

Stability analysis of time-delayed DC motor speed control system

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Abstract: In this paper, the stability of time-delayed DC motor speed control systems is analyzed. The measurement devices and communication links used by networked control systems, cause a significant amount of time delays. The stability boundary of the system in terms of the time delay is theoretically determined and an expression is obtained to compute the delay margin in terms of system parameters. The delay margin is defined as the maximum amount of time delay for which the DC motor speed control system is marginally stable. The results indicate that the system becomes unstable if the time delay exceeds the delay margin at a given set of parameters. Theoretical delay margin results are verified using the time-domain simulations of MATLAB/Simulink.

Key words: DC motor speed control, time delay, stability, delay margin

1. Introduction

The time delay in feedback control has become an important issue in recent years with the extensive use of networked control systems (NCSs) [1–7]. The time delays observed in NCSs, known as network-induced delays, consist of sensor-to-controller delay and controller-to-actuator delay. Even though the advances in communication networks have reduced the magnitude of the networked-induced delays significantly, they still cannot be ignored when designing a control system. Time delays have adverse effects on the control system dynamics and may cause closed-loop instabilities.

The DC motor control system is a typical example of control systems in which the undesirable impacts of time delays on the system dynamic are observed [5]. DC motor control systems are stable systems in general when time delays are not considered. However, inevitable time delays may destabilize the closed-loop system when the DC motor is controlled through a network [5]. For this reason, time delays must be considered in the process of a controller design, and methods need to be developed to compute the delay margin defined as the maximum amount of time delay for a stable operation. The description of the system stability boundary in terms of time delay also helps us design an appropriate controller for cases in which uncertainty in network-induced delays is unavoidable. To the best of our knowledge, the stability of networked control DC motor speed control systems has not been comprehensively analyzed, and in particular, the description of the stability boundary in terms of the delay margin for a broad range of controller gains has not been reported in the literature.

In stability analysis of NCSs, it is common practice to use continuous-time models of the controller, plant, and network-induced delays [4,6–8]. Various methods have been reported in the literature to estimate delay margins of continuous systems with time delays. All existing methods aim to compute the time delay value at

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which the system characteristic equation has purely imaginary roots. There are mainly the following 5 distinct approaches in the literature: i) the Schur–Cohn method (Hermite matrix formation) [9–11]; ii) a direct method to eliminate exponential terms in the characteristic equation [12]; iii) the matrix pencil Kronecker sum method [9–11,13]; iv) Kronecker multiplication and elementary transformation [14]; and v) Rekasius substitution [15–18]. Each of these methods has certain advantages and disadvantages depending on the time-delayed system under study. One can find a detailed comparison of these methods in [19], illustrating their strengths and weakness.

In our previous work [20], the Rekasius substitution method [15,17], which is a frequency domain approach, was effectively used to analyze the stability of the time-delayed DC motor speed control system and to compute delay margins over a large range of controller gains. In this paper, we present an alternative frequency domain theoretical method reported in [12] to estimate the delay margins of the DC motor speed control system. The proposed method first transforms the characteristic equation having exponential terms into a regular polynomial without any exponential terms. This method does not utilize any approximation to remove the exponential terms in the characteristic equation. For that reason, it is an exact method and the real positive roots of the new polynomial are exactly the same as the imaginary roots of the original characteristic equation having exponential terms. Moreover, by using the new polynomial, one can effectively investigate the delay dependency of the system stability and the sensitivities of crossing roots (root tendency) with respect to the time delay. This is the significant contribution of the proposed method. An analytical expression is then developed to compute delay margins in terms of controller gains and DC motor parameters. The comparison of the proposed method with other methods utilizing Rekasius substitution given in [15,17] indicates that this approach is more effective and easy to use and to implement. In addition, there is no need to introduce a pseudo-delay or to use the Routh–Hurwitz stability criterion in employing this method in stability analysis. In our previous work, we successfully applied this method to the stability analysis of time-delayed electric power systems and generator excitation control systems to determine delay margins [21,22]. In this paper, the delay margins are computed for a wide range of controller gains and compared with those obtained by using the Rekasius substitution method [20]. Finally, the accuracy of theoretical delay margin results is verified using MATLAB/Simulink [23].

