

Inverse covariance principal component analysis for power system stability studies

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Abstract: The dominant poles (eigenvalues) of system matrices are used extensively in determining the power system stability analysis. The challenge is to find an accurate and efficient way of computing these dominant poles, especially for large power systems. Here we present a novel way for finding the system stability based on inverse covariance principal component analysis (ICPCA) to compute the eigenvalues of large system matrices. The efficacy of the proposed method is shown by numerical calculations over realistic power system data and we also prove the possibility of using ICPCA to determine the eigenvalues closest to any damping ratio and repeated eigenvalues. Our proposed method can also be applied for stability analysis of other engineering applications.

Key words: Eigenvalues, poles, power systems

1. Introduction

The eigenvalues of system matrices are used extensively in the analysis of power system stability to preserve the stable operation of the system disturbance [1,2]. It is also considered a very important problem for system planning and steady-state operation [3–6].

The application of the transfer function approach is a fundamental tool for system control and stability analysis [7,8]. Power system oscillations occur in frequency ranges from 0 up to less than 3 Hz [1]. The eigenvalues computed in this frequency band are of practical concern. The development of partial eigensolution methods on a transfer function basis is highly appreciated [9–11]. However, eigensolution problems involve high-dimensional state matrices [7,12] in the order of 20,000 or more state variables.

Model reduction is an important issue in power system dynamics. Reduced transfer function models of the power system are very effective for the design of power system stabilizers [13]. The reduced model can be built based on the calculation of the residues after finding the dominant poles [14,15].

In the case of eigenvalues that are not well separated, it is common in the literature to implement the shift-and-invert transformation combined with one of the eigenvalue solvers to enhance the convergence. The application of this kind of transformation requires solving a linear system at each iterative step. In this paper, a new methodology for determining the eigenvalues and the associated eigenvectors is proposed using an inverse covariance decomposition of the data matrix. The proposed algorithm is compared with the standard eigenvalue decomposition (EVD) based on a matrix decomposition strategy [13]. It outperforms the EVD methodology,

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not only in the central processing unit (CPU) time, but also in the identification of the real part of complex eigenvalues (unstable poles) necessary in obtaining full system transient stability.

This paper is organized as follows. In Section 2, we present the dynamical systems and eigenvalue problems, where the statement of the problem is included. In Section 3, the proposed algorithm is presented for computing dominant eigenvalues, computing dominant poles through inverse covariance principal component analysis (ICPCA) expansion, and computing ICPCA expansion. In Section 4, we present numerical examples and simulation results that prove the effectiveness of the proposed method, and, finally, the conclusions of the work are presented in Section 5.

2. Mathematical model and related eigenvalue problem

A linear dynamic system can be modeled in state-space form [4] with the variables L, N, H, and D, as follows:

$$x(k+1) = Lx(k) + Nu(k), \quad (1)$$

$$y(k) = H^T x(k) + Du(k).$$

Here, $\{L\} \in R^{n \times n}$, $N \in R^{n \times m}$, and $H^T \in R^{n \times p}$. Moreover, $x(k)$ is the state vector having R^n , L is the state transition matrix, N is the control matrix, and $u(k)$ is the control input vector of dimensions R^m . $y(k)$ is the output observation matrix of dimensions R^p and D has $R^{n \times m}$ dimensions. Eq. (1) can be reduced in the form of a transformation function, $T : H$, with dimensions $R^{p \times m}$:

$$T(s) = H^T (sI - L)^{-1} N + D. \quad (2)$$

The roots of Eq. (2) are called the poles of the system and the number of poles is equal to the number of the states. Moreover, the values of s satisfying Eq. (2) are called the eigenvalues of the state matrix L . If $L \in R^{n \times n}$ is a square matrix, it will have n eigenvalues.

Numerical methods for determining the eigenvalues of the square matrix have been developed to a high degree of accuracy. Eigenvalues of L are distinct when there are no 2 equal eigenvalues. Moreover, the location of the system poles in the complex plane determines the stability of the system matrix [15,16]. For a stable system, all of the poles have negative real parts, i.e. all of the eigenvalues lie in the left-hand half of the complex s -plane. For an unstable system, at least 1 pole has a positive real part. If the system has an eigenvalue with 0 real parts, the system response will be oscillatory and the system is called critically stable.

