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Design parameters and uncertainty bound estimation functions for adaptive-robust control of robot manipulators

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Abstract

In this paper, a parameter and uncertainty bound estimation functions for adaptive-robust control of robot manipulators are developed. A Lyapunov function is defined and parameters and uncertainty bound estimation functions are developed based on the Lyapunov function. Thus, stability of an uncertain system is guaranteed and uniform boundedness of the tracking error is achieved. As distinct from previous parameter and bound estimation laws, the parameters and uncertainty bounds are updated as a function of a combination of trigonometric function depending on robot parameters and tracking error. Based on the same Lyapunov function, a robust control law is also defined and the stability of the uncertain system is proved under the same set of conditions. Simulation results are given to illustrate the tracking performance of the proposed adaptive-robust controller.

Key Words: *Robust control, adaptive control, robot manipulators, parameter uncertainty, adaptive-robust control, stability analysis*

1. Introduction

Robust control laws are used for parametric uncertainty, unmodeled dynamics, and other sources of uncertainties. The Corless-Leitmann [1] approach is a popular approach used for designing robust controllers for robot manipulators. In an early application of the Corless-Leitmann approach to robot manipulators [2,3], uncertainty bounds are required to derive the controller, and the uncertainty bound depends not only on the inertia parameters but also on the reference trajectory and manipulator state vector. Consequently, it is difficult to compute the uncertainty bound precisely. Spong [4] proposed a new robust controller for robot manipulators using the Lyapunov theory that guarantees the stability of uncertain systems. In this approach, the Leitmann [5] or Corless-Leitmann [1] approach is used for designing the robust controller. One of the advantages of Spong's approach [4] is that uncertainty in the parameter is needed to derive the robust controller and the uncertainty bound parameters depend only on the inertia parameters of the robots. However, disturbance and unmodeled dynamics are not considered in Spong's algorithm. Danesh et al. [6] developed Spong's approach [4] in such a manner that the control scheme was made robust not only to uncertain inertia parameters but also to unmodeled dynamics and disturbances. Koo and Kim [7] and Spong [8] introduced an adaptive scheme of the uncertainty bound on parameters for robust control of robot manipulators. Yaz [9] proposed a robust control law based on Spong's study [4] and the global exponential stability of an uncertain system was guaranteed. Spong's robust controller [4] was extended by Liu and Goldenberg in [10], where parameterized and unparameterized model uncertainties were treated and a compensator was designed for each of 2 uncertainty groups. Uncertainty bound estimation laws were designed for robust controllers [11-15] in order to improve the tracking performance of uncertain systems. Comparative studies of robust controllers were given by Jaritz and Spong [16] and Liu and Goldenberg [17]. Papers about robust control of robot manipulators were surveyed in [18,19].

In this paper, a parameter and bound estimation functions are developed for adaptive-robust control of robot manipulators in order to improve the tracking performance of an uncertain system. For this purpose, the previous robust controllers [4,12-15] were developed in such a manner that both the parameter and uncertainty bound were made adaptive for robustness to uncertainty. Inertia parameters and the uncertainty bound on parameters are adaptive in adaptive-robust control laws [20-22], but inertia parameters are assumed to be known initially and inertia parameters exist in control laws. The disadvantages of the previous adaptive-robust control laws [20-22] are eliminated here, such that robot inertia parameters do not exist in control laws and robot inertia parameters are uncertain as they would be in robust control strategy [4]. The Lyapunov theory, based on the Corless-Leitmann approach [1], was used to design the adaptive-robust control law, and uniform boundedness error convergence was achieved. In addition, a parameter and uncertainty bound estimation laws were designed for the adaptive-robust control of robot manipulators. Apart from previous studies, the parameter and uncertainty bound were updated as a combination of trigonometric function depending on robot parameters and tracking error. For comparison and explanation, a robust control law is also proposed here, based on the adaptive-robust control law, and the stability of an uncertain system is proved under the same set of conditions. Numerical results of the proposed adaptive-robust control law and the proposed robust control law are given. After simulation results, it was seen that the tracking performance of the uncertain system was improved by the proposed adaptive-robust controller, and proper estimation of parameters and uncertainty bounds were achieved.

2. Stability analysis and definition of adaptive-robust control laws

In the absence of friction or other disturbances, the dynamic model of an n-link manipulator can be written as follows [23].

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau \tag{1}$$

Here, q denotes generalized coordinates; τ is the n-dimensional vector of applied torques (or forces); $M(q)$ is the n × n symmetric, positive, definite inertia matrix; $C(q,\dot{q})\dot{q}$ is the n-dimensional vector of centripetal and Coriolis terms; and G(q) is the n-dimensional vector of gravitational terms. Eq. (1) can also be expressed in the following form.

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = Y(q,\dot{q},\ddot{q})\pi \tag{2}$$

Here, π is a p-dimensional vector of robot parameters and Y is an n × p matrix that is a function of joint position, velocity, and acceleration. For any specific trajectory, the desired position, velocity, and acceleration vectors are q_d, $\dot{q}_d$, and $\ddot{q}_d$. The measured actual position and velocity errors are $\tilde{q} = q - q_d$ and $\dot{\tilde{q}} = \dot{q} - \dot{q}_d$. Using this information, $\dot{q}_r$ and $\ddot{q}_r$ are defined as:

$$\dot{q}_r = \dot{q}_d - \Lambda\tilde{q} \text{ and } \ddot{q}_r = \ddot{q}_d - \Lambda\dot{\tilde{q}}, \tag{3}$$

where Λ is a positive definite matrix. The following nominal control law is then considered.

$$\begin{aligned} \tau_0 &= M_0(q)\ddot{q}_r + C_0(q,\dot{q})\dot{q}_r + G_0(q) - K\sigma \\ &= Y(q,\dot{q},\dot{q}_r,\ddot{q}_r)\pi_0 - K\sigma \end{aligned} \tag{4}$$

Here, $\pi_0 \in R^p$ represents the fixed parameters in the dynamic model and $K\sigma$ is the vector of proportional-derivative action. σ is given as [4]:

$$\sigma = \dot{q} - \dot{q}_r = \dot{\tilde{q}} + \Lambda\tilde{q}. \tag{5}$$

It is assumed that there exists an unknown bound on parametric uncertainty such that:

$$\tilde{\pi} = (\pi_0 - \pi) \leq \rho, \tag{6}$$

where $\rho \in R^p$ is the upper uncertainty bound on the parametric uncertainty, assumed to be unknown. $\hat{\pi}$ is the estimation of the parameter, $\hat{\rho}(t)$ is the estimation of the uncertainty bound, and $\hat{\pi}$ and $\hat{\rho}(t)$ should be estimated with the estimation law to control the system properly. Considering $\hat{\pi}$ and $\hat{\rho}(t)$, a new parameter error vector, $\tilde{\theta}$, is defined as:

$$\tilde{\theta} = \hat{\pi} - \hat{\rho}(t). \tag{7}$$

Let us define control inputs $u(t)_1$ and $u(t)_2$ in terms of the nominal control vector τ_0 as:

$$\tau = \tau_0 + Y(q,\dot{q},\dot{q}_r,\ddot{q}_r)(u(t)_1 + u(t)_2) = Y(q,\dot{q},\dot{q}_r,\ddot{q}_r)(\pi_0 + u(t)_1 + u(t)_2) - K\sigma, \tag{8}$$

where $u(t)_1$ and $u(t)_2$ are additional control inputs that will be designed to achieve robustness to parametric uncertainty. Substituting Eq. (8) into Eq. (1), the following is yielded after some algebra.

$$\begin{aligned} M(q)\dot{\sigma} + C(q,\dot{q})\sigma + K\sigma &= Y(q,\dot{q},\dot{q}_r,\ddot{q}_r)(\pi_0 - \pi + u(t)_1 + u(t)_2) \\ &= Y(q,\dot{q},\dot{q}_r,\ddot{q}_r)(\tilde{\pi} + u(t)_1 + u(t)_2) \end{aligned} \tag{9}$$

In order to define the adaptive-robust controllers, the following 2 theorems are given.

Theorem 1 Let $\alpha_1, \alpha_2, \ldots \alpha_p$; $\beta_1, \beta_2, \ldots \beta_p$; and $\lambda_1, \lambda_2, \ldots \lambda_p \in R$. The estimate of parameter $\hat{\pi}$ and the uncertainty bound on parameter $\hat{\rho}(t)$ are updated in time as follows.

$$\begin{bmatrix} \hat{\pi}_1 \\ \hat{\pi}_2 \\ \cdots \\ \hat{\pi}_p \end{bmatrix} = \begin{bmatrix} \beta_1^2\alpha_1 \frac{\cos(\int \alpha_1 Y^T\sigma dt)_1 \arctan(\sin(\int \alpha_1 Y^T\sigma dt)_1)}{\sin^2(\int \alpha_1 Y^T\sigma dt)_1 + 1} \\ \beta_2^2\alpha_2 \frac{\cos(\int \alpha_2 Y^T\sigma dt)_2 \arctan(\sin(\int \alpha_2 Y^T\sigma dt)_2)}{\sin^2(\int \alpha_2 Y^T\sigma dt)_2 + 1} \\ \cdots \\ \beta_p^2\alpha_p \frac{\cos(\int \alpha_p Y^T\sigma dt)_p \arctan(\sin(\int \alpha_p Y^T\sigma dt)_p)}{\sin^2(\int \alpha_p Y^T\sigma dt)_p + 1} \end{bmatrix} ; \quad \hat{\rho}(t) = \begin{bmatrix} \lambda_1 \frac{(\beta_1\alpha_1)\cos(\int \alpha_1 Y^T\sigma dt)_1}{\sin^2(\int \alpha_1 Y^T\sigma dt)_1 + 1} \\ \lambda_2 \frac{(\beta_2\alpha_2)\cos(\int \alpha_2 Y^T\sigma dt)_2}{\sin^2(\int \alpha_2 Y^T\sigma dt)_2 + 1} \\ \cdots \\ \lambda_p \frac{(\beta_p\alpha_p)\cos(\int \alpha_p Y^T\sigma dt)_p}{\sin^2(\int \alpha_p Y^T\sigma dt)_p + 1} \end{bmatrix} \tag{10}$$

The additional control input in the control law of Eq. (8) is defined as follows.

$$u(t)_1 = -\hat{\pi}; \quad (u(t)_2)_i = \begin{cases} -\frac{(Y^T\sigma)_i}{|(Y^T\sigma)_i|}(\rho_i - \hat{\rho}(t)_i) & if \quad \left|(Y^T\sigma)_i\right| > \varepsilon_i \\ -\frac{(Y^T\sigma)_i}{\varepsilon_i}(\rho_i - \hat{\rho}(t)_i) & if \quad \left|(Y^T\sigma)_i\right| \le \varepsilon_i \end{cases} \tag{11}$$

If the control input of Eq. (11) is substituted into the control law of Eq. (8) for the control of the model manipulator, then the control law of Eq. (8) is continuous and the closed-loop system is uniformly ultimate bounded.

