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Lyapunov's direct method for stabilization of the Furuta pendulum

Türker TÜRKER*, **Haluk GÖRGÜN**, **Galip CANSEVER**
Department of Control and Automation Engineering, Yıldız Technical University, İstanbul-TURKEY
e-mails: {turker,gorgun,cansever}@yildiz.edu.tr

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Abstract

This paper presents a novel stabilization result for the Furuta pendulum for a large region of attraction including almost all of the upper half plane. The solution is obtained via constructing a Lyapunov function after set of coordinate changes. Then, a set of differential equations are solved to achieve asymptotic stability which is proved in accordance with La Salle's invariance principle. The effectiveness of the proposed stabilization method is illustrated with simulation studies.

Key Words: *Lyapunov based control, furuta pendulum*

1. Introduction

The Furuta pendulum was first introduced in 1991 [1]. The system consists of an actuated arm fixed on a rotating shaft and an unactuated pendulum which is pivoted at the end of the arm. The rotating axis of the pendulum is perpendicular to the rotating axis of the arm. The pendulum can only be rotated by applying torque on the arm. Since the lack of direct control input to the pendulum, the Furuta pendulum is an underactuated mechanical system. This particular pendulum system is an excellent benchmark problem for control studies and it is one of the most complex two-degree-of-freedom underactuated mechanical systems [2]. The stabilization problem of the Furuta pendulum is simply to bring the pendulum to the unstable equilibrium point, namely upward position, by rotating the arm whose angular position is also supposed to be brought to an exact position that is assigned before hand.

The stabilization problem of the Furuta pendulum has attracted the attention of many researchers and numerous control schemes have been proposed in the literature. Although many different control schemes of linear control techniques have been tested on the system, in all, the system is only stabilized in a small interval in the neighborhood of the unstable equilibrium point. Due to the lack of (direct) control input to the pendulum, the system is not fully feedback linearizable, and resulting nonlinear control design for Furuta pendulum becomes more difficult compared to fully actuated mechanical systems. Nevertheless, in [3] a partial

*Corresponding author: Department of Control and Automation Engineering, Yıldız Technical University, İstanbul-TURKEY

feedback linearization procedure is introduced for underactuated mechanical systems which can also be applied to Furuta pendulum system.

In the literature, the control studies for Furuta pendulum are generally analyzed in two main problem groups. The first one is based on the swing-up the pendulum from its hanging position while the other aims the stabilization of the pendulum in a local interval [2].

In [4], a swing-up control strategy using a subspace projected from the whole state space is proposed, and robustness of the proposed method is shown by means of experimental studies. Some properties of simple strategies for swinging-up the pendulum based on energy control is studied in [5] and it is concluded with that pendulum swing-up is closely related to the ratio between the acceleration of the pivot and the acceleration due to gravity. Another swing-up approach is proposed in [6] which is based on the speed-gradient method. The obtained control law was supported by means of simulation and experimental studies. Also an interesting method for swing-up the Furuta pendulum is introduced in [7]. In this method, the swing-up is provided by bringing the system to a homoclinic orbit.

On the other hand, in [8] a change of coordinates is introduced to simplify the matching equations derived from total energy shaping control methods. Although a solution is obtained for Furuta pendulum, the region of attraction of the system depends on physical system parameters. A similar result is given in [9] where the controller is designed based on a Lyapunov function. Olfati-Saber [10] gives a methodology to obtain a normal form for a class of underactuated mechanical systems and applies fixed point backstepping method to the Furuta pendulum. More recently, Lyapunov's direct method has also been used to stabilize a class of underactuated mechanical systems in [11, 12] with the example of Furuta pendulum. In this approach, however, the designed control input applied to the system is dynamic and the stability analysis is skipped. In [13], a procedure is given to obtain a static feedback controller for the ball and beam system which is inspired from the approach proposed in [11].

Inspired from the direct Lyapunov methods discussed above, this paper addresses the problem of achieving the stabilization of Furuta pendulum on unstable equilibrium point with a static feedback controller. Accordingly the stabilizing controller is produced via partial feedback linearization and Lyapunov's direct method with suitable and simplifying coordinate changes. Then, asymptotic stabilization is provided for the system with the region of attraction including almost all points in the upper half-plane of the pendulum with no dependence on the physical parameters. This result represents the main contribution of this paper comparing to other similar studies [2, 8, 9] in which the local asymptotic stabilization is obtained for the Furuta pendulum with the region of attraction depending on the physical parameters of the system. It should be mentioned that the physical parameters of the Furuta pendulum are considered to be exactly known in the controller design procedure.

The remainder of this paper is organized as follows. Section 2 is devoted to derive the dynamic equations of motion of the pendulum on a cart system. The control problem is formulated in Section 3. Controller design procedure and stability analysis are discussed in Section 4. Following, in Section 5, illustrative simulation studies is presented. Lastly, Section 6 concludes the paper.