2. Time-delayed DC motor speed control system

This section briefly describes the dynamics of the time-delayed DC motor speed control system. Figure 1 illustrates the block diagram of the system. The dynamics of a DC motor driving a load are described by a differential equation of the mechanical system and volt-ampere equations of the armature circuit [5].

$$\begin{aligned}
 J \frac{d\omega_m}{dt} + B\omega_m + T_l &= T_e = K i_a \\
 u(t) = v_a &= e_a + R_a i_a + L_a di_a/dt
 \end{aligned}
 \tag{1}$$

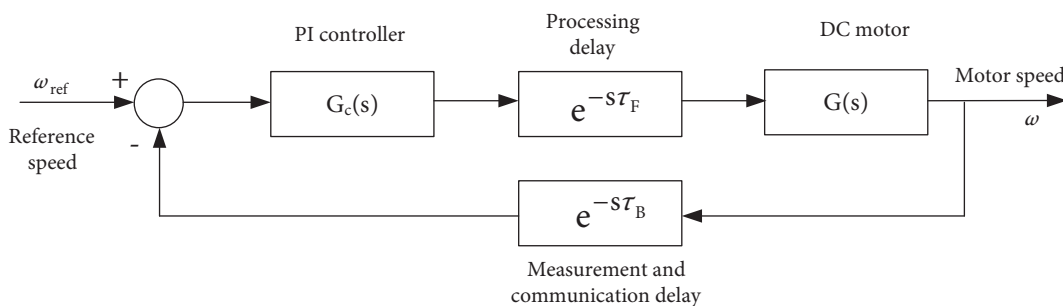


Figure 1. Block diagram of time-delayed DC motor speed control system.

Here, $u = v_a$ represents the armature winding input voltage; i_a , R_a , and L_a denote the current, resistance, and inductance of the armature circuit; respectively; $e_a = K_a \omega_m$ is the back-electromotive-force (EMF) voltage (i.e. generated speed voltage); ω_m is the angular speed of the motor; T_e and T_l are the electromagnetic torque developed by the motor and the mechanical load torque opposing direction; J is the combined moment of inertia of the load and the rotor; B is the equivalent viscous friction constant of the load and the motor; and K and K_a are the torque constant and the back-EMF constant, respectively. The transfer function of the DC motor could be easily obtained as follows:

$$G(s) = \frac{\frac{K}{JL_a}}{s^2 + \left(\frac{R_a}{L_a} + \frac{B}{J}\right)s + \frac{R_a B + K K_a}{JL}} \quad (2)$$

The proportional-integral (PI) controller is defined by the following transfer function:

$$G_c(s) = K_P + \frac{K_I}{s}, \quad (3)$$

where K_P and K_I represent the proportional and integral gains, respectively. The PI controller is used to shape the speed response so as to reach the desired value by adjusting the rate of angular speed rise after a step change and the settling time after initial overshoot.

As shown in Figure 1, all time delays in the feedback loop are lumped together into a feedback delay (τ_B) between the output and the controller. This delay represents the measurement and communication delays (sensor-to-controller delay). The controller processing and communication delay (τ_F) (controller-to-actuator delay) is placed in the feedforward part between the controller and the DC motor. The system characteristic polynomial is easily determined from

$$\Delta(s, \tau) = 1 + G_C(s)G(s)e^{-s\tau} = 0 \quad (4)$$

as

$$\Delta(s, \tau) = P(s) + Q(s)e^{-s\tau} = 0, \quad (5)$$

where $\tau = \tau_B + \tau_F$ and $P(s), Q(s)$ are polynomials in s with real coefficients. These polynomials are given as

$$\begin{aligned} P(s) &= p_3 s^3 + p_2 s^2 + p_1 s \\ Q(s) &= q_1 s + q_0 \end{aligned}, \quad (6)$$

where

$$\begin{aligned} p_3 &= 1, p_2 = \frac{R_a}{L_a} + \frac{B}{J}, p_1 = \frac{R_a B + K K_a}{JL}, p_0 = 0 \\ q_1 &= \frac{K K_P}{JL_a}, q_0 = \frac{K K_I}{JL_a} \end{aligned} \quad (7)$$

The location of the roots of the characteristic Eq. (5) must be determined to analyze the stability of the DC motor speed control system. In the following section, the proposed theoretical method is explained in detail and a formula to compute delay margins is derived.

3. Stability analysis

3.1. Delay margin definition

In stability analysis of time-delayed systems, it is essential to find conditions on the delay such that the system will be stable. Similar to the delay-free system (i.e. $\tau = 0$), the stability of the system is determined by

locations of the roots of the characteristic equation defined by Eq. (5). The characteristic equation in Eq. (5) clearly indicates that roots are functions of the time delay τ due to an exponential term. It is expected that locations of some roots will change as τ varies. For asymptotic stability, it is required that all roots of Eq. (5) be located in the left half of the complex plane. In other words,

$$\Delta(s, \tau) \neq 0, \forall s \in C^+, \tag{8}$$

where C^+ denotes the right half plane of the complex plane.

The existence of the time delay τ may give rise to 2 types of asymptotic stability situations depending on system parameters [10,12]:

- i) Delay-independent stability: The system defined by characteristic Eq. (5) is delay-independent stable if the stability condition of Eq. (8) is satisfied for all positive and finite values of the delay, $\tau \in [0, \infty)$.
- ii) Delay-dependent stability: The system defined by characteristic Eq. (5) is delay-dependent stable if the condition of Eq. (8) is satisfied for some values of delays only in an interval, $\tau \in [0, \tau^*)$.