Theorem 2

The additional control input in the control law of Eq. (8) is defined as follows.

$$u(t)_1 = -\hat{\pi} + \hat{\rho}(t); \quad (u(t)_2)_i = \begin{cases} -\frac{(Y^T\sigma)_i}{|(Y^T\sigma)_i|}\rho_i & if \quad \left|(Y^T\sigma)_i\right| > \varepsilon_i \\ -\frac{(Y^T\sigma)_i}{\varepsilon_i}\rho_i & if \quad \left|(Y^T\sigma)_i\right| \le \varepsilon_i \end{cases} \tag{12}$$

If the control input of Eq. (12) is substituted into the control law of Eq. (8) for the control of the model manipulator, then the control law of Eq. (8) is continuous and the closed-loop system is uniformly ultimate bounded.

Proof of theorem 1

In order to prove the theorem, a Lyapunov function candidate is defined as follows.

$$V(\sigma, \tilde{q}, \Phi_1, \Phi_2, \tilde{\theta}) = \frac{1}{2}\sigma^T M(q)\sigma + \frac{1}{2}\tilde{q}^T B\tilde{q} + \frac{1}{2}\tilde{\theta}^T(\Phi_1^2 + I)^2\Phi_2^2\tilde{\theta}; \; V(\sigma, \tilde{q}, \Phi_1, \Phi_2, \tilde{\theta}) \ge 0 \tag{13}$$

Here, B $\in$ R$^{n\times n}$ is a positive diagonal matrix, and Φ_1 and Φ_2 are chosen as a p $\times$ p-dimensional diagonal matrix changing in time. The time derivative of V along the system of Eq. (9) is:

$$\begin{aligned} \dot{V} \;=\; & \sigma^T M(q)\dot{\sigma} + \sigma^T \tfrac{1}{2}\dot{M}(q)\sigma + \tilde{q}^T B\dot{\tilde{q}} + \tilde{\theta}^T(2(\Phi_1^2 + I)\dot{\Phi}_1\Phi_1\Phi_2^2 \\ & + (\Phi_1^2 + I)^2\Phi_2\dot{\Phi}_2)\tilde{\theta} + \tilde{\theta}^T(\Phi_1^2 + I)^2\Phi_2^2)\dot{\tilde{\theta}} \end{aligned}, \tag{14}$$

and then:

$$\begin{aligned} \dot{V} \;=\; & \sigma^T[\tfrac{1}{2}\dot{M}(q) - C(q,\dot{q})]\sigma - \sigma^T K\sigma + Y(q,\dot{q},\dot{q}_r,\ddot{q}_r)(\tilde{\pi} + u(t)_1 + u(t)_2) + \tilde{q}^T B\dot{\tilde{q}} \\ & + \tilde{\theta}^T(2(\Phi_1^2 + I)\dot{\Phi}_1\Phi_1\Phi_2^2 + (\Phi_1^2 + I)^2\Phi_2\dot{\Phi}_2)\tilde{\theta} + \tilde{\theta}^T(\Phi_1^2 + I)^2\Phi_2^2)\dot{\tilde{\theta}} \end{aligned}. \tag{15}$$

Taking $B = 2\Lambda K$, and using the property $\sigma^T[\dot{M}(q) - 2C(q,\dot{q})]\sigma = 0 \forall\sigma \in R^n$ [24,25], Eq. (14) becomes:

$$\begin{aligned} \dot{V} \;=\; & -\dot{\tilde{q}}^T K\dot{\tilde{q}} - \tilde{q}^T\Lambda K\Lambda\tilde{q} + \sigma^T Y(u(t)_1 + u(t)_2) + \sigma^T Y\tilde{\pi} \\ & + \tilde{\theta}^T(2(\Phi_1^2 + I)\dot{\Phi}_1\Phi_1\Phi_2^2 + (\Phi_1^2 + I)^2\Phi_2\dot{\Phi}_2)\tilde{\theta} + \tilde{\theta}^T(\Phi_1^2 + I)^2\Phi_2^2)\dot{\tilde{\theta}} \end{aligned}. \tag{16}$$

As seen from Eq. (16), there is a relationship between the control inputs u(t)$_1$ and u(t)$_2$ and functions Φ_1 and Φ_2. There may be some sort of functions Φ_1 and Φ_2 for control inputs u(t)$_1$ and u(t)$_2$ that satisfy $\dot{V} \le 0$ in Eq. (16). However, Φ_1 and Φ_2 are unknown and there is no certain rule for the determination of Φ_1 and Φ_2

for control inputs u(t)$_1$ and u(t)$_2$ that satisfies $\dot{V} \leq 0$. We use the system state parameters and mathematical insight to search for an appropriate function of Φ_1 and Φ_2 to prove the theorem. For this purpose, we define Φ_1 and Φ_2 as a time-dependent p-dimensional diagonal matrix, such that:

$$\phi_1 = diag(\sin(\alpha_i \int Y^T \sigma dt)_i), \quad \phi_2 = diag(\frac{1}{(\beta_i \alpha_i)(\cos(\alpha_i \int Y^T \sigma dt)_i)}), \tag{17}$$

where $i = 1, 2.., p$. The time derivatives of Φ_1 and Φ_2 are as follows.

$$\dot{\phi}_1 = diag(\cos(\alpha_i \int Y^T \sigma dt))_i (\alpha_i Y^T \sigma)_i) \quad \dot{\phi}_2 = diag(\frac{\sin(\alpha_i \int Y^T \sigma dt)_i}{(\beta_i \alpha_i)\cos^2(\alpha_i \int Y^T \sigma dt)_i}(\alpha_i Y^T \sigma)_i) \tag{18}$$

From Eq. (10), $\tilde{\theta} = \hat{\pi} - \hat{\rho}(t)$ is defined as follows.

$$\tilde{\theta} = \hat{\pi} - \hat{\rho}(t) = \begin{bmatrix} (\beta_1^2 \alpha_1) \frac{\cos(\int \alpha_1 Y^T \sigma dt)_1 \arctan(\sin(\int \alpha_1 Y^T \sigma dt)_1) - \lambda_1 \beta_1 \alpha_1 \cos(\int \alpha_1 Y^T \sigma dt)_1}{\sin^2(\int \alpha_1 Y^T \sigma dt)_1 + 1} \\ (\beta_2^2 \alpha_2) \frac{\cos(\int \alpha_2 Y^T \sigma dt)_2 \arctan(\sin(\int \alpha_2 Y^T \sigma dt)_2) - \lambda_2 \beta_2 \alpha_2 \cos(\int \alpha_2 Y^T \sigma dt)_2}{\sin^2(\int \alpha_2 Y^T \sigma dt)_2 + 1} \\ .... \\ (\beta_p^2 \alpha_p) \frac{\cos(\int \alpha_p Y^T \sigma dt)_p \arctan(\sin(\int \alpha_p Y^T \sigma dt)_p) - \lambda_p \beta_p \alpha_p \cos(\int \alpha_p Y^T \sigma dt)_p}{\sin^2(\int \alpha_p Y^T \sigma dt)_p + 1} \end{bmatrix}$$

$$= \begin{bmatrix} (\beta_1^2 \alpha_1) \frac{\cos(\int \alpha_1 Y^T \sigma dt)_1 [\arctan(\sin(\int \alpha_1 Y^T \sigma dt)_1) - \lambda_1/\beta_1]}{\sin^2(\int \alpha_1 Y^T \sigma dt)_1 + 1} \\ (\beta_2^2 \alpha_2) \frac{\cos(\int \alpha_2 Y^T \sigma dt)_2 [\arctan(\sin(\int \alpha_2 Y^T \sigma dt)_2) - \lambda_2/\beta_1]}{\sin^2(\int \alpha_2 Y^T \sigma dt)_2 + 1} \\ .... \\ (\beta_p^2 \alpha_p) \frac{\cos(\int \alpha_p Y^T \sigma dt)_p [\arctan(\sin(\int \alpha_p Y^T \sigma dt)_p) - \lambda_p/\beta_p]}{\sin^2(\int \alpha_p Y^T \sigma dt)_p + 1} \end{bmatrix} \tag{19}$$

The time derivative of $\tilde{\theta}$ is as follows.

$$\begin{aligned} \dot{\tilde{\theta}}_i &= -(\beta_i^2 \alpha_i) \frac{\sin(\int \alpha_i Y^T \sigma dt)_i [\arctan(\sin(\int \alpha_i Y^T \sigma dt)_i) - \lambda_i/\beta_i]}{\sin^2(\int \alpha_i Y^T \sigma dt)_i + 1} (\alpha_i Y^T \sigma)_i + \frac{\beta_i^2 \alpha_i^2 \cos^2(\int \alpha_i Y^T \sigma dt)_i}{(\sin^2(\int \alpha_i Y^T \sigma dt)_i + 1)^2} (Y^T \sigma)_i \\ &\quad -(\beta_i^2 \alpha_i) \frac{\cos(\int \alpha_i Y^T \sigma dt)_i [\arctan(\sin(\int \alpha_i Y^T \sigma dt)_i) - \lambda_i/\beta_i]}{(\sin^2(\int \alpha_i Y^T \sigma dt)_i + 1)^2} 2 \sin(\int \alpha_i Y^T \sigma dt)_i \cos(\int \alpha_i Y^T \sigma dt)_i (\alpha_i Y^T \sigma)_i \end{aligned} \tag{20}$$