2. Dynamics

Figure 1 illustrates the Furuta pendulum considered in this study. Lagrangian of the system can be derived as

$$L = \frac{1}{2}\left((I_1 + m_1 l_1^2)\dot{q}_1^2 + 2m_1 l_1 L_2 \cos(q_1)\dot{q}_1\dot{q}_2 + (I_2 + m_2 l_2^2 + m_1 L_2^2 + m_1 l_1^2 \sin^2(q_1))\dot{q}_2^2\right) - m_1 l_1 g \cos(q_1), \quad (1)$$

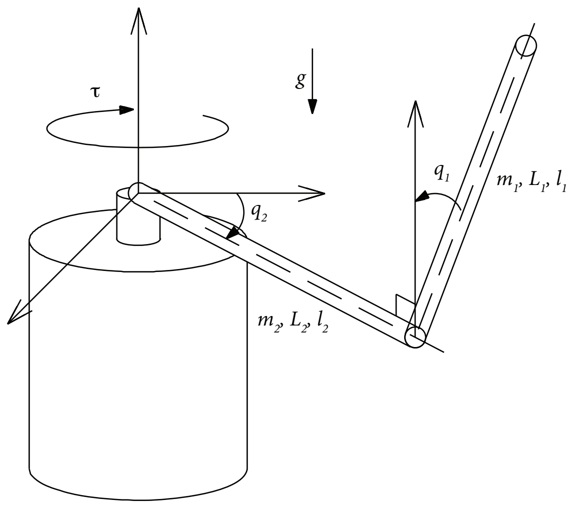

Figure 1. A schematic of the Furuta Pendulum, together with its physical parameters.

where q_1, q_2 are the pendulum angle from the vertical, and the arm angle, respectively. Quantities $\dot{q}_1, \dot{q}_2$ are the angular velocities of the pendulum and the arm, and g is gravitational acceleration. The rest of the system parameters are given in Table 1. Dynamic equations of motion for the Furuta pendulum can be obtained by applying Euler-Lagrange equations as

$$\theta_1\ddot{q}_1 + \theta_2\cos(q_1)\ddot{q}_2 - \theta_3\sin(q_1)\cos(q_1)\dot{q}_2^2 - \theta_4 g\sin(q_1) = 0 \tag{2}$$

$$\theta_2\cos(q_1)\ddot{q}_1 + (\theta_5 + \theta_3\sin^2(q_1))\ddot{q}_2 - \theta_2\sin(q_1)\dot{q}_1^2 + 2\theta_3\sin(q_1)\cos(q_1)\dot{q}_1\dot{q}_2 = \tau, \tag{3}$$

where τ, the control input, is the applied torque to the shaft of the arm and θ_i (i=1,...,5) are positive system parameters defined as $\theta_1 = I_1 + m_1 l_1^2$, $\theta_2 = m_1 l_1 L_2$, $\theta_3 = m_1 l_1^2$, $\theta_4 = m_1 l_1$, $\theta_5 = I_2 + m_2 l_2^2 + m_1 L_2^2$.

3. Problem formulation

This section is devoted to the partial feedback linearization procedure and changes of coordinates to simplify the control design. The first step of the partial feedback linearization (see [3]) is to calculate $\ddot{q}_2$ from (2) as

$$\ddot{q}_2 = -\frac{\theta_1}{\theta_2\cos(q_1)}\ddot{q}_1 + \frac{\theta_3\sin(q_1)}{\theta_2}\dot{q}_2^2 + \frac{\theta_4 g\sin(q_1)}{\theta_2\cos(q_1)}. \tag{4}$$

Note that, to obtain (4), $\cos(q_1)$ should be non-zero which sets the region of attraction to the upper half-plane of the pendulum. After plugging (4) into (3) and rearranging (2) and (3),

$$\frac{\theta_1}{\theta_2\cos(q_1)}\ddot{q}_1 + \ddot{q}_2 - \frac{\theta_3}{\theta_2}\sin(q_1)\dot{q}_2^2 - \frac{\theta_4 g\sin(q_1)}{\theta_2\cos(q_1)} = 0 \tag{5}$$

$$\begin{aligned}(\theta_2\cos(q_1) + \frac{\theta_1(\theta_5+\theta_3\sin^2(q_1))}{\theta_2\cos(q_1)})\ddot{q}_1 - \theta_2\sin(q_1)\dot{q}_1^2 + 2\theta_3\sin(q_1)\cos(q_1)\dot{q}_1\dot{q}_2 \\ + \frac{\theta_3(\theta_5+\theta_3\sin^2(q_1))\sin(q_1)}{\theta_2}\dot{q}_2^2 + \frac{\theta_4(\theta_5+\theta_3\sin^2(q_1))g\sin(q_1)}{\theta_2\cos(q_1)} = \tau.\end{aligned} \tag{6}$$

Table 1. List of parameters and values assigned them in the mechanical system.