When the system is delay-dependent stable, some of the characteristic roots move if time delay τ is increased starting from $\tau = 0$. Figure 2 shows how some roots move with respect to the time delay. It is clear from Figure 2 that the system is assumed to be stable for $\tau = 0$, known as a delay-free system. This is a valid assumption because the speed control system is stable for practical values of system parameters when the total time delay is ignored. Figure 2 illustrates that a pair of complex roots start moving in the left half complex plane as time delay τ increases. At a certain finite delay value τ^* , roots cross the imaginary axis and pass to the right half plane if the delay is further increased. The time delay value for which the roots are located on the imaginary axis is known as the delay margin for stability. In other words, the system will be stable for any given delay less than this margin, $\tau < \tau^*$.

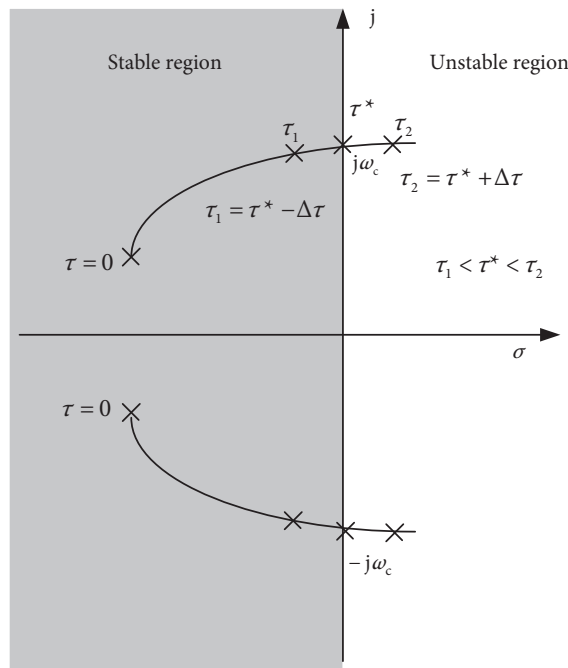


Figure 2. Illustration of eigenvalue movement with respect to time delay.

The first step in the stability analysis of the DC motor speed control system is to determine whether the system for any given set of parameters is delay-independent stable or not, and if not, to calculate the delay margin τ^* . The stability problem of interest is defined as follows:

Given: Time-delayed DC motor speed control system or its characteristic Eq. (5).

Determine: If the system stability depends on the time delay or not; if the system is delay-dependent stable, compute the delay margin.

The following subsection presents a direct method that allows us to evaluate the delay dependency of stability and enables us to develop an analytical formula for delay margin computations [12].

3.2. Solution method

It is well known that all of the roots of the characteristic equation of Eq. (5) must be located in the left half complex plane for a stable system. In the single delay case, the main objective of the stability analysis is to compute delay margin values τ^* for various system parameters. Eq. (5) clearly indicates that the characteristic equation, $\Delta(s, \tau) = 0$, is an implicit function of s and τ . For simplicity, it is assumed that a delay-free system is stable. In other words, all roots of $\Delta(s, 0) = 0$ are in the left half plane. Suppose that the characteristic equation $\Delta(s, \tau) = 0$ has a root on the imaginary axis at $s = j\omega_c$ for some finite values of the time delay τ . Because of the complex conjugate symmetry of complex roots, the equation $\Delta(-s, \tau) = 0$ will also have the same root at $s = j\omega_c$ for the same value of the time delay τ . Consequently, the problem now reduces to finding values of time delay τ such that both $\Delta(s, \tau) = 0$ and $\Delta(-s, \tau) = 0$ have a common root at $s = j\omega_c$. This result could be stated as follows:

$$\begin{aligned} P(s) + Q(s)e^{-s\tau} &= 0 \\ P(-s) + Q(-s)e^{s\tau} &= 0 \end{aligned} \quad (9)$$

The exponential terms in Eq. (9) could be easily eliminated and the following new characteristic equation is obtained:

$$P(s)P(-s) - Q(s)Q(-s) = 0. \quad (10)$$

The replacement s by $j\omega_c$ in Eq. (10) leads to the following polynomial in ω_c^2 [12,21]:

$$W(\omega_c^2) = P(j\omega_c)P(-j\omega_c) - Q(j\omega_c)Q(-j\omega_c) = 0. \quad (11)$$

The final form of the new characteristic equation in terms of system parameters is obtained by substituting $P(s)$ and $Q(s)$ polynomials given in Eqs. (6) and (7) into Eq. (11).

$$W(\omega_c^2) = t_6\omega_c^6 + t_4\omega_c^4 + t_2\omega_c^2 + t_0 = 0 \quad (12)$$

The corresponding coefficients in Eq. (12) are given as

$$\begin{aligned} t_6 &= p_3^2, t_4 = p_2^2 - 2p_1p_3, \\ t_2 &= p_1^2 - q_1^2, t_0 = -q_0^2 \end{aligned} \quad (13)$$