Multiplying the first term of Eq. (20) by $\frac{\cos(\int \alpha_i Y^T \sigma dt)_i}{\cos(\int \alpha_i Y^T \sigma dt)_i}$, the result is as follows.

$$\begin{aligned} \dot{\tilde{\theta}}_i &= -(\beta_i^2 \alpha_i) \frac{\sin(\int \alpha_i Y^T \sigma dt)_i [\arctan(\sin(\int \alpha_i Y^T \sigma dt)_i) - \lambda_i/\beta_i]}{\sin^2(\int \alpha_i Y^T \sigma dt)_i + 1} (\alpha_i Y^T \sigma)_i \frac{\cos(\int \alpha_i Y^T \sigma dt)_i}{\cos(\int \alpha_i Y^T \sigma dt)_i} \\ &\quad + \frac{\beta_i^2 \alpha_i^2 \cos^2(\int \alpha_i Y^T \sigma dt)_i}{(\sin^2(\int \alpha_i Y^T \sigma dt)_i + 1)^2} (Y^T \sigma)_i \\ &\quad -(\beta_i^2 \alpha_i) \frac{\cos(\int \alpha_i Y^T \sigma dt)_i [\arctan(\sin(\int \alpha_i Y^T \sigma dt)_i) - \lambda_i/\beta_i]}{(\sin^2(\int \alpha_i Y^T \sigma dt)_i + 1)^2} 2 \sin(\int \alpha_i Y^T \sigma dt)_i \cos(\int \alpha_i Y^T \sigma dt)_i (\alpha_i Y^T \sigma)_i \end{aligned} \tag{21}$$

Eq. (21) can be arranged depending on $\tilde{\theta}$, as follows.

$$\begin{aligned} \dot{\tilde{\theta}}_i &= -\frac{\sin(\int \alpha_i Y^T \sigma dt)_i}{\cos(\int \alpha_i Y^T \sigma dt)_i}(\alpha_i Y^T \sigma)_i \tilde{\theta}_i + \frac{\beta_i^2 \alpha_i^2 \cos^2(\int \alpha_i Y^T \sigma dt)_i}{(\sin^2(\int \alpha_i Y^T \sigma dt)_i + 1)^2}(Y^T \sigma) \\ &\quad - \frac{2\cos(\int \alpha_i Y^T \sigma dt)_1 \sin \int \alpha_i Y^T \sigma dt)_1}{\sin^2(\int \alpha_i Y^T \sigma dt)_i + 1}(\alpha_i Y^T \sigma)_i \tilde{\theta}_i \end{aligned} \tag{22}$$

If we substitute Φ_1, Φ_2, $\dot{\Phi}_1, \dot{\Phi}_2$, $\tilde{\theta}$, and $\dot{\tilde{\theta}}$ into Eq. (16), the last terms will be as follows.

$$\begin{aligned} &(\tilde{\theta}^T(2(\Phi_1^2 + I)\dot{\Phi}_1\Phi_1\Phi_2^2 + (\Phi_1^2 + I)^2\Phi_2\dot{\Phi}_2)\tilde{\theta} + \tilde{\theta}^T(\Phi_1^2 + I)^2\Phi_2^2)\dot{\tilde{\theta}})_i = \\ &\tilde{\theta}_i 2(\sin^2(\alpha_i \textstyle\int Y^T \sigma dt)_i + 1) cos(\alpha_i \textstyle\int Y^T \sigma dt)_i (\sin(\alpha_i \textstyle\int Y^T \sigma dt)_i)(\frac{1}{\beta_i \alpha_i \cos(\alpha \int Y^T \sigma dt)_i})^2 (\alpha_i Y^T \sigma)_i \tilde{\theta}_i \\ &+ \tilde{\theta}_i (\sin^2(\alpha_i \textstyle\int Y^T \sigma dt)_i + 1)^2 \frac{\sin(\alpha_i \int Y^T \sigma dt)_i}{(\beta_i^2 \alpha_i^2)\cos^3(\alpha_i \int Y^T \sigma dt)_i}(\alpha_i Y^T \sigma)_i \tilde{\theta}_i \\ &- \tilde{\theta}_i (\frac{\sin^2(\alpha_i \int Y^T \sigma dt)_i + 1}{(\beta_i \alpha_i)\cos(\alpha_i \int Y^T \sigma dt)_i})^2 \frac{\sin(\int \alpha_i Y^T \sigma dt)_i}{\cos(\int \alpha_i Y^T \sigma dt)_i}(\alpha_i Y^T \sigma)_i \tilde{\theta}_i \\ &+ \tilde{\theta}_i (\frac{\sin^2(\alpha_i \int Y^T \sigma dt)_i + 1}{(\beta_i \alpha_i)\cos(\alpha_i \int Y^T \sigma dt)_i})^2 \frac{\beta_i^2 \alpha_i^2 \cos^2(\int \alpha_i Y^T \sigma dt)_i}{(\sin^2(\int \alpha_i Y^T \sigma dt)_i + 1)^2}(Y^T \sigma)_i \\ &- \tilde{\theta}_i (\frac{\sin^2(\alpha_i \int Y^T \sigma dt)_i + 1}{(\beta_i \alpha_i)\cos(\alpha_i \int Y^T \sigma dt)_i})^2 \frac{2\cos(\int \alpha_i Y^T \sigma dt)_i \sin(\int \alpha_i Y^T \sigma dt)_i}{\sin^2(\int \alpha_i Y^T \sigma dt)_i + 1}(\alpha_i Y^T \sigma)\tilde{\theta}_i \end{aligned} \tag{23}$$

As seen from Eq. (23), the first and last terms and the second and third terms are canceled out by each other, and the fourth term is equal to $Y^T\sigma$. As a result, Eq. (23) is equal to $\tilde{\theta}^T Y^T \sigma$; that is,

$$\tilde{\theta}^T(2(\Phi_1^2 + I)\dot{\Phi}_1\Phi_1\Phi_2^2 + (\Phi_1^2 + I)^2\Phi_2\dot{\Phi}_2)\tilde{\theta} + \tilde{\theta}^T(\Phi_1^2 + I)^2\Phi_2^2)\dot{\tilde{\theta}} = \tilde{\theta}^T Y^T \sigma. \tag{24}$$

As a result, the time derivative of the Lyapunov function is written as follows.

$$\begin{aligned} \dot{V} &= -\dot{\tilde{q}}^T K \dot{\tilde{q}} - \tilde{q}^T \Lambda K \Lambda \tilde{q} + \sigma^T Y(u(t)_1 + u(t)_2) + \sigma^T Y \tilde{\pi} + \sigma^T Y \tilde{\theta} \\ &= -\dot{\tilde{q}}^T K \dot{\tilde{q}} - \tilde{q}^T \Lambda K \Lambda \tilde{q} + \sigma^T Y(u(t)_1 + u(t)_2) + \sigma^T Y \tilde{\pi} + \sigma^T Y(\hat{\pi} - \hat{\rho}(t)) \end{aligned} \tag{25}$$

If we substitute $u(t)_1$ and $u(t)_2$ from Eq. (11) into Eq. (25), the result will be as follows.

$$\begin{aligned} \dot{V} &= -\dot{\tilde{q}}^T K \dot{\tilde{q}} - \tilde{q}^T \Lambda K \Lambda \tilde{q} - \sigma^T Y \hat{\pi} + \sigma^T Y u(t)_2 + \sigma^T Y(\tilde{\pi} - \hat{\rho}(t)) + \sigma^T Y \hat{\pi} \\ &= -\dot{\tilde{q}}^T K \dot{\tilde{q}} - \tilde{q}^T \Lambda K \Lambda \tilde{q} + \sigma^T Y(\tilde{\pi} - \hat{\rho}(t)) - (Y^T \sigma)^T \begin{bmatrix} \frac{(Y^T\sigma)_1}{|(Y^T\sigma)_1|}(\rho_1 - \hat{\rho}(t)_1) \\ \cdots \\ \frac{(Y^T\sigma)_p}{|(Y^T\sigma)_p|}(\rho_p - \hat{\rho}(t)_p) \end{bmatrix} \end{aligned} \tag{26}$$

Eq. (26) can then be written as:

$$\dot{V} = -x^T Q x + \sigma^T Y \left(\begin{bmatrix} \tilde{\pi}_1 - \hat{\rho}(t)_1 \\ \cdots \\ \tilde{\pi}_p - \hat{\rho}(t)_p \end{bmatrix} - \begin{bmatrix} \frac{(Y^T\sigma)_1}{|(Y^T\sigma)_1|}(\rho_1 - \hat{\rho}(t)_1) \\ \cdots \\ \frac{(Y^T\sigma)_p}{|(Y^T\sigma)_p|}(\rho_p - \hat{\rho}(t)_p) \end{bmatrix} \right) \tag{27}$$

where $x^T = [\tilde{q}^T, \dot{\tilde{q}}^T]$ and Q = diag[Λ^TKΛ, K]. Based on the Leitmann approach [1], it can be shown that $\dot{V} \leq 0$ for $||x|| >$ w where

$$w^2 = \varepsilon\rho(t)/2\lambda_{\min}(Q), \tag{28}$$

and where λ_{min} (Q) denotes the minimum eigenvalue of Q. For the second term in Eq. (27), if $||Y^T\sigma|| > \varepsilon$, then:

$$\sigma^T Y \begin{bmatrix} \tilde{\pi}_1 \\ \cdots \\ \tilde{\pi}_p \end{bmatrix} - (Y^T\sigma)^T \begin{bmatrix} \frac{(Y^T\sigma)_1}{|(Y^T\sigma)_1|}\rho_1 \\ \cdots \\ \frac{(Y^T\sigma)_p}{|(Y^T\sigma)_p|}\rho_p \end{bmatrix} \leq ||\sigma^T Y||(||\tilde{\pi}|| - ||\rho||) \leq 0. \tag{29}$$