Description	Symbol	Value	Unit
Mass of the pendulum	m_1	67.9×10^{-3}	kg
Mass of the arm	m_2	0.2869	kg
Length of the pendulum	L_1	0.14	m
Length of the arm	L_2	0.235	m
Distance of the center of mass of the pendulum to pendulum's pivot point	l_1	0.07	m
Distance of the center of mass of the arm to arm's pivot point	l_2	0.1175	m
Inertia of the pendulum	I_1	5.5452×10^{-5}	kg·m^2
Inertia of the arm	I_2	1.9×10^{-3}	kg·m^2

It should be noted that $\ddot{q}_2$ has been canceled out in (6). Notably, partially linearizing control input can be selected as follows:

$$\begin{aligned}\tau = {} & -\theta_2 \sin(q_1)\dot{q}_1^2 + 2\theta_3 \sin(q_1)\cos(q_1)\dot{q}_1\dot{q}_2 + \frac{\theta_3(\theta_5 + \theta_3 \sin^2(q_1))\sin(q_1)}{\theta_2}\dot{q}_2^2 \\ & + \frac{\theta_4(\theta_5 + \theta_3 \sin^2(q_1))g\sin(q_1)}{\theta_2 \cos(q_1)} + (\theta_2\cos(q_1) + \frac{\theta_1(\theta_5 + \theta_3\sin^2(q_1))}{\theta_2\cos(q_1)})u_1, \end{aligned} \tag{7}$$

where u_1 is the new control input.

We now define the coordinate transformation (see [14]),

$$q_r = \gamma(q_1) + q_2, \tag{8}$$

where $\gamma(q_1)$ is to be defined. The first and the second time derivatives of the new coordinate q_r can be calculated as

$$\dot{q}_r = \gamma'(q_1)\dot{q}_1 + \dot{q}_2, \tag{9}$$

$$\ddot{q}_r = \gamma'(q_1)\ddot{q}_1 + \ddot{q}_2 + \frac{d}{dt}(\gamma'(q_1))\dot{q}_1, \tag{10}$$

where $\gamma'(q_1)$ denotes the derivative of $\gamma(q_1)$ with respect to q_1. After defining $\gamma'(q_1) = \frac{\theta_1}{\theta_2\cos(q_1)}$, $\gamma(q_1)$ can be calculated as

$$\gamma(q_1) = \frac{2\theta_1}{\theta_2}\text{arctanh}(\tan(\frac{q_1}{2})). \tag{11}$$

Equations (5) and (6) can be rearranged by using (7), and introducing a new coordinate, $q_s = q_1$, we get

$$\ddot{q}_r - \frac{\theta_3\sin(q_s)}{\theta_2}\dot{q}_r^2 + \frac{2\theta_1\theta_3\sin(q_s)}{\theta_2^2\cos(q_s)}\dot{q}_r\dot{q}_s - \left(\frac{\theta_1^2\theta_3\sin(q_s)}{\theta_2^3\cos^2(q_s)} + \frac{\theta_1\sin(q_s)}{\theta_2\cos^2(q_s)}\right)\dot{q}_s^2 - \frac{\theta_4 g\sin(q_1)}{\theta_2\cos(q_s)} = 0 \tag{12}$$

$$\ddot{q}_s = u_1. \tag{13}$$

In order to simplify analysis further, we introduce the following change of coordinates (see [13]):

$$\underbrace{\begin{bmatrix} p_r \\ p_s \end{bmatrix}}_{\bar{p}} = \underbrace{\begin{bmatrix} \lambda_r(q_s) & 0 \\ 0 & \lambda_s(q_s) \end{bmatrix}}_{\Lambda(q_s)} \underbrace{\begin{bmatrix} \dot{q}_r \\ \dot{q}_s \end{bmatrix}}_{\dot{\bar{q}}}, \tag{14}$$

where λ_r and λ_s denote non-zero functions which will be assigned. By using (14) and its time derivative, and setting [1]

$$u_1 = \frac{1}{\lambda_s}\left(-\frac{\lambda_s'}{\lambda_s^2}p_s^2 + c_3 p_r + c_4 p_s + g_2 + u_2\right) \tag{15}$$

where c_3, c_4 and g_2 will be assigned and u_2 is the new control input, then the closed-loop system can be written as

$$\begin{aligned} \dot{\bar{q}} &= \Lambda^{-1}\bar{p} \quad (16) \\ \dot{\bar{p}} &= C\bar{p} + G + Fu_2 \quad (17) \end{aligned}$$

where

$$C = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}, G = \begin{bmatrix} \frac{\lambda_r \theta_4 g \sin(q_s)}{\theta_2 \cos(q_s)} \\ g_2 \end{bmatrix}, F = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