It is obvious from Eq. (12) that the exponential terms in the system characteristic equation given in Eq. (5) are now eliminated without using any approximation. For that reason, the positive real roots of Eq. (12) coincide with the imaginary roots of Eq. (5) exactly. The computation of the real roots of Eq. (12) is easier than the computation of the imaginary roots of Eq. (5). This is a great advantage since various methods are available to

determine the real roots of Eq. (12). The delay-dependent stability of the DC motor speed control system could then be easily analyzed by obtaining the roots of Eq. (12). Depending on these roots, the following situations may be observed:

- i) The new polynomial of Eq. (12) does not possess any positive real roots. In this case, the characteristic equation of Eq. (5) will not have any roots on the $j\omega$ -axis. Consequently, the DC motor speed control system will be delay-independent stable.
- ii) The polynomial of Eq. (12) may have at least one positive real root. In this case, the characteristic equation of Eq. (5) will have at least a pair of complex roots on the $j\omega$ -axis. As a result, the DC motor speed control system will be delay-dependent stable.

For a positive real root ω_c , the corresponding value of delay margin τ^* can be easily computed using Eq. (5) as follows [12]:

$$\tau^* = \frac{1}{\omega_c} \text{Tan}^{-1} \left(\frac{\text{Im} \left\{ \frac{P(j\omega_c)}{Q(j\omega_c)} \right\}}{\text{Re} \left\{ -\frac{P(j\omega_c)}{Q(j\omega_c)} \right\}} \right) + \frac{2r\pi}{\omega_c}; \quad r = 0, 1, 2, \dots, \infty. \tag{14}$$

It must be noted that the new characteristic polynomial of Eqs. (11) or (12) will have only a finite number of positive real roots for all $\tau \in \mathfrak{R}^+$. The set of these real roots is given as follows:

$$\{\omega_c\} = \{\omega_{c1}, \omega_{c2}, \dots, \omega_{cq}\}. \tag{15}$$

The number of real roots is affected by both the system order n and the coefficients of the polynomials $P(s)$ and $Q(s)$. Moreover, for each real positive root ω_{cm} , $m = 1, 2, \dots, q$, the corresponding delay margin could be computed by Eq. (14). These delay margins τ_m^* constitute a set of infinitely many delay margins which are periodically spaced. Let us call this set

$$\{\tau_m^*\} = \{\tau_{m1}^*, \tau_{m2}^*, \dots, \tau_{m,\infty}^*\} \quad m = 1, 2, \dots, q, \tag{16}$$

where $\tau_{m,r+1} - \tau_{m,r} = \frac{2\pi}{\omega_c}$ represents the period of repetition. Finally, the system delay margin will be the minimum of τ_m^* , $m = 1, 2, \dots, q$:

$$\tau^* = \min(\tau_m^*). \tag{17}$$

Once the set of real positive roots of Eq. (12) is computed, it is easy to compute the corresponding delay margin for each real positive root ω_{cm} by using Eq. (14). The substitution of polynomials $P(s = j\omega_{cm})$ and $Q(s = j\omega_{cm})$ given in Eqs. (6) and (7) into Eq. (14) results in the following expression:

$$\tau_m^* = \frac{1}{\omega_{cm}} \text{Tan}^{-1} \left(\frac{q_0 p_1 \omega_{cm} + (q_1 p_2 - q_0 p_3) \omega_{cm}^3}{(p_2 q_0 - q_1 p_1) \omega_{cm}^2 + q_1 p_3 \omega_{cm}^4} \right) + \frac{2r\pi}{\omega_{cm}}; \quad r = 0, 1, 2, \dots, \infty. \tag{18}$$

For any positive roots of Eq. (11), it is also required to investigate whether the root of Eq. (5) crosses the imaginary axis with increasing τ at $s = j\omega_c$. The sign of $\text{Re} \left[\frac{ds}{d\tau} \right]$ determines the existence of such crossing. For such crossing to occur, it is necessary that characteristic roots must cross the imaginary axis with a non-zero velocity or

$$\text{Re} \left[\frac{ds}{d\tau} \right]_{s=j\omega_{ck}} \neq 0, \tag{19}$$

where $Re(\bullet)$ represents the real part of a complex variable. The sign of root sensitivity is generally called root tendency (RT) [17].

$$RT|_{s=j\omega_c} = \text{sgn} \left\{ Re \left[\frac{ds}{d\tau} \right]_{s=j\omega_c} \right\} \quad (20)$$