From the Cauchy-Schwarz inequality and the assumption of $||Y^T\sigma|| < \varepsilon$, it will be:

$$\begin{aligned} \sigma^T Y u(t)_2 + \sigma^T Y(\tilde{\pi} - \hat{\rho}(t)) &\leq (Y^T\sigma)^T \left(\|\tilde{\pi} - \hat{\rho}(t)\| \frac{Y^T\sigma}{\|Y^T\sigma\|} - u(t)_2 \right) \\ &\leq (Y^T\sigma)^T \left(\|\rho - \hat{\rho}(t)\| \frac{Y^T\sigma}{\|Y^T\sigma\|} - \frac{Y^T\sigma}{\varepsilon} \|\rho - \hat{\rho}(t)\| \right). \end{aligned} \tag{30}$$

This last term achieves a maximum value of $\varepsilon\rho$(t)/4 when $||Y^T\sigma|| = \varepsilon/2$. We thus have:

$$\dot{V} \leq -x^T Q x + \varepsilon\rho(t)/4. \tag{31}$$

Note that ρ(t) is bounded and given as $\rho(t) = \|\rho - \hat{\rho}(t)\|$. The rest of the proof can be seen in [4,7]. The resulting block diagram of the control law is given in Figure 1.

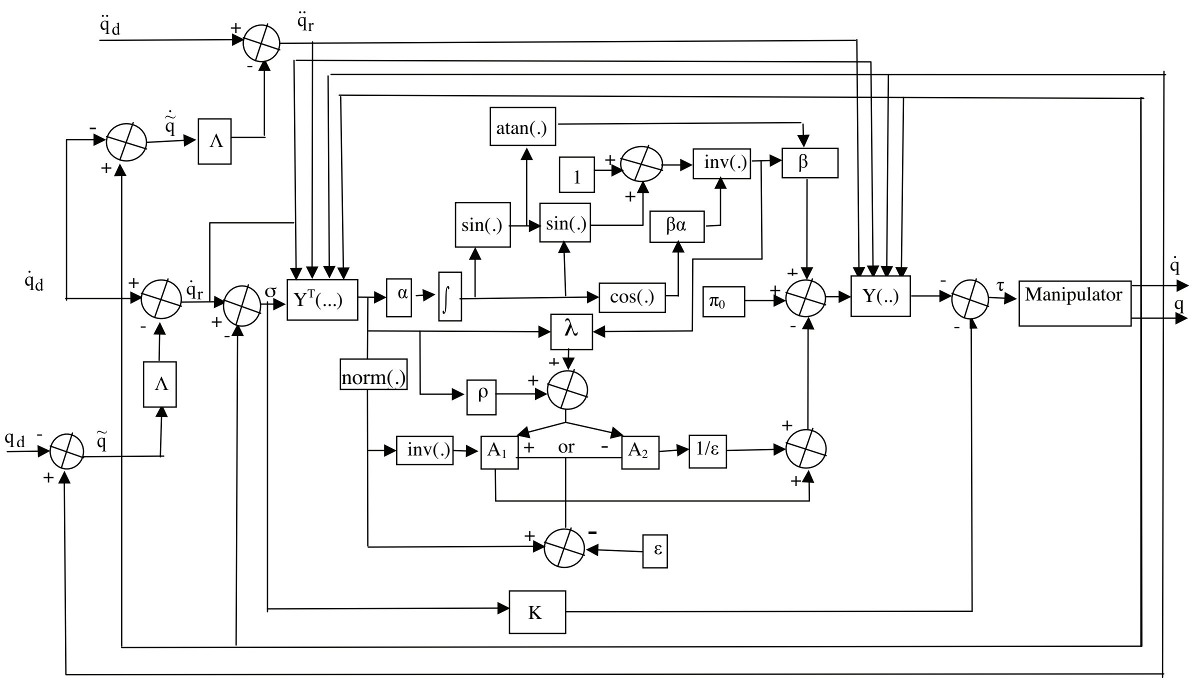

Figure 1. Block diagram of the adaptive-robust control law of Eq. (8) with Eqs. (10) and (11).

Proof of theorem 2

In order to prove the second theorem, Eq. (25) is used. If we substitute $u(t)_1$ from Eq. (12) into Eq. (25), the result will be as follows.

$$\begin{aligned} \dot{V} &= -\dot{\tilde{q}}^T K\dot{\tilde{q}} - \tilde{q}^T \Lambda K \Lambda \tilde{q} + \sigma^T Y(u(t)_1 + u(t)_2) + \sigma^T Y\tilde{\pi} + \sigma^T Y\tilde{\theta} \\ &= -\dot{\tilde{q}}^T K\dot{\tilde{q}} - \tilde{q}^T \Lambda K \Lambda \tilde{q} + \sigma^T Y(-\hat{\pi} + \hat{\rho}(t)) + \sigma^T Y\tilde{\pi} + \sigma^T Y(\hat{\pi} - \hat{\rho}(t)) \\ &= -\dot{\tilde{q}}^T K\dot{\tilde{q}} - \tilde{q}^T \Lambda K \Lambda \tilde{q} + \sigma^T Y u(t)_2 + \sigma^T Y\tilde{\pi} \\ &\leq -x^T Q x + \sigma^T Y \left(\tilde{\pi} - \frac{Y^T\sigma}{\|Y^T\sigma\|}\delta \right) \end{aligned} \tag{32}$$

Here, $\delta = \|\rho\|$. The result obtained in Eq. (32) is the same as would be in [4], and the proof is given in [4].

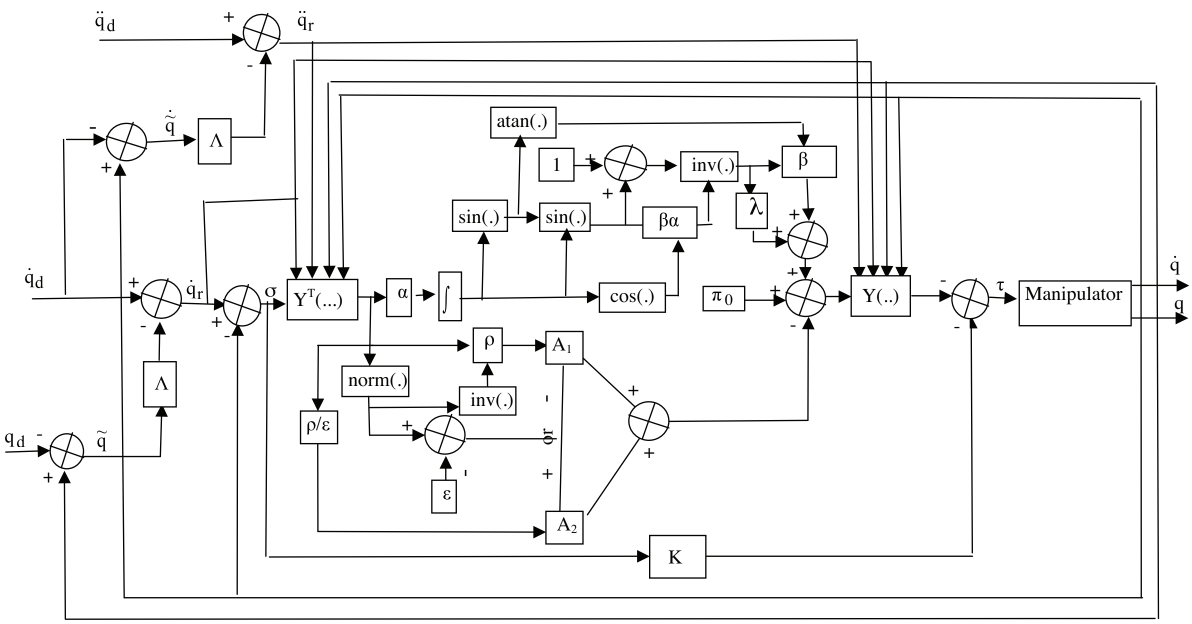

Figure 2. Block diagram of the adaptive-robust control law of Eq. (9) with Eqs. (10) and (12).

Since control input $u(t)_2$ has 2 different inputs, A_1 and A_2, we select a different control input depending on ε. A_1 and A_2 have 2 numbers, such as a 1 and a 0. When $||Y^T\sigma|| - \varepsilon > 0$, a 1 is present in A_1 and a 0 is present in A_2, and the first control input is in effect. When $||Y^T\sigma|| - \varepsilon \leq 0$, a 0 is present in A_1 and a 1 is present in A_2, and the second control input is in effect. Hence, A_1 and A_2 are simple switches that set the mode of the additional control input to be used.

3. Stability analysis and definition of a robust control law

For comparison and explanation, the derivation of a robust control law is considered. For this purpose, the following robust control law is defined in terms of the nominal control vector τ_0, defined in Eq. (4) as in [4].

$$\tau = \tau_0 + Y(q, \dot{q}, \dot{q}_r, \ddot{q}_r)u(t) = Y(q, \dot{q}, \dot{q}_r, \ddot{q}_r)(\pi_0 + u(t)) - K\sigma \tag{33}$$

In order to define a robust control law, the following theorem is given.

