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Learning control of robot manipulators in the presence of additive disturbances*

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Abstract

In this paper, a learning controller for robot manipulators is developed. The controller is proven to yield in a semi-global asymptotic result in the presence of additive input and output disturbances. Lyapunov-based techniques are used to guarantee that the tracking error is asymptotically driven to zero. Numerical simulation results are presented to demonstrate the viability of the proposed learning controller.

Key Words: *Learning control, disturbance rejection, Lyapunov–based methods.*

1. Introduction

In model-based control, the control law is developed based on a model that is free of noise, disturbances, and unmodeled dynamics. However, in practice, the physical system model is usually imprecise due to the effects of the above mentioned discrepancies. A control law designed for a disturbance free system model may not compensate for the disturbances or may even go unstable in the case of small disturbances. Iterative learning control (ILC) can compensate for repeating disturbances, simply by learning from previous iterations. However, a combination of a feedback controller along with ILC is usually considered to reject nonrepeating disturbances [2].

Some of the past research efforts were focused on designing adaptive learning control laws for the robot manipulators in the presence of structured and unstructured uncertainties [3], [4], [5], [6], [7]. In [3], [4], [5], Tayebi *et al.* designed adaptive iterative learning controllers for tracking control of robot manipulators under the assumption that the desired trajectory was periodic. In [6], Sun *et al.* presented an adaptive repetitive learning controller for tracking control of robot manipulators in the presence of unstructured uncertainties. In [7], Yang *et al.* decomposed the dynamic model of the robot manipulator into repetitive and nonrepetitive parts, and designed an adaptive robust iterative learning controller.

The focus of some of the previous research was designing learning controllers to overcome the effects of the disturbances. In [8], an iterative learning controller combined with a feedback controller for a class

* Preliminary results have appeared in [1].

of constrained mechanical systems in the presence of bounded unknown disturbances was presented. In [9], Liu *et al.* suggested a learning controller based on a disturbance observer for a class of nonlinear systems in the presence of unknown disturbances. In [10], Jiang *et al.* proposed an iterative learning controller for a class of minimum-phase multi-input multi-output nonlinear systems with an unknown control gain matrix in the presence of repeating uncertainties. In [11], Dixon *et al.* proposed a learning-based estimate to achieve asymptotic tracking in the presence of a nonlinear periodic disturbance. In [12] and [13], Horowitz *et al.* proposed a repetitive update rule with use of kernel and influence functions, however the usage of these functions tends to be rather complicated. In [14], a command-based iterative learning algorithm was developed to compensate for friction, disturbance and noise effects. In [15] and [16], the researchers suggested a repetitive control scheme. In [14], [15] and [16], the researchers used the so-called Q-filter to enhance the robustness of the controllers, however this prevents the tracking errors to be driven to zero. For work related to learning control techniques the reader is referred to [2], [17], [18] and the references therein.

In this paper, a learning controller for robot manipulators is designed under the assumption that the desired trajectory is periodic. The dynamic model of the robot manipulator is assumed to be uncertain and subject to additive input and output disturbances. In the design of the learning controller, the robust control component in [19] is combined with a nonlinear learning control component to compensate for the uncertain system dynamics and a semi-global asymptotic tracking result is obtained in the presence of additive input and output disturbances [1]. The only assumption imposed on the disturbances is that they were assumed to be twice continuously differentiable and have bounded time derivatives up to second order. In the controller design, Lyapunov-based techniques are used to guarantee that the tracking error is asymptotically driven to zero, and numerical simulation results are presented to demonstrate the performance of the proposed learning controller.

2. Dynamic model

The dynamic model for an n-joint, revolute, direct-drive robot manipulator is considered to be of the following form

$$M(q)(\ddot{q}+d_1)+B(q,\dot{q})=\tau+d_2 \tag{1}$$

where $q(t), \dot{q}(t), \ddot{q}(t)\in\mathbb{R}^n$ denote the joint positions, velocities, and accelerations, respectively, $M(q)\in\mathbb{R}^{n\times n}$ represents the inertia effects, $B(q,\dot{q})\in\mathbb{R}^n$ represents the dynamic effects due to Centripetal and Coriolis forces, gravity and dynamic friction, $d_1(t), d_2(t)\in\mathbb{R}^n$ are unknown nonlinear disturbances, $\tau(t)\in\mathbb{R}^n$ represents the control input vector. It is assumed that the inertia matrix and the other dynamic effects are uncertain. The system model in (1) is assumed to satisfy the following assumptions.