The transfer function $T(s)$ can be stated as follows:

$$T(s) = \frac{R_1}{s - \lambda_1} + \frac{R_2}{s - \lambda_2} + \dots + \frac{R_n}{s - \lambda_n}, \quad (3)$$

where R_n are the residues and λ_n are the poles of the transfer function $T(s)$. The dominant pole is the one associated with a residue with a large $\frac{R_n}{Re\lambda_n}$. The magnitude plot of $T(s)$ shows that peaks occur at frequencies close to the imaginary part of the dominant poles. Letting $k < n$, the following approximation is obtained:

$$T(s) = \frac{R_1}{s - \lambda_1} + \frac{R_2}{s - \lambda_2} + \dots + \frac{R_k}{s - \lambda_k}. \quad (4)$$

Hence, the problem can be formulated in a linear single-input single-output, time-invariant, and dynamical system in order to find $k \ll n$ dominant poles.

In small-signal stability analysis, one is interested in assessing whether the system, linearized around an operating point, is dynamically stable or unstable under small disturbances [17,18]. The states of the system in the time domain can be formulated as a linear combination of the terms $x_i e^{\lambda_i t}$. The presence of eigenvalues with the real part, i.e. $\lambda_i \in Re > 0$, will result in states that grow exponentially with time [19]. In this case, the operating point, or more generally the system, is called unstable. This paper mainly presents a robust algorithm for the small-signal stability problem.

Given the system matrix L , the nontrivial problem is to compute the finite eigenvalues λ_i (and corresponding eigenvectors), and solving for λ_i gives:

$$\det(L - \lambda I) = 0, \quad (5)$$

where the n solutions of Eq. (5) are the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of the $n \times n$ matrix L and $\lambda_i \neq \lambda_j, i \neq j$; they may be real or complex in the form of $\sigma \pm j\omega$. The complex eigenvalues are always in conjugate pairs if L is real. In a power system, the stability of the operating points ($\delta o, \omega o$) may be analyzed by studying the eigenvalues, where δo is the normal rotor angle in electrical radians and ωo is the synchronous angular velocity in electrical rad/s [1,17].

Usually, one is not only interested in the eigenvalues with positive real parts, but also in the eigenvalues closest to the imaginary axis. In small-signal stability analysis, these eigenvalues are characterized as having a small damping ratio [20].

Let $\lambda_i = \sigma_i \pm j\omega_i$ be the i th eigenvalue of the state matrix L . The real parts σ_i of the eigenvalues give the damping, and the imaginary part ω_i gives the frequency of oscillation. The relative damping ratio ζ is expressed as follows [21]:

$$\zeta = -\frac{\sigma}{\sqrt{(\sigma^2 + \omega^2)}}. \quad (6)$$

This ratio (ζ) determines the decaying property of the oscillation; it also gives an indication for the amount of overshoot and oscillation that the response undergoes. In power systems, the oscillatory modes of a damping ratio of less than 3% are considered to be critical [1,12]. The problem can be reformulated as given in the system matrix L , to compute the finite eigenvalues (and corresponding eigenvectors) with the smallest damping ratio. Moreover, the eigenvalues and their corresponding eigenvectors are usually calculated via standard EVD-based dimensionality reduction methods, which are not sufficiently suitable for the small stability analysis [14].

3. Algorithm

In this algorithm, we introduce a method to find the dominant poles, through finding the eigenvalues of the system. A scalar transfer function $T(s)$ could be made equal to its transpose as follows:

$$T(s) = H^T (s_k I - L)^{-1} N = N^T (s_k I - L)^{-T} H. \quad (7)$$

In matrix form, we have:

$$\begin{bmatrix} s_k I - L & -N \\ H^T & 0 \end{bmatrix} \begin{bmatrix} X(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} (s_k I - L)^{-T} & c \\ -N^T & o \end{bmatrix} \begin{bmatrix} V(s) \\ U(s) \end{bmatrix}, \quad (8)$$

where the vectors $X(s)$ and $V(s)$ represent the Laplace transformations for the vectors L and L^T , and $X(s)$ and $V(s)$ tend to converge to the right and left eigenvectors of L .