An expression for the root tendency could be easily derived by taking the derivative of Eq. (5) with respect to τ :

$$\frac{ds}{d\tau} = \frac{Q(s)se^{-s\tau}}{P'(s) + Q'(s)e^{-s\tau} - Q(s)\tau e^{-s\tau}}, \quad (21)$$

where $P'(s)$ and $Q'(s)$ denote the first-order derivative of $P(s)$ and $Q(s)$ with respect to s , respectively. Using Eq. (5), the above expression could be rewritten as:

$$\frac{ds}{d\tau} = -s \left[\frac{P'(s)}{P(s)} - \frac{Q'(s)}{Q(s)} + \tau \right]^{-1}. \quad (22)$$

In order to obtain the root tendency, Eq. (22) needs to be evaluated at $s = j\omega_c$.

$$\begin{aligned} RT|_{s=j\omega_c} &= -\text{sgn} \left[Re \left(j\omega_c \left(\frac{P'(j\omega_c)}{P(j\omega_c)} - \frac{Q'(j\omega_c)}{Q(j\omega_c)} + \tau \right)^{-1} \right) \right] \\ &= -\text{sgn} \left[Re \left(\frac{1}{j\omega_c} \left(\frac{P'(j\omega_c)}{P(j\omega_c)} - \frac{Q'(j\omega_c)}{Q(j\omega_c)} + \tau \right) \right) \right] \\ &= \text{sgn} \left[Im \left(\frac{1}{\omega_c} \left(\frac{Q'(j\omega_c)}{Q(j\omega_c)} - \frac{P'(j\omega_c)}{P(j\omega_c)} \right) \right) \right] \end{aligned} \quad (23)$$

It must be emphasized that the root tendency given by Eq. (23) is independent from the time delay τ . This implies that even though there is an infinite number of values of τ associated with each value of ω_c that makes $\Delta(j\omega_c, \tau) = 0$, the behavior of the roots at these points will always be the same. The RT expression of Eq. (23) could be further simplified so that the RT information can be deduced trivially from the polynomial $W(\omega_c^2)$ [12]. Recall that for $s = j\omega_c$ we have $W(\omega_c^2) = 0$. Then, from Eq. (11), we have $Q(j\omega_c) = \frac{P(j\omega_c)P(-j\omega_c)}{Q(-j\omega_c)}$. Thus,

$$\begin{aligned} RT|_{s=j\omega_c} &= \text{sgn} \left[Im \left(\frac{1}{\omega_c} \left(\frac{Q'(j\omega_c)Q(-j\omega_c)}{P(j\omega_c)P(-j\omega_c)} - \frac{P'(j\omega_c)}{P(j\omega_c)} \right) \right) \right] \\ &= \text{sgn} \left[Im \left(\frac{1}{\omega_c} \left(\frac{Q'(j\omega_c)Q(-j\omega_c) - P'(j\omega_c)P(-j\omega_c)}{P(j\omega_c)P(-j\omega_c)} \right) \right) \right] \\ &= \text{sgn} \left[Im \left(\frac{1}{\omega_c} (Q'(j\omega_c)Q(-j\omega_c) - P'(j\omega_c)P(-j\omega_c)) \right) \right], \end{aligned} \quad (24)$$

since $P(j\omega_c)P(-j\omega_c) = |P(j\omega_c)|^2 > 0$. Finally, using the property $Im(z) = \frac{z-\bar{z}}{2j}$, for any complex number z , we have

$$RT|_{s=j\omega_c} = \text{sgn} \frac{1}{2j\omega_c} [(Q'(j\omega_c)Q(-j\omega_c) - Q(j\omega_c)Q'(-j\omega_c) - P'(j\omega_c)P(-j\omega_c) + P(j\omega_c)P'(-j\omega_c))], \quad (25)$$

which finally results in

$$RT|_{s=j\omega_c} = \text{sgn} [W'(\omega_c^2)], \tag{26}$$

where the prime denotes differentiation with respect to ω_c^2 . The evaluation of the root tendency by Eq. (26) is one of the most significant features of the proposed method. This expression provides a simple criterion to find the direction of transition of the roots at $s = j\omega_c$ as τ increases from $\tau_1 = \tau^* - \Delta\tau$ to $\tau_2 = \tau^* + \Delta\tau$, $0 < \Delta\tau \ll 1$, as shown in Figure 2. If $RT = +1$ for a root $s = j\omega_c$, then this root crosses the imaginary axis from the stable left half plane to the unstable right half plane or vice versa if $RT = -1$.