Assumption 1 *The nonlinear functions, $M(\cdot)$ and $B(\cdot)$, are continuously differentiable up to their second order time derivatives (i.e., $M(\cdot), B(\cdot)\in\mathcal{C}^2$).*

Assumption 2 *The additive disturbances, $d_1(t)$ and $d_2(t)$, are assumed to be continuously differentiable and bounded up to their second order time derivatives (i.e., $d_i(t)\in\mathcal{C}^2$ and $d_i(t), \dot{d}_i(t), \ddot{d}_i(t)\in\mathcal{L}_\infty$, $\forall i=1,2$).*

Assumption 3 *The subsequent development utilizes the property that the inertia matrix is positive definite, symmetric and satisfies the following inequalities [20]*

$$\underline{m}\left\|\xi\right\|^2\leq\xi^T M(\cdot)\xi\leq\bar{m}(\cdot)\left\|\xi\right\|^2\quad\forall\xi\in\mathbb{R}^n \tag{2}$$

where $\underline{m} \in \mathbb{R}$ *is a positive bounding constant,* $\bar{m}(q) \in \mathbb{R}$ *is a positive, globally invertible, nondecreasing function of its argument, and* $\|\cdot\|$ *denotes the Euclidean norm.*

To facilitate the subsequent design, the output tracking error $e_1(t) \in \mathbb{R}^n$ is defined as

$$e_1 \triangleq q_d - q \tag{3}$$

where $q_d(t) \in \mathbb{R}^n$ is the desired trajectory satisfying the following properties

$$q_d^{(i)}(t+T) = x_d^{(i)}(t) \quad , \quad \forall i = 0, 1, ..., 4 \tag{4}$$

$$q_d^{(i)}(t) \in \mathcal{L}_\infty \quad , \quad \forall i = 0, 1, ..., 4 \tag{5}$$

where $T \in \mathbb{R}^+$ is its period.

The control design objective is to develop a learning control law that ensures $\|e_1(t)\| \to 0$, while keeping all signals remain bounded under the closed-loop system. To achieve the tracking control objective, the subsequent development is derived based on the assumption that the joint position and velocity measurements are available for control development.

3. Development of the learning control law

To facilitate the control development, the filtered tracking error signals, $e_2(t)$, $r(t) \in \mathbb{R}^n$ are defined as follows

$$e_2 \triangleq \dot{e}_1 + \lambda_1 e_1 \tag{6}$$

$$r \triangleq \dot{e}_2 + \lambda_2 e_2 \tag{7}$$

where λ_1, $\lambda_2 \in \mathbb{R}^{n \times n}$ are constant, diagonal, positive definite, gain matrices. After taking the time derivative of (7) and premultiplying by $M(\cdot)$, the following expression can be derived

$$M\dot{r} = M(\dddot{q}_d + \lambda_1 \ddot{e}_1 + \lambda_2 \dot{e}_2) + \dot{M}\ddot{q} + \dot{B} - \dot{\tau} - \dot{d}_2 + M\dot{d}_1 + \dot{M}d_1 \tag{8}$$

where the first time derivative of (1) and (3) were utilized. The expression in (8) can be arranged as

$$M\dot{r} = -\frac{1}{2}\dot{M}r - e_2 - \dot{\tau} + N - \dot{d}_2 + M\dot{d}_1 + \dot{M}d_1 \tag{9}$$

where the auxiliary function $N(q, \dot{q}, \ddot{q}, t) \in \mathbb{R}^n$ is defined as

$$N \triangleq M(\dddot{q}_d + \lambda_1 \ddot{e}_1 + \lambda_2 \dot{e}_2) + \dot{M}\left(\ddot{q} + \frac{1}{2}r\right) + e_2 + \dot{B}. \tag{10}$$

