all other nodes. From (15), g_{ij} will be non-zero for all i and $j = j_0$, i.e., G will have a positive column, which yields the desired result. □

Example 2 *To illustrate the first part of Lemma 3, consider the averaging matrix in (8), with the corresponding* *adjacency matrix*

$$G = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \tag{16}$$

i.e., $\mathbb{W} = \{W\}$*, and* $\mathbb{G} = \{G\}$*. For* $K = 3$*, we have*

$$G^K = G^3 = \begin{bmatrix} 4 & 5 & 3 & 1 \\ 5 & 7 & 6 & 3 \\ 3 & 6 & 7 & 5 \\ 1 & 3 & 5 & 4 \end{bmatrix}, \tag{17}$$

which is complete.

For the second part of Lemma 3, consider the matrices W_1 *and* W_2 *in (12–13), i.e.,* $\mathbb{W} = \{W_1, W_2\}$*, and* $\mathbb{G} = \{G_1, G_2\}$ *with*

$$G_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \ G_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \tag{18}$$

Let $K = 3$*, and consider* $\mathbb{G}_3 = \{G_1^3, G_1^2G_2, G_1G_2G_1, G_1G_2^2, G_2G_1^2, G_2G_1G_2, G_2^2G_1, G_2^3\}$*. It is not difficult to verify that all of the eight matrices in* $\mathbb{G}_3$ *have a positive column (hence the matrices in* $\pi_3 = \{W_1^3, W_1^2W_2, W_1W_2W_1, W_1W_2^2, W_2W_1^2, W_2W_1W_2, W_2^2W_1, W_2^3\}$ *are all scrambling).*

Theorem 2 *Given a set of matrices* $\mathbb{W}$ *satisfying Assumption 1, suppose that one of the following statements holds for the associated set* $\mathbb{G}$*:*
(i) Every $G \in \mathbb{G}$ *is connected, or*
(ii) For every graph G *of* $\mathbb{G}$*, there is a vertex from which there is a path to all other vertices.*

Then there exists some positive integer $K \leq n-1$ *so that* $\|x(t)_{|\Delta}\|_\infty$ *is a CLF for all systems with matrices in* π_K *and hence consensus is achieved.*

Proof Suppose that every graph in $\mathbb{G}$ is connected. Then by Lemma 3(i), there exists an integer $K \leq n-1$ for which every element of $\mathbb{G}_K$ is complete, which, by definition of a complete graph, implies that all the matrices in π_K are strictly positive (and also scrambling). Therefore, $\|x(t)_{|\Delta}\|_\infty$ is common to all systems with the system matrices in π_K, and consensus is achieved.

Under hypothesis (ii), it follows from Lemma 3(ii) that all the matrices in π_K have a positive column and are scrambling by definition, hence the conclusion. □

Remark 4 *Theorem 2 relates consensus to the existence of a CLF for a set of systems whose system matrices are in* π_K *for some positive integer* $K \leq n-1$*. In the worst case, this implies that one has to check whether the matrices in* π_{n-1} *are scrambling or not, which is straightforward.*

Example 3 *The network topologies associated with the system matrices* W_1 *and* W_2 *in (12–13) satisfy the conditions of Theorem 2, hence consensus is achieved.*

3.2. Extension to arbitrarily switching kopologies

For a particular sequence (rather than arbitrary switching) of stochastic matrices, it is not necessary that each corresponding graph be connected. However, it is necessary that the graphs associated with a product of matrices satisfy the assumptions in Theorem 2 over regular intervals. That is why the following conditions stated in [2] suffice to achieve consensus.

Corollary 1 *(Propositions 1-2 [2]) Consensus is achieved if either (i) the associated graph for* $\Pi_{t=t_0}^{\infty} W(t)$ *is connected for all* $t_0 \geq 1$, *or (ii) there exists* $T \geq 0$ *such that for all* $t \geq 1$, *there is a node connected all other nodes across* $[t, t+T]$.

Remark 5 *Under the jointly connected conditions given in [1, 2], it follows from Theorem 2 that consensus is reached. In [1, 2], the non-existence of a common quadratic Lyapunov function has been pointed out and alternative methods have been proposed to study the convergence of the averaging based consensus algorithm. In this paper, we demonstrate that there is a CLF for an associated set of systems from which results for averaging based consensus algorithms can be obtained.*

3.3. Extensions to time-varying and interval matrices

The above results are extended to time-varying and interval matrices that satisfy Assumption 1 by constructing a set of extreme matrices W_1, W_2, ..., W_N from $W(t) = [w_{ij}(t)]$. For a set of interval matrices, $\tilde{W}_l = [\tilde{w}_{l,ij}]$, where $\tilde{w}_{l,ij} \in [\underline{w}_{l,ij}, \overline{w}_{l,ij}]$, $l = 1, 2, \ldots, N$, the extreme matrices to be considered are simply $W_l = [\underline{w}_{l,ij}]$. As an example consider

$$W(t) \in \left\{ \begin{bmatrix} 1 - w_{1,12} & w_{1,12} \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ w_{2,21} & 1 - w_{2,21} \end{bmatrix} \right\}$$

where $w_{1,12} = [0.2, 0.5]$ and $w_{2,21} = [0.1, 0.8]$. Then

$$W_1 = \begin{bmatrix} 0.5 & 0.2 \\ 0 & 1 \end{bmatrix}, W_2 = \begin{bmatrix} 1 & 0 \\ 0.1 & 0.2 \end{bmatrix},$$

for which consensus is achieved. Further reduction in the number of extreme matrices is possible by eliminating matrices that correspond to the same graphs.

For time-varying matrix sets, the extremal matrices are computed by taking the infimum over the components of $w_{ij}(t)$ over time. For instance, if $W(t)$ assumes

$$W(t) \in \left\{ \begin{bmatrix} 1 - 2^{-t-1} & 2^{-t-1} \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2^{-t-1} & 1 - 2^{-t-1} \end{bmatrix} \right\}.$$

then we take

$$W_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is not necessary to consider the second matrix

$$W_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix},$$

as it has the same graph as W_1. Since conditions of Theorem 2 are not satisfied, it follows that consensus may not be guaranteed for this class of matrices.

4. Concluding remarks

In this paper, we have studied the convergence properties of distributed synchronization algorithms in a deterministic setting. The relation between synchronization and the existence of a common norm for the averaging matrices used has been established. It is important to note the existence of a node (possibly different over different time intervals) distributing information to other nodes of the network either directly or indirectly, so that conditions of Lemma 3 are met and the network could be synchronized [1, 2]. On the other hand, contrary to what is pointed out in [1, 2], the non-existence of a common quadratic Lyapunov function is not critical in the convergence proofs of these algorithms. What is crucial is the existence of a CLF for an expanded set of matrices having a CLF, even if the original set of averaging matrices might not have one. The results can also be extended to time-varying and interval matrices that satisfy Assumption 1 by constructing a set of extreme matrices.

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