For the DC motor speed control system, the root tendency for each crossing frequency could be easily found by the following equation obtained by taking the derivative of the polynomial given by Eq. (12) with respect to ω_c^2 .

$$RT|_{s=j\omega_c} = \text{sgn} [W'(\omega_c^2)] = 3t_6\omega_c^4 + 2t_4\omega_c^2 + t_2 \tag{27}$$

At this point, it is necessary to compare the proposed method with the one presented in [15,17]. The method of [17] first utilizes Rekasius substitution [15] to remove the exponential terms in the characteristic equation of Eq. (5). The elimination of the exponential terms is achieved by using the Rekasius substitution, an exact substitution given by

$$e^{-s\tau} = \frac{1 - T.s}{1 + T.s}, \tag{28}$$

where $T \in \Re$ represents a pseudo-delay. With the help this substitution, the characteristic equation of Eq. (5) is transformed into a new polynomial without having any exponential terms similar to one given in Eq. (11). Moreover, purely imaginary roots of this new polynomial, which could be computed by the Routh stability criterion, are the same as those of the characteristic equation. Finally, the corresponding delay margin is determined by [17]:

$$\tau^* = \frac{2}{\omega_c} [Tan^{-1}(\omega_c T) \pm r\pi], \quad r = 0, 1, 2, \dots \tag{29}$$

It must be noted that both methods are based on the elimination of the exponential terms in the characteristic equation using different substitutions that are both exact. Moreover, both methods aim to obtain a new polynomial not including any exponential terms. In the proposed method, the real roots of this new polynomial, if any exist, coincide with the imaginary roots ω_c of the original characteristic equation exactly. The comparison of the method of [17] with the proposed one clearly shows that the method of [17] needs the introduction of a pseudo-delay T and an additional step, the Routh–Hurwitz stability criterion, to compute the pseudo-delay T and the imaginary roots of the characteristic equation ω_c . Additionally, it must be stated that the proposed method enables us to easily determine whether the system is delay-dependent or delay-independent stable.

4. Theoretical and simulation results

4.1. Theoretical results

In this section, for various values PI controller gains, delay margins are determined using Eq. (18). The accuracy of theoretical delay margin results are confirmed by using MATLAB/Simulink. The DC motor parameters used in this paper is as follows [5]: $J = 42.6 \times 10^{-6}$ kg m², $L_a = 170$ mH, $R_a = 4.67$ Ω, $B = 47.3 \times 10^{-3}$ N m s/rad, $K = 14.7 \times 10^{-3}$ N m/A, $K_a = 14.7 \times 10^{-3}$ V s/rad.

First, we choose typical PI controller gains ($K_P = 0.3$; $K_I = 1.0 \text{ s}^{-1}$) to demonstrate the delay margin computation. The process of the delay margin computation consists of the following 4 steps:

Step 1: Determine the characteristic equation of the time-delayed DC motor speed control system using Eqs. (5)–(7). This equation is found to be:

$$\Delta(s, \tau) = (s^3 + 28.583s^2 + 60.404s) + (608.948s + 2029.826)e^{-s\tau} = 0.$$

Note that for $\tau = 0$ the characteristic equation has roots at $s_{1,2} = -12.547 \pm j20.600$; $s_3 = -3.489$. The delay-free system is stable since these roots are located in the left half complex plane.

Step 2: Obtain the $W(\omega_c^2)$ polynomial using Eq. (12) or (13), and find its real positive roots ω_{cm} , if it exists. The polynomial is found as:

$$W(\omega_c^2) = \omega_c^6 + 696.2\omega_c^4 - 3.672 \times 10^5 \omega_c^2 - 4.120 \times 10^6 = 0.$$

This polynomial has only 1 positive real root, $\omega_c = 18.944 \text{ rad/s}$.

Step 3: Calculate the delay margin for each positive root found in Step 2 using Eq. (18) and select the minimum of those as the system delay margin. The delay margin is found as $\tau^* = 0.04713 \text{ s}$.

Step 4: Determine the root tendency (RT) for each positive root ω_{cm} using Eq. (27). The RT for $\omega_c = 18.944 \text{ rad/s}$ is computed as $RT = +1$. This RT indicates that a pair of complex roots moves from the stable left half plane to the unstable right half plane, crossing the $j\omega$ -axis at $s = \pm j18.944 \text{ rad/s}$ for $\tau^* = 0.04713 \text{ s}$. As a result, the system becomes unstable.

For a complete theoretical analysis, the effect of PI controller gains on the delay margin is also investigated. For this reason, the values of PI controller gains are chosen in the ranges of $K_P = 0.1 - 0.9$ and $K_I = 0.1 - 3.0 \text{ s}^{-1}$ and delay margins are computed. The Table shows delay margins for various values of PI controller gains. The following observations could be made about delay margin results. It is clear from the Table that the delay margin decreases when the integral controller gain is increased for fixed K_P values. The effect of K_P on the delay margin has 2 tendencies for a fixed K_I . When K_I belongs to the interval of $K_I = 0.1 - 1.3 \text{ s}^{-1}$, the delay margin decreases when K_P is increased. On the contrary, when K_I lies in the range of $K_I = 1.4 - 3.0 \text{ s}^{-1}$, an increase is observed in the delay margin with an increase in K_P when K_P is smaller, whereas it decreases when K_P is larger.