To facilitate the subsequent analysis, (9) can be rearranged as follows

$$M\dot{r} = -\frac{1}{2}\dot{M}r - e_2 - \dot{\tau} + \tilde{N} + N_d + \psi \tag{11}$$

where auxiliary functions $\tilde{N}(q, \dot{q}, \ddot{q}, t)$, $N_d(t)$, $\psi(t) \in \mathbb{R}^n$ are defined as follows

$$\tilde{N} \triangleq \left(N + M\dot{d}_1 + \dot{M}d_1\right) - \left(N_d + M_d\dot{d}_1 + \dot{M}_d d_1\right) \tag{12}$$

$$N_d \triangleq N|_{q=q_d,\dot{q}=\dot{q}_d,\ddot{q}=\ddot{q}_d} \tag{13}$$

$$\psi \triangleq -\dot{d}_2 + M_d\dot{d}_1 + \dot{M}_d d_1 \tag{14}$$

and $M_d(t) \in \mathbb{R}^{n\times n}$ is defined as follows

$$M_d \triangleq M|_{q=q_d}. \tag{15}$$

Remark 1 *By utilizing the Mean Value Theorem [21] along with Assumptions 1 and 2, the following upper bound can be developed*

$$\left\|\tilde{N}(\cdot)\right\| \leq \rho(\|z\|)\|z\| \tag{16}$$

where $z(t) \in \mathbb{R}^{3n}$ *is defined as follows*

$$z \triangleq \begin{bmatrix} e_1^T & e_2^T & r^T \end{bmatrix}^T \tag{17}$$

and $\rho(\cdot) \in \mathbb{R}$ *is a globally invertible, nondecreasing, nonnegative function.*

Remark 2 *After utilizing (5) and Assumption 2 along with (14) and its first time derivative, it is clear that* $\psi(t),\ \dot{\psi}(t) \in \mathcal{L}_\infty$.

Remark 3 *After utilizing (5) and (10) along with (13) and its first time derivative, it is clear that* $N_d(t)$, $\dot{N}_d(t) \in \mathcal{L}_\infty$.

Remark 4 *After utilizing (4), it is clear that* $N_d(t)$ *satisfies the following equation*

$$N_d(t+T) = N_d(t). \tag{18}$$

Based on (11), the control input is designed as

$$\tau \triangleq (K + I_n)\left[e_2(t) - e_2(t_0) + \lambda_2\int_{t_0}^{t} e_2(\sigma)\,d\sigma\right] + (C_1 + C_2)\int_{t_0}^{t} Sgn(e_2(\sigma))\,d\sigma + \hat{W}_d(t) \tag{19}$$

where K, C_1, $C_2 \in \mathbb{R}^{n\times n}$ are constant, diagonal, positive definite, gain matrices, $I_n \in \mathbb{R}^{n\times n}$ is the standard identity matrix, and $\mathrm{Sgn}(\cdot)$ is the vector signum function defined as follows

$$Sgn(\xi) \triangleq \begin{bmatrix} sgn(\xi_1) & \cdots & sgn(\xi_n) \end{bmatrix}^T, \forall\xi = \begin{bmatrix} \xi_1 & \cdots & \xi_n \end{bmatrix}^T. \tag{20}$$

In (19), $\hat{W}_d(t) \in \mathbb{R}^n$ is an auxiliary function designed as

$$\hat{W}_d(t) \triangleq k_L\left[e_2(t) - e_2(t_0) + \lambda_2\int_{t_0}^{t} e_2(\sigma)\,d\sigma\right] + \hat{W}_d(t-T) \tag{21}$$

where $k_L \in \mathbb{R}$ is a positive control gain. It should be noted that since $\hat{W}_d(t_0) = 0_{n\times 1}$ it follows that $u(t_0) = 0_{n\times 1}$ where $0_{n\times 1} \in \mathbb{R}^{n\times 1}$ is a zero vector. The auxiliary function $\hat{N}_d(t) \in \mathbb{R}^n$ is defined as