Theorem 3

Let $\alpha_1, \alpha_2, \ldots \alpha_p$ and $\beta_1, \beta_2, \ldots \beta_p \in$ R. The estimation of the uncertainty bound on the parametric uncertainty $\hat{\rho}(t)$ is updated in time as follows.

$$\begin{bmatrix} \hat{\rho}(t)_1 \\ \hat{\rho}(t)_2 \\ .... \\ \hat{\rho}(t)_p \end{bmatrix} = \begin{bmatrix} \beta_1^2 \alpha_1 \frac{\cos(\int \alpha_1 Y^T \sigma dt)_1 \arctan(\sin(\int \alpha_1 Y^T \sigma dt)_1)}{\sin^2(\int \alpha_1 Y^T \sigma dt)_1 + 1} + \rho_1 \\ \beta_2^2 \alpha_2 \frac{\cos(\int \alpha_2 Y^T \sigma dt)_2 \arctan(\sin(\int \alpha_2 Y^T \sigma dt)_2)}{\sin^2(\int \alpha_2 Y^T \sigma dt)_2 + 1} + \rho_2 \\ \cdots \\ \beta_p^2 \alpha_p \frac{\cos(\int \alpha_p Y^T \sigma dt)_p \arctan(\sin(\int \alpha_p Y^T \sigma dt)_p)}{\sin^2(\int \alpha_p Y^T \sigma dt)_p + 1} + \rho_n \end{bmatrix} \tag{34}$$

The additional control input in the control law of Eq. (33) is as follows.

$$(u(t))_i = \begin{cases} -\frac{(Y^T\sigma)_i}{|(Y^T\sigma)_i|} \hat{\rho}(t)_i & if \quad \left|(Y^T\sigma)_i\right| > \varepsilon_i \\ -\frac{(Y^T\sigma)_i}{\varepsilon_i} \hat{\rho}(t)_i & if \quad \left|(Y^T\sigma)_i\right| \le \varepsilon_i \end{cases} \tag{35}$$

If the control input of Eq. (35) is substituted into the control law of Eq. (33) for the control of the model manipulator, then the control law of Eq. (33) is continuous and the closed-loop system is uniformly ultimate bounded.

Proof

It is assumed that there exists an unknown bound on parametric uncertainty such that

$$\pi_0 - \pi \le \rho \text{ and } \|\pi_0 - \pi\| \le \delta. \tag{36}$$

Since $\rho \in$ R^p is assumed to be unknown, ρ should be estimated with an estimation law to control the system properly. $\hat{\rho}(t)$ shows that the estimate of ρ and $\tilde{\rho}$ is the estimation error.

$$\tilde{\rho} = \rho - \hat{\rho}(t) \tag{37}$$

Substituting Eq. (33) into Eq. (1), the following is yielded after some algebra:

$$M(q)\dot{\sigma} + C(q,\dot{q})\sigma + K\sigma = Y(q,\dot{q},\dot{q}_r,\ddot{q}_r)(\tilde{\pi} + u(t)). \tag{38}$$

In order to prove the theorem, the following Lyapunov function is considered.

$$V(\sigma,\tilde{q},\Phi_1,\Phi_2,\tilde{\rho} = \frac{1}{2}\sigma^T M(q)\sigma + \frac{1}{2}\tilde{q}^T B\tilde{q} + \frac{1}{2}\tilde{\rho}^T(\Phi_1^2 + I)^2\Phi_2^2\tilde{\rho}; \quad V(\sigma,\tilde{q},\Phi_1,\Phi_2,\tilde{\rho}) \ge 0 \tag{39}$$

Here, B $\in$ R$^{n\times n}$ is a positive diagonal matrix, and Φ_1 and Φ_1 are chosen as a p $\times$ p-dimensional diagonal matrix changing in time. The time derivative of V along the system of Eq. (38) is as follows:

$$\begin{aligned} \dot{V} &= \sigma^T M(q)\dot{\sigma} + \sigma^T \tfrac{1}{2}\dot{M}(q)\sigma + \tilde{q}^T B\dot{\tilde{q}} + \tilde{\rho}^T(2(\Phi_1^2 + I)\dot{\Phi}_1\Phi_1\Phi_2^2 \\ &\quad + (\Phi_1^2 + I)^2\Phi_2\dot{\Phi}_2)\tilde{\rho} + \tilde{\rho}^T(\Phi_1^2 + I)^2\Phi_2^2)\dot{\tilde{\rho}} \end{aligned}, \tag{40}$$

and then:

$$\begin{aligned}\dot{V} &= \sigma^T[\tfrac{1}{2}\dot{M}(q) - C(q,\dot{q})]\sigma - \sigma^T K\sigma + Y(q,\dot{q},\dot{q}_r,\ddot{q}_r)(\tilde{\pi} + u(t)) + \tilde{q}B\dot{\tilde{q}} \\ &+\tilde{\rho}^T(2(\Phi_1^2+I)\dot{\Phi}_1\Phi_1\Phi_2^2 + (\Phi_1^2+I)^2\Phi_2\dot{\Phi}_2)\tilde{\rho} + \tilde{\rho}^T(\Phi_1^2+I)^2\Phi_2^2)\dot{\tilde{\rho}}\end{aligned} \quad . \tag{41}$$

Taking $B = 2\Lambda K$ and using the property $\sigma^T[\dot{M}(q) - 2C(q,\dot{q})]\sigma = 0 \forall\sigma \in R^n$ [24,25], Eq. (41) becomes:

$$\begin{aligned}\dot{V} &= -\dot{\tilde{q}}^T K\dot{\tilde{q}} - \tilde{q}^T\Lambda K\Lambda\tilde{q} + \sigma^T Y(u(t) + \sigma^T Y\tilde{\pi}) \\ &+\tilde{\rho}^T(2(\Phi_1^2+I)\dot{\Phi}_1\Phi_1\Phi_2^2 + (\Phi_1^2+I)^2\Phi_2\dot{\Phi}_2)\tilde{\rho} + \tilde{\rho}^T(\Phi_1^2+I)^2\Phi_2^2)\dot{\tilde{\rho}}\end{aligned} \quad . \tag{42}$$

From Eq. (34), $\tilde{\rho} = \rho - \hat{\rho}(t)$ is defined as follows.

$$\tilde{\rho} = \rho - \hat{\rho}(t) = \begin{bmatrix} -(\beta_1^2\alpha_1)\frac{\cos(\int\alpha_1 Y^T\sigma dt)_1 \arctan(\sin(\int\alpha_1 Y^T\sigma dt)_1)}{\sin^2(\int\alpha_1 Y^T\sigma dt)_1+1} \\ -(\beta_2^2\alpha_2)\frac{\cos(\int\alpha_2 Y^T\sigma dt)_2 \arctan(\sin(\int\alpha_2 Y^T\sigma dt)_2)}{\sin^2(\int\alpha_2 Y^T\sigma dt)_2+1} \\ \dots \\ -(\beta_p^2\alpha_p)\frac{\cos(\int\alpha_p Y^T\sigma dt)_p \arctan(\sin(\int\alpha_p Y^T\sigma dt)_p)}{\sin^2(\int\alpha_p Y^T\sigma dt)_p+1} \end{bmatrix} \tag{43}$$

The time derivative of $\tilde{\rho}$ is as follows.

$$\begin{aligned}\dot{\tilde{\rho}}_i &= (\beta_i^2\alpha_i)\frac{\sin(\int\alpha_i Y^T\sigma dt)_i \arctan(\sin(\int\alpha_i Y^T\sigma dt)_i)}{\sin^2(\int\alpha_i Y^T\sigma dt)_i+1}(\alpha_i Y^T\sigma)_i - \frac{\beta_i^2\alpha_i^2\cos^2(\int\alpha_i Y^T\sigma dt)_i}{(\sin^2(\int\alpha_i Y^T\sigma dt)_i+1)^2}(Y^T\sigma)_i \\ &+(\beta_i^2\alpha_i)\frac{\cos(\int\alpha_i Y^T\sigma dt)_i \arctan(\sin(\int\alpha_i Y^T\sigma dt)_i)}{(\sin^2(\int\alpha_i Y^T\sigma dt)_i+1)^2} 2\sin(\int\alpha_i Y^T\sigma dt)_i\cos(\int\alpha_i Y^T\sigma dt)_i(\alpha_i Y^T\sigma)_i\end{aligned} \tag{44}$$

Multiplying the first term of Eq. (44) by $\frac{\cos(\int\alpha_i Y^T\sigma dt)_i}{\cos(\int\alpha_i Y^T\sigma dt)_i}$, the result is as follows.

$$\begin{aligned}\dot{\tilde{\rho}}_i &= (\beta_i^2\alpha_i)\frac{\sin(\int\alpha_i Y^T\sigma dt)_i \arctan(\sin(\int\alpha_i Y^T\sigma dt)_i)}{\sin^2(\int\alpha_i Y^T\sigma dt)_i+1}(\alpha_i Y^T\sigma)_i\frac{\cos(\int\alpha_i Y^T\sigma dt)_i}{\cos(\int\alpha_i Y^T\sigma dt)_i} \\ &-\frac{\beta_i^2\alpha_i^2\cos^2(\int\alpha_i Y^T\sigma dt)_i}{(\sin^2(\int\alpha_i Y^T\sigma dt)_i+1)^2}(Y^T\sigma)_i \\ &+(\beta_i^2\alpha_i)\frac{\cos(\int\alpha_i Y^T\sigma dt)_i \arctan(\sin(\int\alpha_i Y^T\sigma dt)_i)}{(\sin^2(\int\alpha_i Y^T\sigma dt)_i+1)^2} 2\sin(\int\alpha_i Y^T\sigma dt)_i\cos(\int\alpha_i Y^T\sigma dt)_i(Y^T\sigma)_i\end{aligned} \tag{45}$$

Eq. (45) can be arranged depending on $\tilde{\rho}$, as follows.

$$\begin{aligned}\dot{\tilde{\rho}}_i &= -\frac{\sin(\int\alpha_i Y^T\sigma dt)_i}{\cos(\int\alpha_i Y^T\sigma dt)_i}(\alpha_i Y^T\sigma)_i\tilde{\rho}_i - \frac{\beta_i^2\alpha_i^2\cos^2(\int\alpha_i Y^T\sigma dt)_i}{(\sin^2(\int\alpha_i Y^T\sigma dt)_i+1)^2}(Y^T\sigma)_i \\ &-\frac{2\cos(\int\alpha_i Y^T\sigma dt)_1\sin\int\alpha_i Y^T\sigma dt)_1}{\sin^2(\int\alpha_i Y^T\sigma dt)_i+1}\tilde{\rho}_i(\alpha_i Y^T\sigma)_i\end{aligned} \tag{46}$$