with

$$\begin{aligned} c_1 &= \frac{\theta_3 \sin(q_s)}{\theta_2 \lambda_r} p_r - \frac{\theta_1 \theta_3 \sin(q_s)}{\theta_2^2 \cos(q_s)\lambda_s} p_s + \frac{\lambda_r'}{2\lambda_r \lambda_s} p_s \\ c_2 &= -\frac{\theta_1 \theta_3 \sin(q_s)}{\theta_2^2 \cos(q_s)\lambda_s} p_r + \frac{\theta_1^2 \theta_3 \sin(q_s)\lambda_r}{\theta_2^3 \cos^2(q_s)\lambda_s^2} p_s + \frac{\theta_1 \sin(q_s)\lambda_r}{\theta_2 \cos^2(q_s)\lambda_s^2} p_s + \frac{\lambda_r'}{2\lambda_r \lambda_s} p_r. \end{aligned}$$

We pursue from here to the procedure which aims to design stabilizing controller.

4. Control design

In order to find a stabilizing controller, the candidate Lyapunov function is of the form

$$V(\bar{q},\bar{p}) = \frac{1}{2}\bar{p}^{\mathrm{T}} K(\bar{q})\bar{p} + \phi(\bar{q}), \tag{18}$$

where $K(\bar{q}) \in \mathbb{R}^{2\times 2}$ is a positive definite and symmetric matrix, $\phi(\bar{q})$ is a scalar function that has to have (at least) a local minimum at the point which the system will be driven to. Taking the time derivative of the Lyapunov function yields

$$\dot{V} = \bar{p}^{\mathrm{T}} K(\bar{q})\dot{\bar{p}} + \frac{1}{2}\bar{p}^{\mathrm{T}}\Lambda^{-1}(\nabla_{\bar{q}}(K(\bar{q})\bar{p})^{\mathrm{T}}))\bar{p} + \bar{p}^{\mathrm{T}}\Lambda^{-1}\nabla_{\bar{q}}\phi. \tag{19}$$

After substituting (17) into (19), a straight forward calculation gives

$$\dot{V} = \bar{p}^{\mathrm{T}}\underbrace{\left(KC + \frac{1}{2}\Lambda^{-1}(\nabla_{\bar{q}}(K(\bar{q})\bar{p})^{\mathrm{T}})\right)}_{W} p + p^{\mathrm{T}}\underbrace{\left(KG + \Lambda^{-1}\nabla_{\bar{q}}\phi\right)}_{Z} + p^{\mathrm{T}} K F u_2. \tag{20}$$

If it is achieved to compose a skew-symmetric $W(\bar{q},p)$ and $Z(\bar{q}) = 0$, then the time derivative of the Lyapunov function turns out to be

$$\dot{V} = p^{\mathrm{T}} K F u_2, \tag{21}$$

[1] After that point, for all expressions that are functions of $\bar{q}$ and $\bar{p}$ or their entries, their dependence will be written only the first time they appeared for the simplicity.

which can easily be made negative semi-definite by choosing an appropriate u_2; so that the system becomes at least stable. Notably, next challenge is assigning appropriate K and Λ such that W is skew-symmetric and $Z = 0$. Under the direction of similar approaches [8, 12, 13], the matrix K is selected as dependent only on q_s. Introduce

$$K(q_s) = \begin{bmatrix} k_1(q_s) & k_2(q_s) \\ k_2(q_s) & k_3(q_s) \end{bmatrix}, \tag{22}$$

then one can compute

$$W = \begin{bmatrix} k_1c_1 + k_2c_3 & k_1c_2 + k_2c_4 \\ k_2c_1 + k_3c_3 + \frac{k_1' p_r}{2\lambda_s} + \frac{k_2' p_s}{2\lambda_s} & k_2c_2 + k_3c_4 + \frac{k_2' p_r}{2\lambda_s} + \frac{k_3' p_s}{2\lambda_s} \end{bmatrix}. \tag{23}$$

The aim here is to construct the skew-symmetric matrix W. Note that c_3 and c_4 in the entries of W are also the components of feedback signal given by equation (15), thus, assigning these parameters properly provides W to be skew-symmetric. Parameters c_3 and c_4 can be calculated from equation (23) to satisfy $[W]_{ii} = 0$ as

$$c_3 = -\frac{k_1}{k_2}c_1 \tag{24}$$

$$c_4 = -\frac{k_2}{k_3}c_2 - \frac{k_2' p_r}{2k_3\lambda_s} - \frac{k_3' p_s}{2k_3\lambda_s}. \tag{25}$$

Once c_3 and c_4 are calculated they can be plugged into the other entries of W, in order to obtain the differential equation $[W]_{12} + [W]_{21} = 0$ that can be constituted as

$$-\frac{(k_1k_3 - k_2^2)}{k_2}c_1 + \frac{(k_1k_3 - k_2^2)}{k_3}c_2 - \frac{k_2'k_2 - k_1'k_3}{2k_3\lambda_s}p_r - \frac{k_3'k_2 - k_2'k_3}{2k_3\lambda_s}p_s = 0. \tag{26}$$