$$\hat{N}_d \triangleq \dot{\hat{W}}_d. \tag{22}$$

By utilizing (22) along with (21), the following can be obtained

$$\hat{N}_d(t) = \hat{N}_d(t-T) + k_L r(t). \tag{23}$$

The time derivative of the control input is obtained as

$$\dot{\tau} = (K + I_n) r + (C_1 + C_2) Sgn(e_2) + \hat{N}_d(t) \tag{24}$$

where (22) was utilized. Finally, after substituting (24) into (11), the closed-loop error system for $r(t)$ is obtained as follows

$$M\dot{r} = -\frac{1}{2}\dot{M}r - e_2 - (K + I_n) r - (C_1 + C_2) Sgn(e_2) + \tilde{N} + \tilde{N}_d + \psi \tag{25}$$

where $\tilde{N}_d(t) \in \mathbb{R}^n$ is an auxiliary function defined as

$$\tilde{N}_d \triangleq N_d - \hat{N}_d. \tag{26}$$

It should be noted that, after utilizing (18) and (23), the following expression can be obtained for $\tilde{N}_d(t)$

$$\tilde{N}_d(t) = \tilde{N}_d(t-T) - k_L r. \tag{27}$$

3.1. Stability analysis

Theorem 1 *The control law in (19) and (21) ensures the boundedness of all system signals within closed-loop operation and the output tracking error and its time derivatives are driven to zero in the sense that* $\|e_1(t)\| \to 0$ *as* $t \to \infty$ *provided that the elements of* K *are selected sufficiently large relative to the system initial conditions and*

$$\lambda_{\min}(\lambda_2) > \frac{1}{2}, \tag{28}$$

$$C_{1i} > \|\psi_i(t)\|_{\mathcal{L}_\infty} + \frac{1}{\lambda_{2i}} \left\|\dot{\psi}_i(t)\right\|_{\mathcal{L}_\infty} \tag{29}$$

where the subscript $i = 1, ..., n$ *denotes the* i*th element of the vector or diagonal matrix.*

Proof See Appendix A.

4. Numerical simulation results

A numerical simulation was performed to demonstrate the viability of the proposed robust learning control algorithm. The following 2-link, revolute robot dynamic model was utilized where $M(\cdot)$ and $B(\cdot)$ are defined as follows [22]

$$M = \begin{bmatrix} 3.12 + 2\sin(q_{a_2}) & 0.75 + \sin(q_{a_2}) \\ 0.75 + \sin(q_{a_2}) & 0.75 \end{bmatrix}, \quad B = \begin{bmatrix} 0.75\dot{q}_{a_2}(2\dot{q}_{a_1} + \dot{q}_{a_2})\sin(q_{a_2}) \\ -0.75(\dot{q}_{a_1})^2 \sin(q_{a_2}) \end{bmatrix}.$$

The additive disturbances were chosen as

$$d_1 = \begin{bmatrix} \cos(3t) + \exp(-0.5t) \\ \sin(2t) + \exp(-0.5t) \end{bmatrix} \quad , \quad d_2 = \begin{bmatrix} \sin(3t) + \exp(-0.5t) \\ \cos(2t) + \exp(-0.5t) \end{bmatrix}.$$

The control gains were chosen as $\lambda_1 = \lambda_2 = I_2$, $C_1 = C_2 = 10I_2$, $K = 100I_2$, and $k_L = 10$ where $I_2 \in \mathbb{R}^{2\times 2}$ denotes the standard identity matrix.

The joint positions and the tracking error are presented in Figure 1 and it is clear that the tracking objective was achieved.