Algorithm 1. Algorithm for the calculation of dominant eigenvalues or poles.

Input: System matrices L, N, H , and an initial estimate of pole s_1 .

Output: Right and left eigenvectors x and v of dominant pole λ

- 1: Put $k = 1$
- 2: **While** no convergence **do**
- 3: Solve for x and u ,
$$\begin{bmatrix} s_k I - L & -N \\ H^T & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
- 4: Solve for v and u ,
$$\begin{bmatrix} s_k I - L & H \\ -N^T & 0 \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
- 5: The new pole estimate is $s_{k+1} = s_k + \frac{u}{v^T x}$
- 6: The convergence of the pole estimate is $|Lx - s_{k+1}x| < \epsilon$
- 7: **end while**

We use multiple moving shifts for s_k to find more than 1 dominant pole. By proposing different initial shifts, the computation of repeated poles can be avoided. In step 8 of Algorithm 2, the generalized eigenvalues λ_i of (Z, G) , i.e. $Zx = \lambda Gx$, can be computed while using the MATLAB command ‘Eig’. Practically, moving shifts will be decreased in number as soon as the convergence of the pole is achieved, keeping the left and right eigenvectors in the matrices V and X .

Algorithm 2. Algorithm for computing dominant poles through ICPCA expansion.

Input: Matrices L, N, H , and an initial estimate of pole s_1 , and the required $poles_r$.

Output: Right and left eigenvectors x and v , dominant pole $\lambda_i r_t l_t i$, where $i = 1 \dots poles_r$.

- 1: Put $poles = 0$, $Rt = Lt = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}$, $k = 1$
- 2: **while** not converged, i.e. $poles < poles_r$ **do**
- 3: Solve for x and, i.e. $x = X(s_k)$,
$$\begin{bmatrix} s_k I - L & -N \\ H^T & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
- 4: Solve for v and, i.e. $v = V(s_k)$
$$\begin{bmatrix} s_k I - L & H \\ -N^T & 0 \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
- 5: $X =$ ICPCA expansion (X, Rt, Lt, x)
- 6: $V =$ ICPCA expansion (V, Rt, Lt, v)
- 7: Compute $G = V^T X$ and $Z = V^T L X$
- 8: Compute eigentriplets of $(Z, G) = (\tilde{\lambda}_i, \tilde{x}_i, \tilde{v}_i)$
- 9: Compute eigentriplets of L as: $\hat{\lambda}_i = \tilde{\lambda}_i \hat{x}_i = X \tilde{x}_i \tilde{v}_i = V \tilde{v}_i i = 1, \dots k$

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10: Collect  $\hat{X} = [\hat{x}_1 \dots \hat{x}_k]$ ,  $\hat{V} = [\hat{v}_1 \dots \hat{v}_k]$ , and  $\hat{\Lambda} = [\hat{\lambda}_1 \dots \hat{\lambda}_k]$ .
11: if  $|A\hat{x}_1 - \hat{\lambda}_1\hat{x}_1| < \epsilon$ , then
12:   Function  $(Rt, Lt, \check{X}, \check{V}, \Lambda) = \text{squeeze}(\hat{x}_1, \hat{v}_1, Rt, Lt, \hat{X}_{2\dots}, \hat{V}_{2\dots}, \hat{\lambda}_1)$ 
13:    $Rt = [Rt, \bar{a}]$   $Lt = [Lt, \bar{v}]$   $\Lambda = [\Lambda, \bar{\lambda}]$ 
14:   if  $\text{imag } \lambda \neq 0$ , then
15:      $Rt = [Rt, \bar{x}]$   $Lt = [Lt, v]$   $\Lambda = [\Lambda, \bar{\lambda}]$ 
16:   end if
17:    $\check{X} = \check{V} = \begin{bmatrix} \vdots \\ \vdots \end{bmatrix}$ 
18:   for  $e = 1, \dots, k - 1$  do
19:      $\check{X} = \text{ICPCA expansion}(\check{X}, Rt, Lt, X_e)$ 
20:      $\check{V} = \text{ICPCA expansion}(\check{X}, Rt, Lt, V_e)$ 
21:   end for
22:    $\text{poles} = \text{poles} + 1$ 
23:    $\hat{\lambda}_1 = \hat{\lambda}_2$ 
24: end if
25:  $k = k + 1$ 
26: New pole estimate  $s_{k+1} = \hat{\lambda}_1$ 
27: end while
    