Plugging c_1 and c_2 into (26) gives

$$\begin{aligned} &\left[(k_1k_3 - k_2^2)\left(-\frac{\theta_3\sin(q_s)}{\theta_2k_2\lambda_r} - \frac{\theta_1\theta_3\sin(q_s)}{\theta_2^2k_3\cos(q_s)\lambda_s} + \frac{\lambda_r'}{2k_3\lambda_r\lambda_s}\right) - \frac{k_2'k_2 - k_1'k_3}{2k_3\lambda_s}\right]p_r \\ &\left[(k_1k_3 - k_2^2)\left(\frac{\theta_1\theta_3\sin(q_s)}{\theta_2^2k_2\cos(q_s)\lambda_s} + \frac{\theta_1\sin(q_s)\lambda_r}{\theta_2k_3\cos^2(q_s)\lambda_s^2} + \frac{\theta_1^2\theta_3\sin(q_s)\lambda_r}{\theta_2^3k_3\cos^2(q_s)\lambda_s^2} - \frac{\lambda_r'}{2k_2\lambda_r\lambda_s}\right) - \frac{k_3'k_2 - k_2'k_3}{2k_3\lambda_s}\right]p_s = 0, \end{aligned} \tag{27}$$

which needs to be satisfied for all p_r and p_s. That means, both the expressions in brackets in (27) have to be equal to zero which constitutes two differential equations with four unknowns k_1, k_2, k_3, λ_r and a free parameter λ_s. Selecting $\lambda_s = \frac{z_1\lambda_r}{\cos(q_s)}$, $k_1 = a + \cos^2(q_s)$ and $k_2 = k_3 = \cos^2(q_s)$ with the constants of $z_1 \neq 0$ and $a > 0$ brings these differential equations into the form of

$$\alpha_1\sin(q_s) + \frac{\cos(q_s)\lambda_r'}{2z_1\lambda_r} = 0 \tag{28}$$

$$\alpha_2\sin(q_s) - \frac{\cos(q_s)\lambda_r'}{2z_1\lambda_r} = 0, \tag{29}$$

where

$$\begin{aligned} \alpha_1 &= -\tfrac{\theta_3}{\theta_2} - \tfrac{\theta_1\theta_3}{z_1\theta_2^2} \\ \alpha_2 &= \tfrac{\theta_1\theta_3}{z_1\theta_2^2} + \tfrac{\theta_1}{z_1^2\theta_2} + \tfrac{\theta_1^2\theta_3}{z_1^2\theta_2^3}. \end{aligned} \tag{30}$$

Note that with $a > 0$, $\det(K) = a\cos^2(q_s)$, which is positive definite $\forall q_s \in (-\pi/2, \pi/2)$. Adding both sides of (28) and (29) yields

$$\alpha_1 + \alpha_2 = 0. \tag{31}$$

This condition can be satisfied by setting z_1 as

$$z_1 = \sqrt{\frac{\theta_1\theta_2^2 + \theta_1^2\theta_3}{\theta_2^2\theta_3}}. \tag{32}$$

After satisfying the condition (31), equation (29) can be rewritten as

$$2z_1\alpha_2\sin(q_s) - \frac{\cos(q_s)\lambda_r'}{\lambda_r} = 0 \tag{33}$$

from which

$$\lambda_r = d_1[\cos(q_s)]^{-2z_1\alpha_2} \tag{34}$$

can be calculated with the nonzero integration constant d_1. With that last result, the W is provided to be skew-symmetric, which means the first term on the right hand side of the derivative of the candidate Lyapunov function in equation (20) vanishes.

The next step in design is to set the second term of equation (20) to zero. To achieve this, the vector Z is needed to be equal to zero for every value of q in the region of attraction. The vector can be given as

$$Z = \begin{bmatrix} k_1 & k_2 \\ k_2 & k_3 \end{bmatrix} \begin{bmatrix} \frac{\lambda_r\theta_4 g\sin(q_s)}{\theta_2\cos(q_s)} \\ g_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{\lambda_r} & 0 \\ 0 & \frac{1}{\lambda_s} \end{bmatrix} \begin{bmatrix} \frac{\partial\phi}{\partial q_r} \\ \frac{\partial\phi}{\partial q_s} \end{bmatrix}. \tag{35}$$

In order to satisfy $Z = 0$, g_2 can be calculated from the first row of (35),

$$g_2 = -\frac{k_1\lambda_r\theta_4 g\sin(q_s)}{k_2\theta_2\cos(q_s)} - \frac{1}{k_2\lambda_r}\frac{\partial\phi}{\partial q_r}. \tag{36}$$