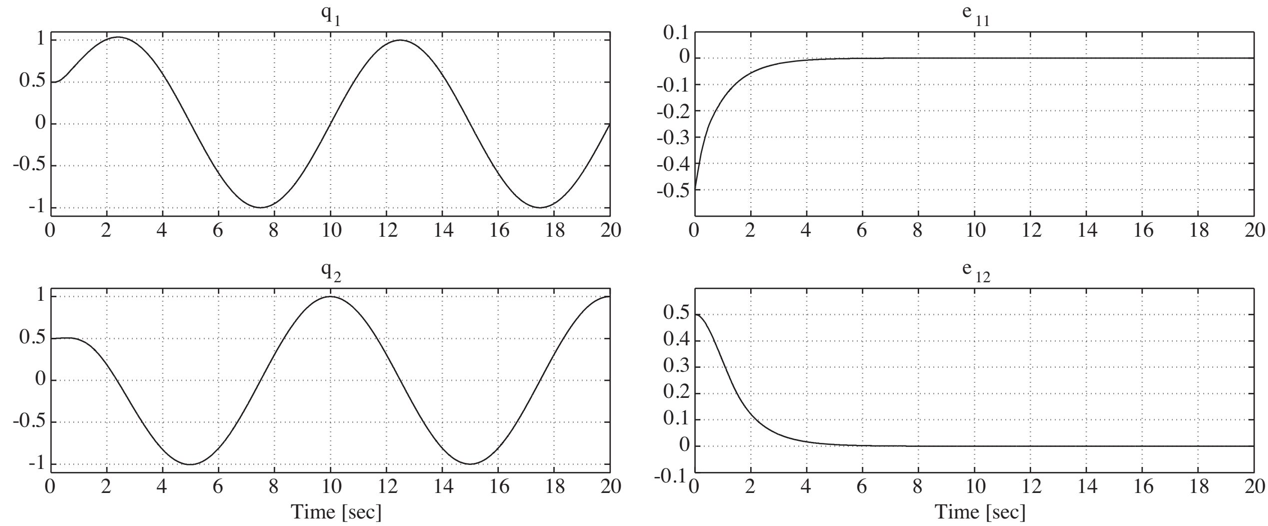

Figure 1. Joint positions $q(t)$ (left) and Tracking error $e_1(t)$ (right)

5. Conclusion

A learning controller was developed for robot manipulators. The controller was proven to yield in a semi-global asymptotic result in the presence of additive disturbances provided that the assumption that the desired trajectory is periodic. Since no assumptions were made on the periodicity of the disturbances, it is clear that the suggested controller compensated for both repeating and nonrepeating disturbances. Lyapunov-based techniques were used to guarantee that the tracking error is asymptotically driven to zero. Numerical simulation results were presented to show the proof of concept.

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A. Proof of Theorem 1

In the subsequent stability analysis the following lemma is utilized.

Lemma 1 *Let the auxiliary functions* $L_1(t)$, $L_2(t) \in \mathbb{R}$ *be defined as follows*

$$L_1 \triangleq r^T(\psi - C_1 Sgn(e_2)) \quad , \quad L_2 \triangleq -\dot{e}_2^T C_2 Sgn(e_2). \tag{30}$$

If C_1 *is selected to satisfy the sufficient condition (29), then*

$$\textstyle\int_{t_0}^{t} L_1(\tau)\, d\tau \leq \zeta_{b1} \quad , \quad \int_{t_0}^{t} L_2(\tau)\, d\tau \leq \zeta_{b2} \tag{31}$$

where ζ_{b1}, $\zeta_{b2} \in \mathbb{R}$ *are positive constants defined as*

$$\zeta_{b1} \triangleq \textstyle\sum_{i=1}^{m} C_{1i}\,|e_{2i}(t_0)| - e_2^T(t_0)\,\psi(t_0) \quad , \quad \zeta_{b2} \triangleq \sum_{i=1}^{m} C_{2i}\,|e_{2i}(t_0)|. \tag{32}$$

Proof See [23] for $L_1(t)$ and [19] for $L_2(t)$.

Proof of Theorem 1 is given as follows.

Proof Let the auxiliary functions $P_1(t)$, $P_2(t) \in \mathbb{R}$ be defined as follows

$$P_1 \triangleq \zeta_{b1} - \textstyle\int_{t_0}^{t} L_1(\tau)\, d\tau \quad , \quad P_2 \triangleq \zeta_{b2} - \int_{t_0}^{t} L_2(\tau)\, d\tau \tag{33}$$

where $L_1(t)$, $L_2(t)$, ζ_{b1} and ζ_{b2} were defined in (30)-(32). The proof of Lemma 1 ensures that $P_1(t)$ and $P_2(t)$ are non-negative. Let $V(s(t), t) \in \mathbb{R}$ denotes the following non-negative function