Moreover, comparison of the delay margins presented in the Table with those obtained by the Rekasius substitution method [20] clearly shows that delay margin differences between the 2 methods are considerably small. It should be mentioned here that the largest difference between delay margins is observed to be around 0.1% for the same range of PI controller gains presented in the Table. More importantly, the proposed method is conservative compared to the Rekasius substitution method.

4.2. Verification of theoretical results

The theoretical delay margin results are verified using MATLAB/Simulink. Figure 3 shows the Simulink model of the time-delayed DC motor speed control system. The box labeled as the DC motor represents the state-space equation model of the DC motor given in Eq. (1). The box shown as the PI controller is the Simulink model of the PI controller given by Eq. (3). Two transportation delay blocks are used to include the measurement/communication delay in the feedback path and the processing delay in the feedforward path. The box's motor speed and scope are used to get the speed data and to display their waveform.

Table. Delay margins obtained by the proposed theoretical method.

K_I (s^{-1})	τ^* (s)				
	$K_P = 0.1$	$K_P = 0.3$	$K_P = 0.5$	$K_P = 0.7$	$K_P = 0.9$
0.1	0.20612	0.05638	0.03167	0.02193	0.01675
0.2	0.18152	0.05540	0.03140	0.02179	0.01671
0.3	0.15808	0.05440	0.03111	0.02167	0.01659
0.4	0.13741	0.05339	0.03083	0.02154	0.01652
0.5	0.11980	0.05237	0.03055	0.02141	0.01645
0.6	0.10497	0.05133	0.03026	0.02128	0.01637
0.7	0.09246	0.05029	0.02997	0.02115	0.01630
0.8	0.08185	0.04924	0.02969	0.02102	0.01623
0.9	0.07280	0.04818	0.02940	0.02088	0.01615
1.0	0.06500	0.04713	0.02910	0.02075	0.01608
1.1	0.05824	0.04608	0.02881	0.02062	0.01600
1.2	0.05233	0.04502	0.02852	0.02049	0.01593
1.3	0.04712	0.04398	0.02822	0.02035	0.01585
1.4	0.04250	0.04294	0.02793	0.02022	0.01578
1.5	0.03838	0.04190	0.02764	0.02009	0.01570
1.6	0.03469	0.04088	0.02734	0.01996	0.01563
1.7	0.03136	0.03986	0.02705	0.01982	0.01555
1.8	0.02835	0.03886	0.02675	0.01969	0.01547
1.9	0.02562	0.03787	0.02645	0.01956	0.01539
2.0	0.02312	0.03689	0.02616	0.01942	0.01532
2.1	0.02083	0.03593	0.02586	0.01929	0.01525
2.2	0.01873	0.03498	0.02556	0.01915	0.01517
2.3	0.01679	0.03405	0.02527	0.01902	0.01509
2.4	0.01499	0.03313	0.02497	0.01888	0.01502
2.5	0.01333	0.03223	0.02468	0.01875	0.01494
2.6	0.01179	0.03135	0.02438	0.01861	0.01487
2.7	0.01035	0.03048	0.02409	0.01848	0.01479
2.8	0.00901	0.02963	0.02379	0.01834	0.01471
2.9	0.00776	0.02879	0.02350	0.01821	0.01464
3.0	0.00658	0.02798	0.02321	0.01807	0.01456

In order to illustrate the verification, PI controller gains are chosen as $K_P = 0.3; K_I = 1.0 s^{-1}$. It is clear from the Table that the delay margin is $\tau^* = 0.04713$ s for these PI gains. This theoretical delay margin result implies that the system will be marginally stable at this delay value. However, using the Simulink model shown in Figure 3, it is found that the system is marginally stable at $\tau^* = 0.04726$ s. Figure 4 shows simulation results for this delay value. The error between the theoretical delay margin and the one obtained by simulation is 0.2758%, which is evidently negligible. It is clear from Figure 4 that persistent oscillations occur, which verifies the marginal stability predicted by the theory. When the time delay is less than $\tau^* = 0.04726$ s, the speed control system is expected to be stable. Figure 5 presents such a stable simulation result for $\tau = 0.046$ s. On the other hand, when the time delay is larger than $\tau^* = 0.04726$ s, the system becomes unstable since it has increasing oscillations, as shown in Figure 6 for $\tau = 0.048$ s.