If we substitute Φ_1, Φ_2, $\dot{\Phi}_1$, $\dot{\Phi}_2$, $\tilde{\rho}$, and $\dot{\hat{\rho}}$ into Eq. (42), the last terms will be as follows.

$$\begin{aligned}
&(\tilde{\rho}^T(2(\Phi_1^2+I)\dot{\Phi}_1\Phi_1\Phi_2^2+(\Phi_1^2+I)^2\Phi_2\dot{\Phi}_2)\tilde{\rho}+\tilde{\rho}^T(\Phi_1^2+I)^2\Phi_2^2)\dot{\tilde{\rho}})_i = \\
&\tilde{\rho}_i 2(\sin^2(\alpha_i \textstyle\int Y^T\sigma dt)_i+1)cos(\alpha_i \int Y^T\sigma dt)_i(\sin(\alpha_i \int Y^T\sigma dt)_i)(\frac{1}{\beta_i\alpha_i\cos(\alpha\int Y^T\sigma dt)_i})^2(\alpha_i Y^T\sigma)_i\tilde{\rho}_i \\
&+\tilde{\rho}_i(\sin^2(\alpha_i \textstyle\int Y^T\sigma dt)_i+1)^2\frac{\sin(\alpha_i\int Y^T\sigma dt)_i}{(\beta_i^2\alpha_i^2)\cos^3(\alpha_i\int Y^T\sigma dt)_i}(\alpha_i Y^T\sigma)_i\tilde{\rho}_i \\
&-\tilde{\rho}_i(\frac{\sin^2(\alpha_i\int Y^T\sigma dt)_i+1}{(\beta_i\alpha_i)\cos(\alpha_i\int Y^T\sigma dt)_i})^2\frac{\sin(\int\alpha_i Y^T\sigma dt)_i}{\cos(\int\alpha_i Y^T\sigma dt)_i}(\alpha_i Y^T\sigma)_i\tilde{\rho}_i \\
&-\tilde{\rho}_i(\frac{\sin^2(\alpha_i\int Y^T\sigma dt)_i+1}{(\beta_i\alpha_i)\cos(\alpha_i\int Y^T\sigma dt)_i})^2\frac{\beta_i^2\alpha_i^2\cos^2(\int\alpha_i Y^T\sigma dt)_i}{(\sin^2(\int\alpha_i Y^T\sigma dt)_i+1)^2}(Y^T\sigma)_i \\
&-\tilde{\rho}_i(\frac{\sin^2(\alpha_i\int Y^T\sigma dt)_i+1}{(\beta_i\alpha_i)\cos(\alpha_i\int Y^T\sigma dt)_i})^2\frac{2\cos(\int\alpha_i Y^T\sigma dt)_i\sin(\int\alpha_i Y^T\sigma dt)_i}{\sin^2(\int\alpha_i Y^T\sigma dt)_i+1}(\alpha_i Y^T\sigma)\tilde{\rho}_i
\end{aligned} \tag{47}$$

As seen from Eq. (47), the first and last terms and the second and third terms cancel each other out, and the fourth term is equal to $-\tilde{\rho}^T Y^T\sigma$. As a result, Eq. (47) is equal to $-\tilde{\rho}^T Y^T\sigma$; that is,

$$\tilde{\rho}^T(2(\Phi_1^2+I)\dot{\Phi}_1\Phi_1\Phi_2^2+(\Phi_1^2+I)^2\Phi_2\dot{\Phi}_2)\tilde{\rho}+\tilde{\rho}^T(\Phi_1^2+I)^2\Phi_2^2)\dot{\tilde{\rho}} = -\tilde{\rho}^T Y^T\sigma. \tag{48}$$

As a result, the time derivative of the Lyapunov function is written as follows.

$$\begin{aligned}
\dot{V} &= -\dot{\tilde{q}}^T K\dot{\tilde{q}} - \tilde{q}^T\Lambda K\Lambda\tilde{q} + \sigma^T Y u(t) + \sigma^T Y\tilde{\pi} + \sigma^T Y\tilde{\rho} \\
&= -\dot{\tilde{q}}^T K\dot{\tilde{q}} - \tilde{q}^T\Lambda K\Lambda\tilde{q} + \sigma^T Y u(t) + \sigma^T Y\tilde{\pi} - \sigma^T Y(\rho-\hat{\rho}(t)) \\
&= \leq -x^T Qx + \sigma^T Y u(t) + \sigma^T Y\hat{\rho}(t)
\end{aligned} \tag{49}$$

Here, $x^T = [\tilde{q}^T, \dot{\tilde{q}}^T]$ and Q = diag[Λ^TKΛ, K]. Based on the Leitmann approach [1], we can show that $\dot{V} \leq 0$ for $||$x$|| >$ w where

$$w^2 = ||\hat{\rho}(t)||/2\lambda_{min}(Q), \tag{50}$$

and where λ_{min}(Q) denotes the minimum eigenvalue of Q. For the second term in Eq. (49), if $||\mathrm{Y}^T\sigma|| > \varepsilon$, then:

$$\leq -x^T Qx + \sigma^T Y\hat{\rho}(t) - \sigma^T Y\begin{bmatrix}\frac{(Y^T\sigma)_1}{|(Y^T\sigma)_1|}\hat{\rho}(t)_1 \\ \dots \\ \frac{(Y^T\sigma)_p}{|(Y^T\sigma)_p|} + \hat{\rho}(t)_p\end{bmatrix} \leq 0. \tag{51}$$

From the Cauchy-Schwarz inequality and the assumption of $||\mathrm{Y}^T\sigma|| < \varepsilon$, we have:

$$\begin{aligned}
\sigma^T Y u(t) + \sigma^T Y(\hat{\rho}(t)) &\leq (Y^T\sigma)^T\left(\|\hat{\rho}(t)\|\frac{Y^T\sigma}{\|Y^T\sigma\|} + u(t)\right) \\
&\leq (Y^T\sigma)^T\left(\|\hat{\rho}(t)\|\frac{Y^T\sigma}{\|Y^T\sigma\|} - \frac{Y^T\sigma}{\varepsilon}\|\hat{\rho}(t)\|\right).
\end{aligned} \tag{52}$$

This last term achieves a maximum value of $\varepsilon||\hat{\rho}(t)||/4$ when $||\mathrm{Y}^T\sigma|| = \varepsilon/2$. We have:

$$\dot{V} \leq -x^T Qx + \varepsilon||\hat{\rho}(t)||/4. \tag{53}$$

Note that $\hat{\rho}(t)$ is bounded and given as $\hat{\rho}(t)$. The rest of the proof can be seen in [4,7]. The resulting block diagram of the control law is given in Figure 3.

For explanation and comparison, the robust control law proposed by Spong [4] is given as follows.

$$u(t) = \begin{cases} -\delta \frac{Y^T\sigma}{\|Y^T\sigma\|} & if \quad \left\|Y^T\sigma\right\| > \varepsilon \\ -\delta \frac{Y^T\sigma}{\varepsilon} & if \quad \left\|Y^T\sigma\right\| \leq \varepsilon \end{cases} \tag{54}$$

Here, δ is an uncertainty bound. Having a single number δ to measure the parametric uncertainty may lead to an overly conservative design or higher than necessary gains. For this reason, different "weights" or gains to the components of u(t)$_i$ are assigned. It can be done as follows. Suppose that $\tilde{\pi}_i$ is measured for the uncertainty for each parameter separately, as in [4].

$$u(t)_i = \begin{cases} -\rho_i \upsilon_i / |\upsilon_i| & if \quad |\upsilon_i| > \varepsilon_i \\ -(\rho_i/\varepsilon_i)\upsilon_i & if \quad |\upsilon_i| \leq \varepsilon_i \end{cases} \tag{55}$$

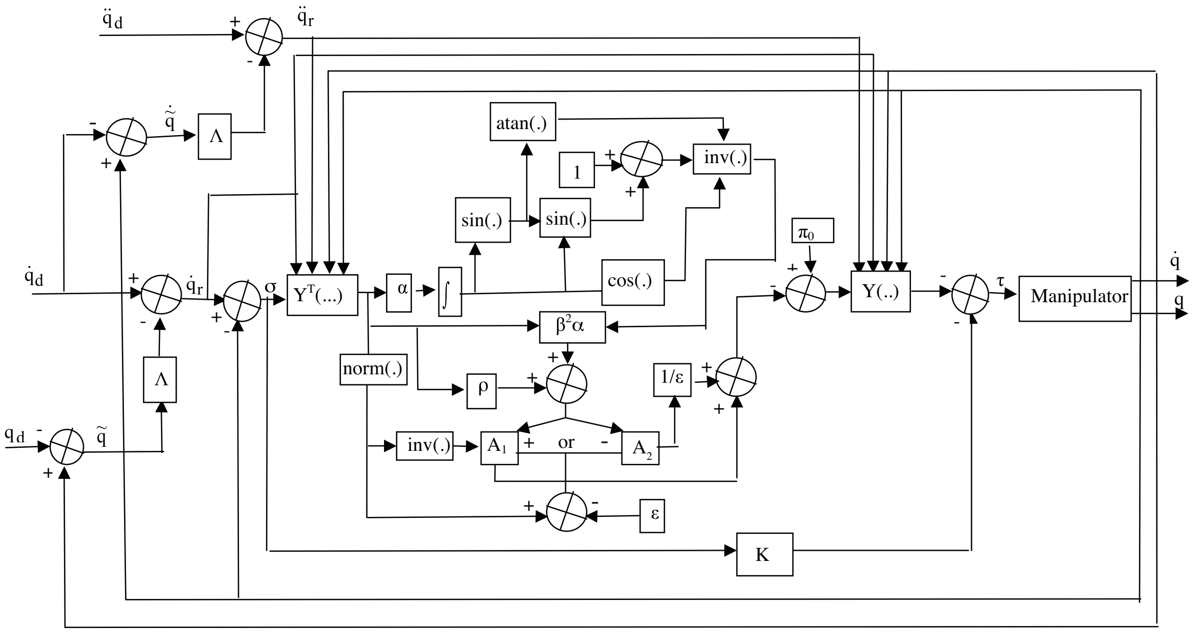

Figure 3. Block diagram of the robust control law of Eq. (33) with Eqs. (34) and (35).