$$V \triangleq \frac{1}{2}e_1^T e_1 + \frac{1}{2}e_2^T e_2 + \frac{1}{2}r^T M r + P_1 + P_2 + V_g \tag{34}$$

where $V_g(t) \in \mathbb{R}$ is a non-negative function defined as

$$V_g \triangleq \frac{1}{2k_L}\int_{t-T}^{t} \tilde{N}_d^T(\sigma)\,\tilde{N}_d(\sigma)\, d\sigma \tag{35}$$

where $s(t) \in \mathbb{R}^{(3n+3)\times 1}$ is defined as follows

$$s \triangleq \begin{bmatrix} z^T & \sqrt{P_1} & \sqrt{P_2} & \sqrt{V_g} \end{bmatrix}^T. \tag{36}$$

After utilizing (2), the expression in (34) can be bounded as

$$W_1(s) \leq V(s,t) \leq W_2(s) \tag{37}$$

where $W_1(s)$, $W_2(s) \in \mathbb{R}$ are defined as follows

$$W_1(s) \triangleq \alpha_1 \|s\|^2 \quad , \quad W_2(s) \triangleq \alpha_2(\|s\|)\,\|s\|^2 \tag{38}$$

and α_1, $\alpha_2(\cdot) \in \mathbb{R}$ are defined as follows

$$\alpha_1 \triangleq \frac{1}{2}\min\{1, \underline{m}\} \ , \ \alpha_2 \triangleq \max\left\{1, \frac{1}{2}\bar{m}(\|s\|)\right\}. \tag{39}$$

After taking the time derivative of (34), the following expression can be obtained

$$\dot{V} = -e_1^T \lambda_1 e_1 - e_2^T \lambda_2 e_2 + e_1^T e_2 - r^T r + r^T \tilde{N} - r^T K r - \frac{k_L}{2} r^T r - e_2^T \lambda_2^T C_2 Sgn(e_2) \quad (40)$$

where (6), (7), (25), (27), and (30) were utilized. By utilizing (16), (28), and the triangle inequality, an upper-bound on (40) can be obtained as follows

$$\begin{aligned} \dot{V} &\leq -\lambda_3 \|z\|^2 - \left(\lambda_{\min}(K) + \frac{k_L}{2}\right) \|r\|^2 + \|r\| \rho(\|z\|) \|z\| - e_2^T \Lambda^T C_2 Sgn(e_2) \\ &\leq -\left(\lambda_4 - \frac{\rho^2(\|z\|)}{4\lambda_{\min}(K)}\right) \|z\|^2 - e_2^T \lambda_2^T C_2 Sgn(e_2) \end{aligned} \quad (41)$$

where $\lambda_3 \triangleq \min\left\{\frac{1}{2}, \lambda_{\min}(\lambda_2) - \frac{1}{2}\right\}$ and $\lambda_4 \triangleq \min\left\{\lambda_3, \frac{k_L}{2}\right\}$. The following inequality can be developed

$$\dot{V} \leq W(s) - e_n^T \Lambda^T C_2 Sgn(e_2) \quad (42)$$

where $W(s) \in \mathbb{R}$ denotes the following non-positive function

$$W(s) \triangleq -\beta_0 \|z\|^2 \quad (43)$$

with $\beta_0 \in \mathbb{R}$ being a positive constant, and provided that $\lambda_{\min}(K)$ is selected according to the following sufficient condition

$$\lambda_{\min}(K) \geq \frac{\rho^2(\|z\|)}{4\lambda_4}. \quad (44)$$

Based on (34)-(39) and (41)-(43), the regions D and S can be defined as follows

$$\mathcal{D} = \left\{s : \|s\| < \rho^{-1}\left(2\sqrt{\lambda_4 \lambda_{\min}(K)}\right)\right\} \quad (45)$$

$$\mathcal{S} = \left\{s \in \mathcal{D} : W_2(s) < \alpha_1 \left(\rho^{-1}\left(2\sqrt{\lambda_4 \lambda_{\min}(K)}\right)\right)^2\right\}. \quad (46)$$