Note that in (36), g_2 is another feedback term which is used for setting Z to zero. Next, partial differential equation is obtained as following by using the second row of equation (35):

$$-\frac{k_1k_3 - k_2^2}{k_2}\frac{\lambda_r\theta_4 g\sin(q_s)}{\theta_2\cos(q_s)} - \frac{1}{\lambda_r}\frac{\partial\phi}{\partial q_r} + \frac{1}{\lambda_s}\frac{\partial\phi}{\partial q_s} = 0. \tag{37}$$

This PDE is equivalent to the second term on the right hand side of equation (20), so satisfying that PDE cancels out the second term in the time derivative of the candidate Lyapunov function. Moreover, the solution to that equation also constructs the second term in candidate Lyapunov function. Therefore, the solution has to have a (local) minimum at the desired equilibrium. After plugging all results in, a solution to that PDE can be calculated as

$$\phi(q_r, q_s) = d_2 + \frac{ad_1^2 z_1\theta_4 g}{\theta_2(4z_1\alpha_1 + 4)}[\cos(q_s)]^{-4z_1\alpha_1 - 3} + d_3\left(\frac{1}{z_1}(q_r - q_r^*) + 2\text{arctanh}\left(\tan\left(\frac{q_s}{2}\right)\right)\right)^2. \tag{38}$$

where q_r^* denotes the desired angular position for the arm angle.

Stability analysis of the designed control scheme can be summarized with the following result.

Proposition 1 *A Furuta pendulum system (2), (3) in closed loop with the control input given by (7) guarantees an asymptotically stable equilibrium at zero with the region of attraction containing the set* $(\bar{q}, \bar{p}) \in ((-\pi/2, \pi/2) \times \mathbb{R}^3)$.

Proof The Lyapunov function (18) is positive definite in the region of attraction with $\phi(\bar{q})$ given in (38) and positive definite and symmetric matrix $K \in \mathbb{R}^{2\times 2}$ defined in (22). Assigning

$$u_2 = -kF^{\mathrm{T}}K\bar{p} \tag{39}$$

for some constant $k > 0$, with the inclusion of the skew-symmetric W and $Z = 0$, the time derivative of the Lyapunov function (20) can be rewritten as

$$\dot{V} = -k\bar{p}^{\mathrm{T}}KFF^{\mathrm{T}}K\bar{p}, \tag{40}$$

which is negative semi-definite. Thus, $V(\bar{q}, \bar{p})$ is non-increasing implying $\bar{q}$ and $\bar{p}$ are bounded in the region of attraction. Note that q is also bounded, since q_r and $\gamma(q_s)$ is bounded in $(\bar{q}, \bar{p}) \in ((-\pi/2, \pi/2) \times \mathbb{R}^3)$.

We invoke La Salle's invariant set principle (see [15]), in order to prove the asymptotic stability of the closed loop system with $q_s \in (-\pi/2, \pi/2)$. Define the set $\Omega_{\bar{c}} \in ((-\pi/2, \pi/2) \times \mathbb{R}^3)$ as

$$\Omega_c = \{(q, p) \in ((-\pi/2, \pi/2) \times \mathbb{R}^3 | V(\bar{q}, \bar{p}) < c)\} \tag{41}$$

with

$$\bar{c} = \sup\{c > 0 : V(\bar{q}, \bar{p}) < c \mid \Omega_c \text{ is bounded}\}. \tag{42}$$

To proceed, we need to show the largest invariant set in $\Omega_{\bar{c}}$. Let's concentrate again on (40) which can be rewritten by using (8) and (14) as

$$\dot{V} = -k(k_2 p_r + k_3 p_s)^2 = -k((k_2\lambda_r\gamma' + k_3\lambda_s)\dot{q}_1 + k_2\lambda_r\dot{q}_2)^2. \tag{43}$$

It can be concluded from (43) that $\dot{V}$ is equal to 0 only when $\dot{q}_2 = \frac{(k_2\lambda_r\gamma' + k_3\lambda_s)}{k_2\lambda_r}\dot{q}_1$. In that expression, the equality $\frac{(k_2\lambda_r\gamma' + k_3\lambda_s)}{k_2\lambda_r} = \frac{2\theta_1}{\theta_2\cos(q_1)}$ is provided which means $\dot{q}_2 = \beta\dot{q}_1$ has to be satisfied with $\beta \neq 0$ for $\dot{V}$ in (43) to be equal to zero. It can be shown using this information that any limit cycle does not exist. Note that any equilibrium point with $q_1 \neq 0$ in the region of attraction can be obtained with only the control input compensates the gravity force, however, that means a force on q_2 which causes $\ddot{q}_2 \neq 0$ and $\dot{q}_2 \neq 0$. So that, $\dot{V} = 0$ with $\dot{q}_1 \neq \dot{q}_2$ can only be true for a moment which implies $\dot{V}$ can not stay equal to 0 when $\dot{q}_1 \neq \dot{q}_2$.