4. Simulation results

For illustration, a 2-link robot manipulator is given in Figure 4 [4]. Parameterization of this robot is given by

$$\pi_1 = m_1 1_{c1}^2 + m_2 l_1^2 + I_1, \quad \pi_2 = m_2 l_{c2}^2 + I_2, \quad \pi_3 = m_2 l_1 l_{c2}, \quad \pi_4 = m_1 l_{c1}, \quad \pi_5 = m_2 l_1, \quad \pi_6 = m_2 l_{c2}. \tag{56}$$

Using the above parameters, the matrix M(q), $C(q,\dot{q})$, and the vector G(q) in Eq. (1) are given as follows.

$$M(q) = \begin{bmatrix} \pi_1 + \pi_2 + 2\pi_3\cos(q_2) & \pi_2 + \pi_3\cos(q_2) \\ \pi_2 + \pi_3\cos(q_2) & \pi_2 \end{bmatrix} C(q,\dot{q}) = \begin{bmatrix} -\pi_3\sin(q_2)\dot{q}_2 & -\pi_3\sin(q_2)(\dot{q}_1 + \dot{q}_2) \\ \pi_3\sin(q_2)\dot{q}_1 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} g(\pi_4 + \pi_5)\cos(q_1) + g\pi_6 \cos(q_1 + q_2) \\ g\pi_6 \cos(q_1 + q_2) \end{bmatrix} \tag{57}$$

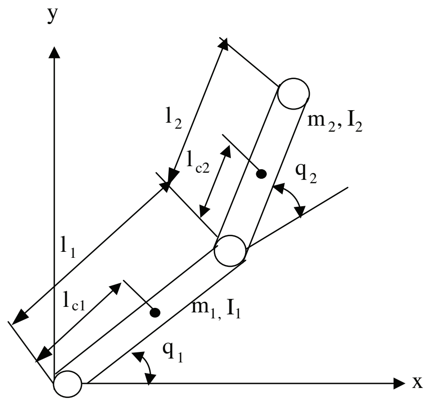

Figure 4. Two-link planar robot [4].

With this parameterization, the dynamic model in Eq. (1) can be written as:

$$Y(q,\dot{q},\ddot{q})\pi = \tau. \tag{58}$$

The component y_{ij} of $Y(q,\dot{q},\ddot{q})$ is given as follows.

$$y_{11} = \ddot{q}_1; \quad y_{12} = \ddot{q}_1 + \ddot{q}_2; \quad y_{13} = \cos(q_2)(2\ddot{q}_1 + \ddot{q}_2) - \sin(q_2)(\dot{q}_2^2 + 2\dot{q}_1\dot{q}_2);$$

$$y_{14} = g_c\cos(q_1); \quad y_{15} = g_c\cos(q_1); \quad y_{16} = g_c\cos(q_1 + q_2); quad y_{21} = 0;$$

$$y_{22} = \ddot{q}_1 + \ddot{q}_2; \quad y_{23} = \cos(q_2)\ddot{q}_1 + \sin(q_2)(\dot{q}_1^2); \quad y_{24} = 0; \quad y_{25} = 0; \quad y_{26} = g_c\cos(q_1 + q_2). \tag{59}$$

$Y(q,\dot{q},\dot{q}_r,\ddot{q}_r)$ in Eq. (4) have the following components.

$$y_{11} = \ddot{q}_{r1}; \quad y_{12} = \ddot{q}_{r1} + \ddot{q}_{r2}; \quad y_{13} = \cos(q_2)(2\ddot{q}_{r1} + \ddot{q}_{r2}) - \sin(q_2)(\dot{q}_1\dot{q}_{r2} + \dot{q}_1\dot{q}_{r2} + \dot{q}_2\dot{q}_{r2}); \quad y_{14} = g_c\cos(q_1);$$

$$y_{15} = g_c\cos(q_1); \quad y_{16} = g_c\cos(q_1 + q_2); \quad y_{21} = 0; \quad y_{22} = \ddot{q}_{r1} + \ddot{q}_{r2}; \quad y_{23} = \cos(q_2)\ddot{q}_{r1} + \sin(q_2)(\dot{q}_1\dot{q}_{r1});$$

$$y_{24} = 0; \quad y_{25} = 0; \quad y_{26} = g_c\cos(q_1 + q_2). \tag{60}$$

For illustrative purposes, let us assume that the parameters of the unloaded manipulator are known; they are given in Table 1. Using the values from Table 1, the ith component of π obtained by means of Eq. (56) is given in Table 2. It is assumed that the parameters m_2, l_{c2}, and I_2 are changed in the intervals of

$$0 \le \Delta m_2 \le 10; \quad 0 \le \Delta l_{c2} \le 0.5; \quad 0 \le I_2 \le \frac{15}{12}. \tag{61}$$

Choosing the mean value for the range of possible π_i values in Eq. (61) yields the nominal parameter vector, and the computed values for the ith component of π_0 are shown in Table 3 [4].

Table 1. Parameters of the unloaded arm [4].

m_1	m_2	l_1	l_2	l_{c1}	l_{c2}	I_1	I_2
10	5	1	1	0.5	0.5	10/12	5/12

Table 2. π_i for the unloaded arm [4].

π_1	π_2	π_3	π_4	π_5	π_6
8.33	1.67	2.5	5	5	2.5

Table 3. Nominal parameter vector π_0 [4].

π_{01}	π_{02}	π_{03}	π_{04}	π_{05}	π_{06}
13.33	8.96	8.75	5	10	8.75

With this choice of nominal parameter vector π_0 and the uncertainty range given by Eq. (61), it is an easy matter to calculate the uncertainty bound δ as follows:

$$\|\tilde{\pi}\|^2 = \sum_{i=1}^{6} (\pi_{i0} - \pi_i)^2 \leq 181.26. \tag{62}$$

Thus, $\delta = \sqrt{181.26} = 13.46$. The uncertainty bounds for each parameter are shown separately in Table 4. The uncertainty bounds ρ_i in Table 4 are simply the difference between the values given in Table 3 and in Table 2, and the value of δ is the Euclidean norm of the vector with components ρ_i [4].

Table 4. Uncertainty bound [4].

ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6
5	7.29	6. 25	0	5	6.25

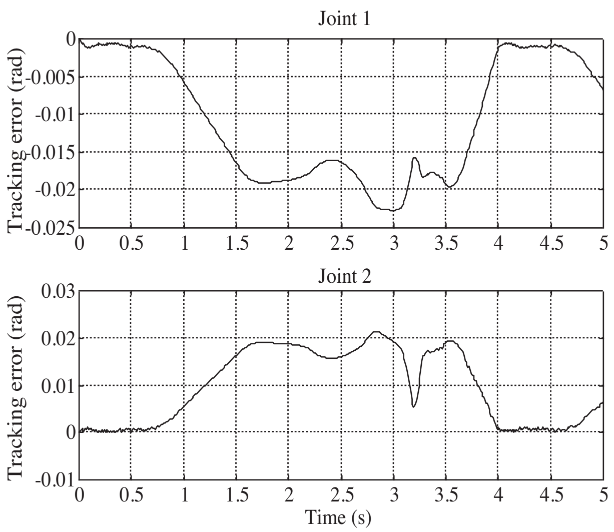

Figure 5. Response using the adaptive-robust control law of Eq. (11) when Λ = diag(15 15), K = diag(50 50), $\alpha = \beta = \lambda = 1$, and $\varepsilon = 0.01$.

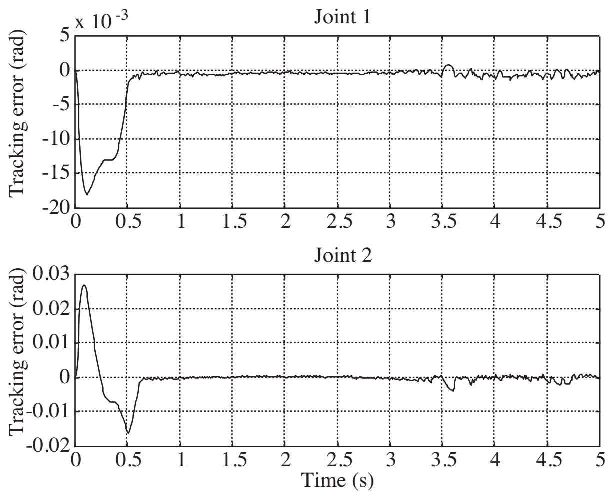

Figure 6. Response using the adaptive-robust control law of Eq. (11) when Λ = diag(15 15), K = diag(50 50), $\alpha = 0.1$, $\beta = 8$, $\lambda = -10$, and $\varepsilon = 0.01$.

For computer simulations, the desired trajectory for both joints are defined as $q_1 = q_2 = 0.5\cos(0.5\pi t) - 0.50$. Simulations were done under maximum uncertainty (worst case) using the control laws of Eqs. (11), (12), (35), and (55). In order to investigate the performance of the new controllers, each control law with the same control parameters, such as K = diag(15 15) and Λ = diag(50 50), was applied to the same model system using the same trajectory. The control parameters Λ and K were chosen to be identical, while α, β, and λ were changed. The obtained results for various α, β, and λ values are plotted in Figures 5-10.