```

A drawback of Algorithm 2 is that the data determined in the current iteration will be discarded at the end of the iteration. However, the preserved data are included in the new pole estimates s_{k+1} . Subspaces V and X include the information about the other dominant eigentriplets [14]. Hence, dominant pole algorithms can be generalized into Algorithm 2. Now, for the shift $s1$, the approximated right and left eigenvector $x1$ and $v1$ will be included in X and V . Next, X and V are expanded orthogonally for the next iteration using Cholesky ICPCA.

Algorithm 3. Algorithm for computing ICPCA expansion.

Input: X, Rt, Lt, x

Output: X and $x_{k+1} = \prod_{e=1}^{\text{poles}} (I - Rt_e Lt_e^T) / (Lt_e^T Rt_e) \cdot x$

```

1: Function  $x = \text{ICPCA}(X, x)$ 
2:  $[L, D] = \text{ldl}(X, \text{lower})$ 
3:  $\Sigma^{-1} = L^T D^{-1} L$ 
4:  $(X^{-1}, D) = \text{eig}(\Sigma^{-1})$ 
5:  $X = \begin{bmatrix} X, x \\ |x| \end{bmatrix}$ 
    
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Table. The 10 most dominant eigenvalues with corresponding damping and frequency for the New England 39-bus system.

No.	Eigenvalue	Magnitude	Equiv. damping	Equiv. freq. (rad/s)
1.	$-4.67e-001 \pm 8.96e+000i$	$8.98e+000$	$-8.04e-001$	$5.46e+002$
2.	$-2.97e-001 \pm 6.96e+000i$	$6.96e+000$	$-7.69e-001$	$5.05e+002$
3.	$-2.49e-001 \pm 3.69e+000i$	$3.69e+000$	$-6.24e-001$	$4.19e+002$
4.	$-9.79e-001 \pm 1.14e+000i$	$1.50e+000$	$-1.76e-001$	$4.63e+002$
5.	$-1.12e-001 \pm 7.10e+000i$	$7.10e+000$	$-7.77e-001$	$5.04e+002$
6.	$-3.70e-001 \pm 8.61e+000i$	$8.62e+000$	$-8.00e-001$	$5.38e+002$
7.	$-2.83e-001 \pm 6.28e+000i$	$6.29e+000$	$-7.51e-001$	$4.90e+002$
8.	$-3.01e-001 \pm 5.79e+000i$	$5.80e+000$	$-7.35e-001$	$4.78e+002$
9.	$-2.82e-001 \pm 7.54e+000i$	$7.54e+000$	$-7.82e-001$	$5.16e+002$
10	$-4.12e-001 \pm 8.78e+000i$	$8.79e+000$	$-8.02e-001$	$5.42e+002$