Note that the region of attraction in (46) can be made arbitrarily large to include any initial condition by increasing $\lambda_{\min}(K)$ (*i.e.*, a semi-global stability result). Specifically, (38) and (46) can be used to calculate the region of attraction as follows

$$W_2(s(t_0)) < \alpha_1 \left(\rho^{-1}\left(2\sqrt{\lambda_4 \lambda_{\min}(K)}\right)\right)^2 \Longrightarrow \|s(t_0)\| < \sqrt{\frac{\alpha_1}{\alpha_2(\|s(t_0)\|)}} \rho^{-1}\left(2\sqrt{\lambda_4 \lambda_{\min}(K)}\right), \quad (47)$$

which can be rearranged as

$$\lambda_{\min}(K) \geq \frac{1}{4\lambda_4} \rho^2 \left(\sqrt{\frac{\alpha_2(\|s(t_0)\|)}{\alpha_1}} \|s(t_0)\|\right). \quad (48)$$

By utilizing (17), (32), and (36) the following explicit expression for $\|s(t_0)\|$ can be derived as follows

$$\|s(t_0)\|^2 = \|e_1(t_0)\|^2 + \|e_2(t_0)\|^2 + \|r(t_0)\|^2 + \zeta_{b1} + \zeta_{b2}. \quad (49)$$

From (34), (42), (46)-(48), it is clear that $V(s,t) \in \mathcal{L}_\infty$ $\forall s(t_0) \in \mathcal{S}$; hence $s(t), z(t) \in \mathcal{L}_\infty$ $\forall s(t_0) \in \mathcal{S}$. From (42), it is clear that $e_2(t) \in \mathcal{L}_1$ $\forall s(t_0) \in \mathcal{S}$. From (7), it is clear that $\dot{e}_2(t) \in \mathcal{L}_\infty$ $\forall s(t_0) \in \mathcal{S}$. The above boundedness statements can be utilized along with (6) and its time derivative to show that $e_1(t)$, $\dot{e}_1(t)$, $\ddot{e}_1(t) \in \mathcal{L}_\infty$ $\forall s(t_0) \in \mathcal{S}$. Using (3), and (5), and their time derivatives, it can be proven that $q(t)$, $\dot{q}(t)$, $\ddot{q}(t) \in \mathcal{L}_\infty$ $\forall s(t_0) \in \mathcal{S}$. Then, it is clear that $M(\cdot)$, $\dot{M}(\cdot)$, $f(\cdot) \in \mathcal{L}_\infty$ $\forall s(t_0) \in \mathcal{S}$. By using these boundedness statements along with (1) it is clear that $\tau(t) \in \mathcal{L}_\infty$ $\forall s(t_0) \in \mathcal{S}$. These boundedness statements can be used along with the time derivative of (43) to prove that $\dot{W}(s(t)) \in \mathcal{L}_\infty$ $\forall s(t_0) \in \mathcal{S}$; hence $W(s(t))$ is uniformly continuous. A direct application of Theorem 8.4 in [21] can be used to prove that $\|z(t)\| \to 0$ as $t \to \infty$ $\forall s(t_0) \in \mathcal{S}$. Then, from (17), it is clear that $\|e_1(t)\|$, $\|e_2(t)\|$, $\|r(t)\| \to 0$ as $t \to \infty$ $\forall s(t_0) \in \mathcal{S}$. Since $\|e_2(t)\|$, $\|r(t)\| \to 0$ as $t \to \infty$ $\forall s(t_0) \in \mathcal{S}$, from (7), it is easy to see that $\|\dot{e}_2(t)\| \to 0$ as $t \to \infty$ $\forall s(t_0) \in \mathcal{S}$. Finally, (6) and its time derivative can be used to prove that $\|\dot{e}_1(t)\|$, $\|\ddot{e}_1(t)\| \to 0$ as $t \to \infty$ $\forall s(t_0) \in \mathcal{S}$. This proves that the tracking control objective was met.