On the other hand, $\dot{q}_1 = \dot{q}_2$ is only satisfied under the velocities of both coordinates are equal to zero which implies also $\ddot{q}_1 = \ddot{q}_2 = 0$. Recalling the dynamic equations (2) and (3) with the control input given by (7), (15) and (39), one can observe that these conditions only true when $q_1 = q_2 = 0$. Finally it is easy to show that from (7), (15) and (39), the control input is also bounded under the considerations above. This concludes the proof. □

5. Simulation results

Three simulation studies are performed on Matlab Simulink® to illustrate the performance of the designed controller. The design parameters of the control signal and the initial conditions are given in Table 2 for each

Table 2. The Initial values and the control design parameters used in the present work's simulation studies.

	$q_1(0)$ (rad)	$q_2(0)$ (rad)	$\dot{q}_1(0)$ (rad/s)	$\dot{q}_2(0)$ (rad/s)	q_2^* (rad)	k	a	d_1	d_3
Simulation 1	$\pi/6$	0	0	0	0	20	5	1	10
Simulation 2	$\pi/3$	0	0	0	$-\pi/3$	20	5	1	10
Simulation 3	$\pi/2-\pi/128$	0	0	0	0	25	5	-10	15

simulation. Figures 2-4 show the results for the change of the pendulum angle, the arm angle and the control signal with respect to time. Some conditions for the effects of the design parameters and the initial values on the performance are observed during the simulation studies.

It is worth emphasizing that, as the pendulum angle approaches to the borders of the interval $(-\pi/2, \pi/2)$ initially, the control signal increases as expected, because of the necessity of the power to stabilize the system. This condition can be observed in Figure 4. Since the pendulum angle is almost perpendicular to the effect of gravity, the control signal takes huge values ($\sim \pm 1.5 \times 10^5$ Nm) at the beginning in order to compensate for gravity. On the other hand, different initial conditions of the arm have only a little effect on the control signal while they have almost no effect on the transient response.

The effects of the design parameters on the transient response can be summarized as follows. The positive constant k is proportional to the settling time while it is inverse proportional to the oscillations, up to a certain value, depending on the system parameters. As k increases after that value, oscillations at the beginning and the settling time increase. A tuning around the value 20 for k gives the best performance for the system considered in this study. The constant d_1 is proportional to the oscillations and the settling time, a selection as $d_1 < -1$ gives a better performance and the response gets better as d_1 decreases. The positive constant d_3 is proportional to the settling time and oscillations. Finally, the positive constant a has the same effect as the constant d_1.

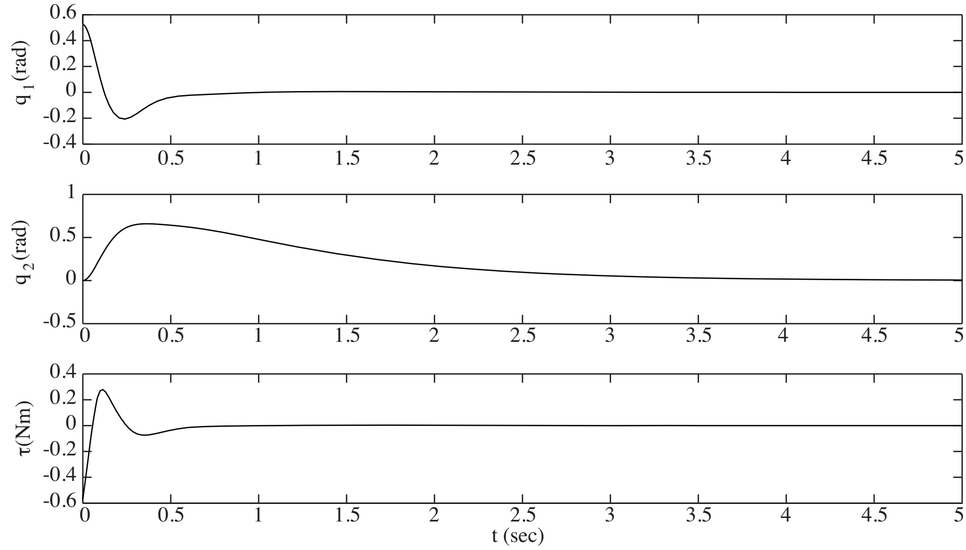

Figure 2. Change of the pendulum angle q_1, the arm angle q_2 and the control input t for Simulation 1.