As seen from Figures 5-10, the tracking performance of the proposed controller from Eq. (11) is a little better than the known control law of Eq. (55) [4], but worse than the robust control law of Eq. (35) for the parameters $\alpha = 1$, $\beta = 1$, and $\lambda = 1$. The tracking performance of the proposed robust control law of Eq. (35) is better than the robust controller of Eq. (55) [4]; however, pure transient behavior and chattering are

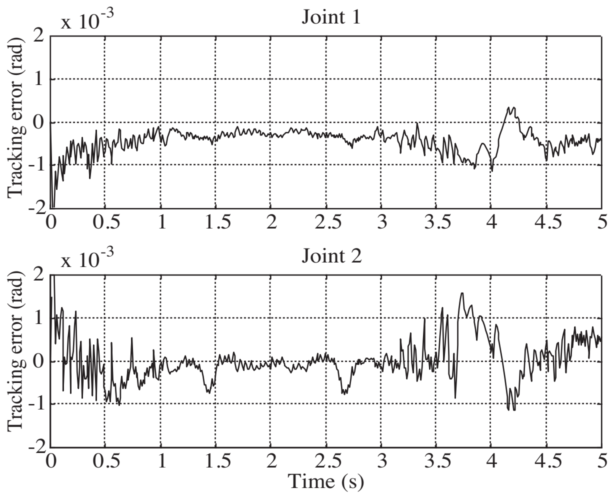

Figure 7. Response using the adaptive-robust control law of Eq. (11) when Λ = diag(15 15), K = diag(50 50), α = 0.1, $\beta = 8$, $\lambda = 10$, and $\varepsilon = 0.01$.

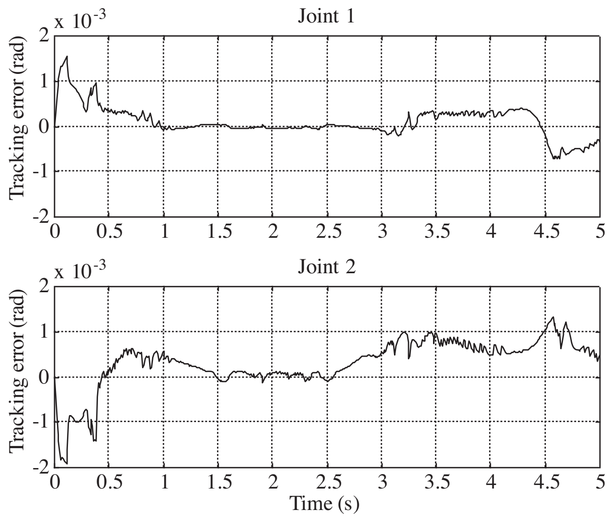

Figure 8. Response using the adaptive-robust control law of Eq. (12) when Λ = diag(15 15), K = diag(50 50), α = 0.1, $\beta = 8$, $\lambda = 10$, and $\varepsilon = 0.01$.

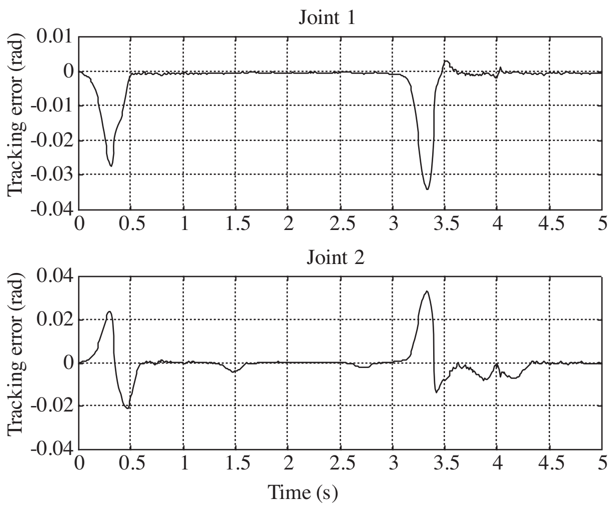

Figure 9. Response using the proposed robust control law of Eq. (35) when Λ = diag(15 15), K = diag(50 50), $\alpha = 1$, $\beta = 6$, and $\varepsilon = 0.01$.

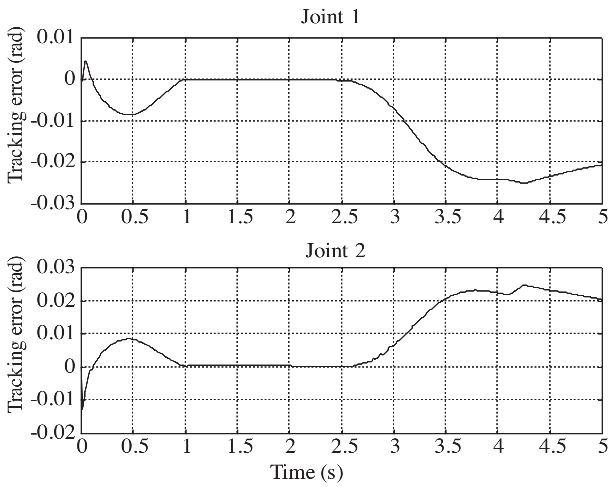

Figure 10. Response using the robust control law of Eq. (55) when Λ = diag(15 15), K = diag(50 50), and ε = 0.01 [4].

observed in the tracking responses. The tracking performance of the proposed adaptive-robust control law is increased if parameters are chosen such that $\alpha = 0.1$, $\beta = 8$, $\lambda = 10$ and $\alpha = 0.1$, $\beta = 8$, $\lambda = -10$. As seen in Figure 8, the tracking error is reduced after 0.5 s to less than 0.0007 rad for the first joint and less than 0.001 rad for the second joint for the proposed adaptive-robust control law with the control parameters of $\alpha = 0.1$, $\beta = 8$, $\lambda = 10$, Λ = diag(15 15), and K = diag(50 50).

5. Conclusion

In this paper, a parameter and bound estimation functions for the adaptive-robust control of robot manipulators were developed in order to improve the tracking performance of an uncertain system. For comparison and explanation, a new robust control law was also developed based on the proposed adaptive-robust control law. The parameters of the adaptive-robust controller and robust controllers were the same and the stability of the uncertain system was proved under the same set of conditions. The proposed robust control law and previous robust control laws [12-15] were the same, except that the uncertainty bound estimation laws were different. Computer simulations were carried out under the same conditions and with the same control parameters of Λ = diag(15 15) and K = diag(50 50), and the results were given in Figures 5-10. As seen from Figures 5-10, the tracking performance of the proposed adaptive-robust control law is better than the proposed robust controller of Eq. (35), and the pure transient behavior and chattering in the robust controller of Eq. (35) were removed. These results also show that the tracking performance of the adaptive-robust controller was better than that of previous robust controllers [12-15], and the tracking performance of previous robust controllers [12,15] was also improved.

6. Discussion

Spong [4] proposed a new robust controller for robot manipulators using the Lyapunov theory that guarantees stability of uncertain systems. In [4], the nominal control parameter π_0 and uncertainty bound parameters ρ are constant, and the constant π_0 and ρ cause pure tracking performance. In order to improve tracking performance, adaptive uncertainty bound parameter control laws were designed and ρ was made adaptive in [11-15]. However, pure transient behavior and chattering were observed in [11-15]. The parameter π_0 and uncertainty bound parameters ρ were made adaptive in [20-23]; however, the robot parameter π exists in the control law. The robot parameter π is assumed to be known and the Corless-Leitmann approach [1] is not used to design the adaptive-robust control laws [20-23].

In order to improve the tracking performance of an uncertain system, an adaptive-robust control law was considered. For this purpose, the previous robust controllers [4,12-15] were developed in such a manner that both the parameters and uncertainty bounds were made adaptive for robustness to uncertainty. The disadvantage of the previous adaptive-robust control laws [20-22] was eliminated, such that the robot parameter π does not exist in the control law and the robot parameter π is uncertain, as it would be in a robust control strategy [4].

As shown in Figures 6-8, the tracking error is very small for the proposed adaptive-robust controllers, and the tracking performance of the uncertain system can be improved for the appropriate values of control parameters α, β, and λ. The designed parameter and bound estimation functions are very effective for improving the tracking performance, and tracking performance is improved by adjusting control parameters α, β, and λ to appropriate values. The functions $\hat{\pi}$ and $\hat{\rho}(t)$ act as compensators; that is, they estimate the most appropriate values of $\hat{\pi}$ and $\hat{\rho}(t)$ in order to reduce the tracking error. Computer simulation results illustrated that proper estimation of $\hat{\pi}$ and $\hat{\rho}(t)$ were achieved, tracking performance was improved, and, as a result, proper estimation of $\hat{\pi}$ and $\hat{\rho}(t)$ improved the tracking performance.

In order to guarantee stability of an uncertain system, a Lyapunov function was defined, including 2 novel functions such as Φ_1 and Φ_2. There may be some sort of functions Φ_1 and Φ_2 for control inputs $\mathrm{u(t)}_1$ and $\mathrm{u(t)}_2$ that satisfy the stability of the uncertain system. However, there is no certain rule for the determination of Φ_1 and Φ_2 for control inputs $\mathrm{u(t)}_1$ and $\mathrm{u(t)}_2$ that satisfies $\dot{V} \leq 0$ in Eq. (15). We used system state parameters and mathematical insight and found the appropriate novel functions Φ_1 and Φ_2 for control inputs $\mathrm{u(t)}_1$ and $\mathrm{u(t)}_2$ that make $\dot{V} \leq 0$. As a result, the stability of the uncertain system is guaranteed and the uniform boundedness error convergence is shown based on the Lyapunov theory and the Corless-Leitmann approach [1]. Based on the adaptive-robust control law, a robust control law was also defined and the stability of the uncertain system was proved under the same set of conditions. The proposed adaptive-robust control law and previous studies [4,12-15] were developed for uncertain parameters, and the friction model does not exist in dynamic models. Danesh et al. [6] developed Spong's approach [4] in such a manner that the control law was made robust not only to uncertain inertia parameters but also to robust unmodeled dynamics and disturbances. Similarly, it is possible to develop the proposed adaptive-robust control law to be robust not only to uncertain inertia parameters but also to unmodeled dynamics and disturbances. This possibility can be considered for further studies.

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