Other control objective is to prove that all signals remain bounded under the closed-loop operation. In that sense, the main objective is to show that $\hat{W}_d(t)$ is bounded. For $\forall t \in (t_0, t_0+T)$, from (21), the following expression can be written for $\hat{W}_d(t)$

$$\hat{W}_d(t) = k_L\left[e_2(t) - e_2(t_0) + \lambda_2 \int_{t_0}^{t} e_2(\sigma)\, d\sigma\right] \tag{50}$$

where the fact that $\hat{W}_d(t) = 0_{n\times 1}$ $\forall t < t_0$ was utilized. Since $e_2(t) \in \mathcal{L}_1 \cap \mathcal{L}_\infty$ $\forall s(t_0) \in \mathcal{S}$ then it can be proven that $\int_{t_0}^{t} e_2(\sigma)\, d\sigma \in \mathcal{L}_\infty$ $\forall s(t_0) \in \mathcal{S}$; thus, $\hat{W}_d(t) \in \mathcal{L}_\infty$ $\forall t \in (t_0, t_0+T)$, $\forall s(t_0) \in \mathcal{S}$. From (21), it can be proven that $\hat{W}_d(t) \in \mathcal{L}_\infty$ $\forall t \in (t_0+T, t_0+2T)$, $\forall s(t_0) \in \mathcal{S}$ (since $\hat{W}_r(t-T) \in \mathcal{L}_\infty$ $\forall t \in (t_0, t_0+T)$, $\forall s(t_0) \in \mathcal{S}$). Similarly, for any finite m, it can be proven that $\hat{W}_d(t) \in \mathcal{L}_\infty$ $\forall t \in (t_0+(m-1)T, t_0+mT)$, $\forall s(t_0) \in \mathcal{S}$ (since $\hat{W}_d(t-T) \in \mathcal{L}_\infty$ $\forall t \in (t_0+(m-2)T, t_0+(m-1)T)$, $\forall s(t_0) \in \mathcal{S}$). For any finite m, it can be concluded that $\hat{W}_d(t) \in \mathcal{L}_\infty$ $\forall t \in (t_0, t_0+mT)$, $\forall s(t_0) \in \mathcal{S}$. At this point, the extended space $\mathcal{L}_{\infty e}$ will be defined to facilitate the subsequent analysis. The extended space contains the functions whose $\mathcal{L}_\infty$ norm may grow to infinity, but only at infinity and it is clear that $\mathcal{L}_\infty \subset \mathcal{L}_{\infty e}$ [24]. When m goes to infinity, from the definition of the extended space $\mathcal{L}_{\infty e}$, it can be proven that $\hat{W}_d(t) \in \mathcal{L}_{\infty e}$ $\forall t$, $\forall s(t_0) \in \mathcal{S}$. From (23), the following expression can be written for $\hat{N}_d(t)$

$$\hat{N}_d(t) = k_L r(t)\, \forall t \in (t_0, t_0+T) \tag{51}$$

where the fact that $\hat{N}_d(t) = 0_{n\times 1}$ $\forall t < t_0$ was utilized. Since $r(t) \in \mathcal{L}_\infty$ $\forall s(t_0) \in \mathcal{S}$, then, from (51), it is clear that $\hat{N}_d(t) \in \mathcal{L}_\infty$ $\forall t \in (t_0, t_0+T)$, $\forall s(t_0) \in \mathcal{S}$. For any finite m, it can be concluded that $\hat{N}_d(t) \in \mathcal{L}_\infty$ $\forall t \in (t_0, t_0+mT)$, $\forall s(t_0) \in \mathcal{S}$. Similarly, when m goes to infinity, it can be proven that $\hat{N}_d(t) \in \mathcal{L}_{\infty e}$ $\forall t$, $\forall s(t_0) \in \mathcal{S}$. Then, from Remark 3 and (26), it is clear that $\tilde{N}_d(t) \in \mathcal{L}_{\infty e}$ $\forall t$, $\forall s(t_0) \in \mathcal{S}$. From (24), it can be proven that $\dot{\tau}(t) \in \mathcal{L}_{\infty e}$ $\forall t$, $\forall s(t_0) \in \mathcal{S}$. Previous boundedness statements can be utilized along with (11) to show that $\dot{r}(t) \in \mathcal{L}_{\infty e}$ $\forall t$, $\forall s(t_0) \in \mathcal{S}$. This proves that the boundedness objective was met. □