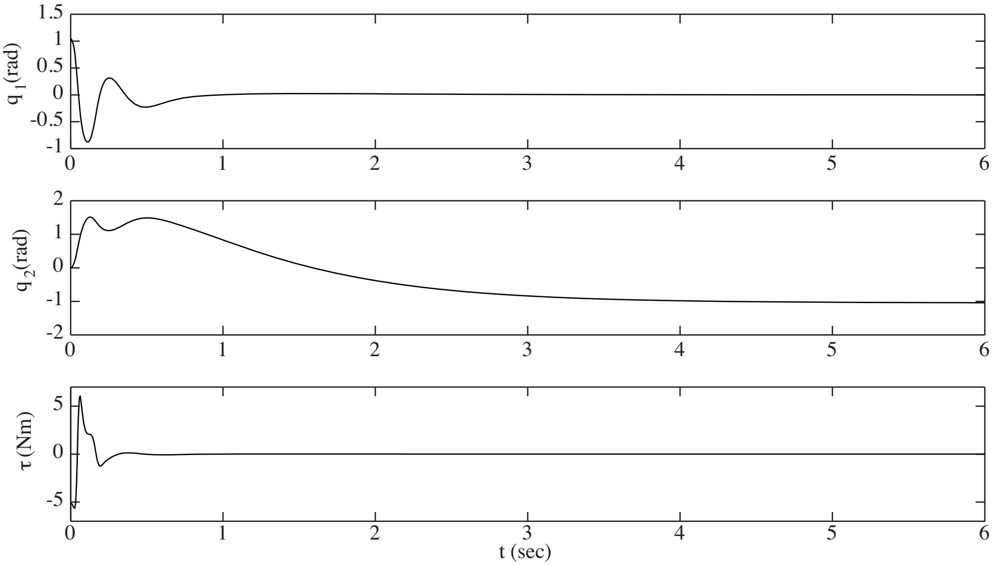

Figure 3. Change of the pendulum angle q_1, the arm angle q_2 and the control input t for Simulation 2.

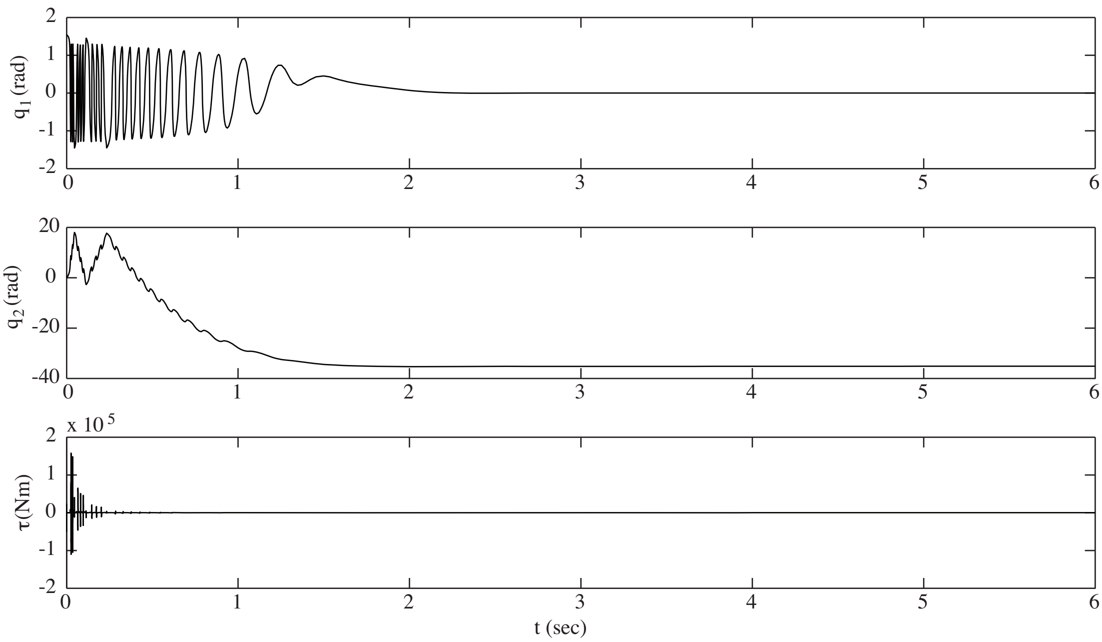

Figure 4. Change of the pendulum angle q_1, the arm angle q_2 and the control input t for Simulation 3.

Comparing Figure 2 and Figure 3, one can realize that the change of the settling time is very small while the change of the control input is very big with an increase on oscillations although there are only differences on the initial values. Also in Figure 4, almost no change on the settling time is observed while there is a big increase on oscillations and the control input. It can be concluded by mentioning that, the effects of the initial values are significant on oscillation and the control signal while the control parameters mostly effect the settling time.

6. Conclusion

A stabilization algorithm based on feedback linearization, coordinate transformations and direct Lyapunov method has been proposed for the Furuta pendulum. It has been proven that the designed controller asymptotically stabilizes the system if the pendulum is initially in the upper half plane with no dependence on physically system parameters. The obtained controller stabilizing the pendulum has been successfully supported by means of the computer simulations and the results were discussed in the related section. It is also shown in simulation studies that the control algorithm achieves asymptotic stabilization even the pendulum is initially very close to the horizontal plane. A generalization of the proposed method and robust controller design are considered as future studies.

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