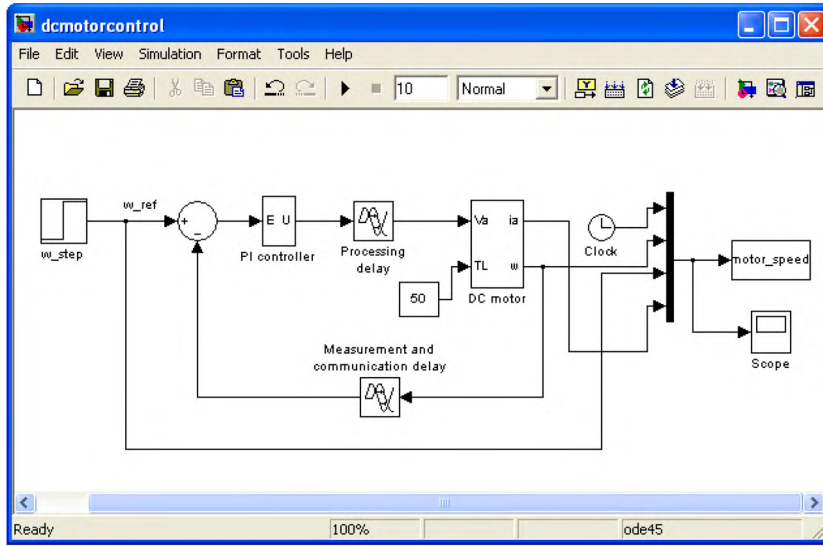


Figure 3. Simulink model of the time-delayed DC motor speed control system.

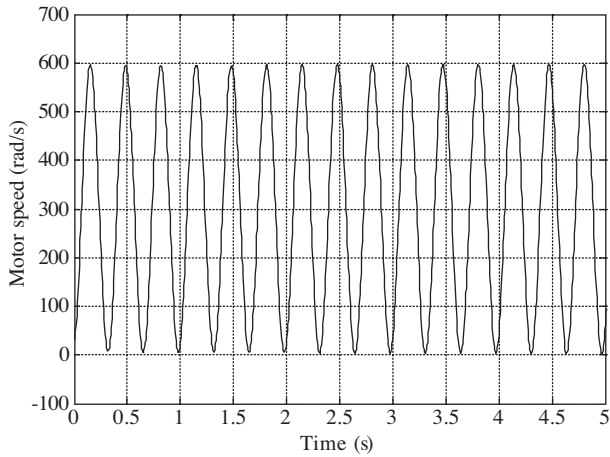


Figure 4. Motor speed for $K_I = 1.0s^{-1}$, $K_P = 0.3$, and $\tau^* = 0.04726$ s: marginally stable operation.

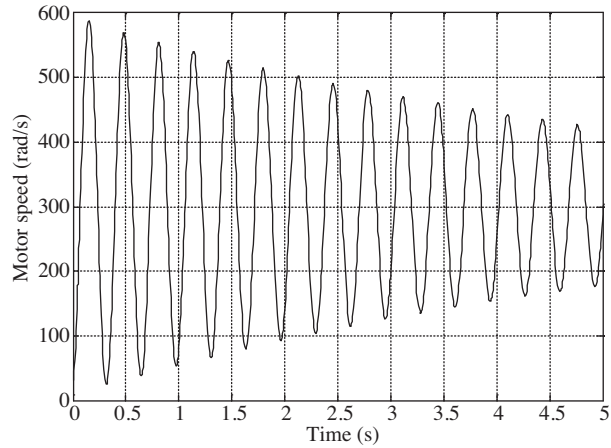


Figure 5. Motor speed for $K_I = 1.0s^{-1}$, $K_P = 0.3$, and $\tau^* = 0.046$ s: stable operation.

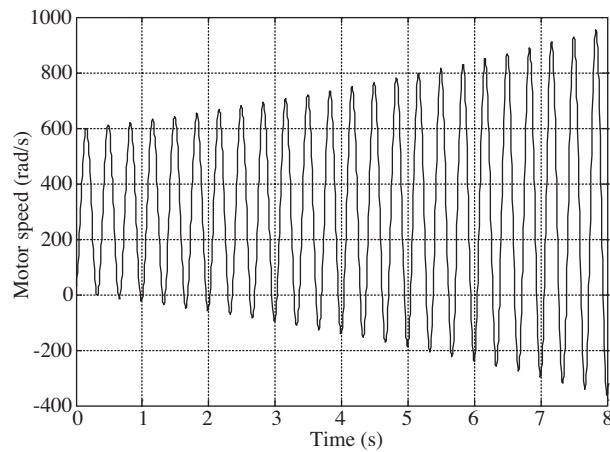


Figure 6. Motor speed for $K_I = 1.0s^{-1}$, $K_P = 0.3$, and $\tau^* = 0.048$ s: unstable operation.

5. Conclusion

This paper has analyzed the closed-loop stability of a DC motor speed control system that contains time delays in feedback and feedforward parts. A theoretical method has been used to compute the delay margin values for stability for various values of the PI controller gains. The accuracy of delay margin results was proven using time domain simulation capabilities of MATLAB/Simulink. It has been observed that the percentage error between the theoretical delay margin results and those determined by simulation are negligible. Therefore, the proposed method is an effective method to estimate delay margins of DC motor speed control systems. The following future work is planned: i) delay margins will be computed by using time domain approaches such as the Razumikhin theorem and Lyapunov–Krasovskii functionals, and results will be compared with those reported in this work; and ii) the delay interval problem will be analyzed.

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