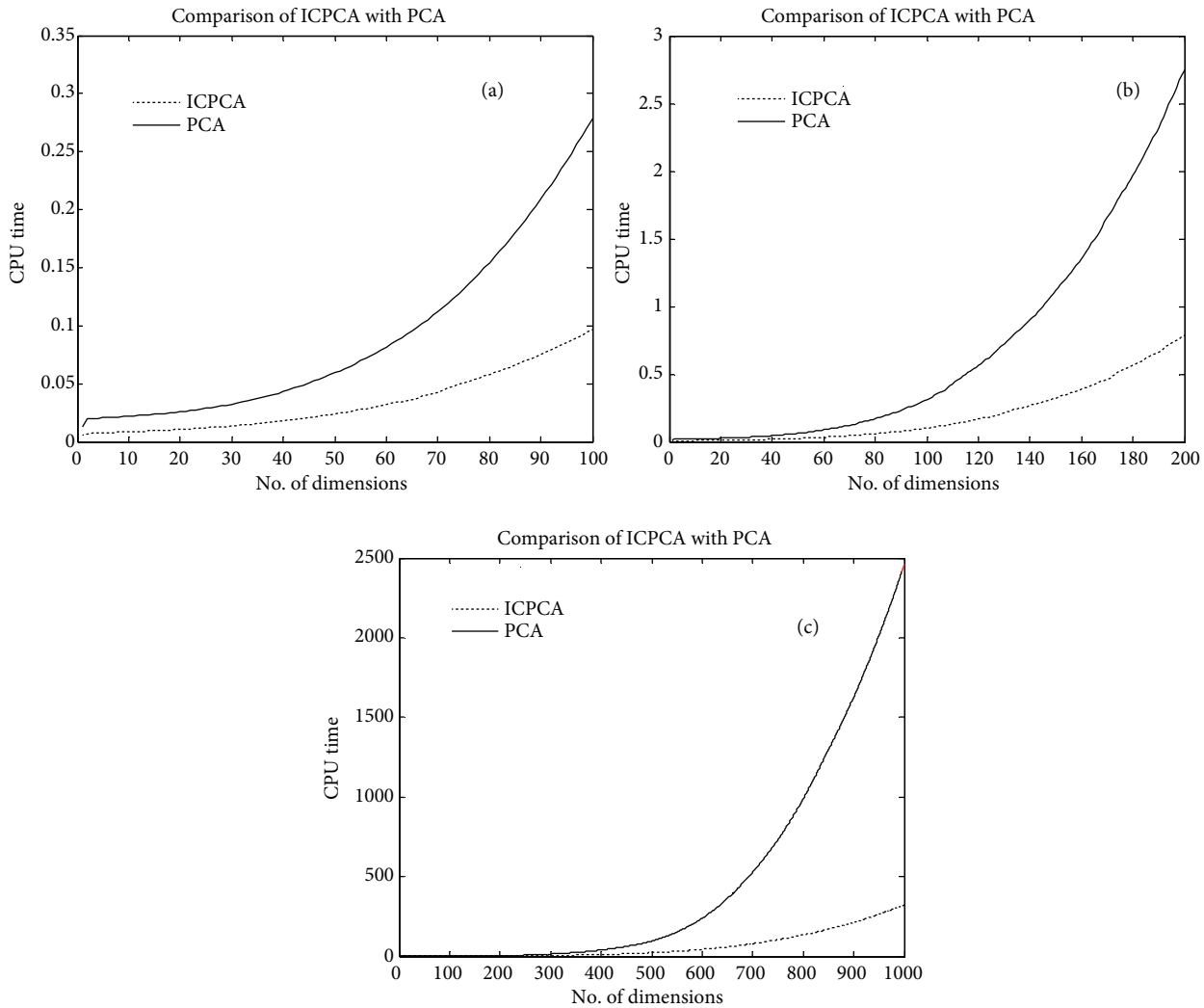


Figure 1. a), b), and c): CPU time for the PCA using the EVD and ICPCA methods as a function of different data dimensions.

4. Numerical examples

We propose using Cholesky inverse covariance $\Sigma^{-1} = L^T D^{-1} L$ to have an estimate of the action-dependent inverse covariance matrix. We apply the Cholesky decomposition only to the inverse covariance matrix sequences. The most dominating eigenvector will be measured first. In a similar way, all of the $h - 1$ basis vectors will be measured in descending order with respect to their dominance.

In this paper, the performance of the ICPCA in terms of the processing time is compared with EVD-based PCA. The results for the CPU time are obtained versus the increase of the data dimensionality. Here, we present the performance measure of a 3000-dimensional matrix generated using a Gaussian random number generator (using MATLAB's function 'randn'). The CPU time required in computing the h leading eigenvectors is a function of the data dimensionality.

The data from 1 to 3000 are given in Figures 1a, 1b, and 1c. It is clear that the CPU time for ICPCA

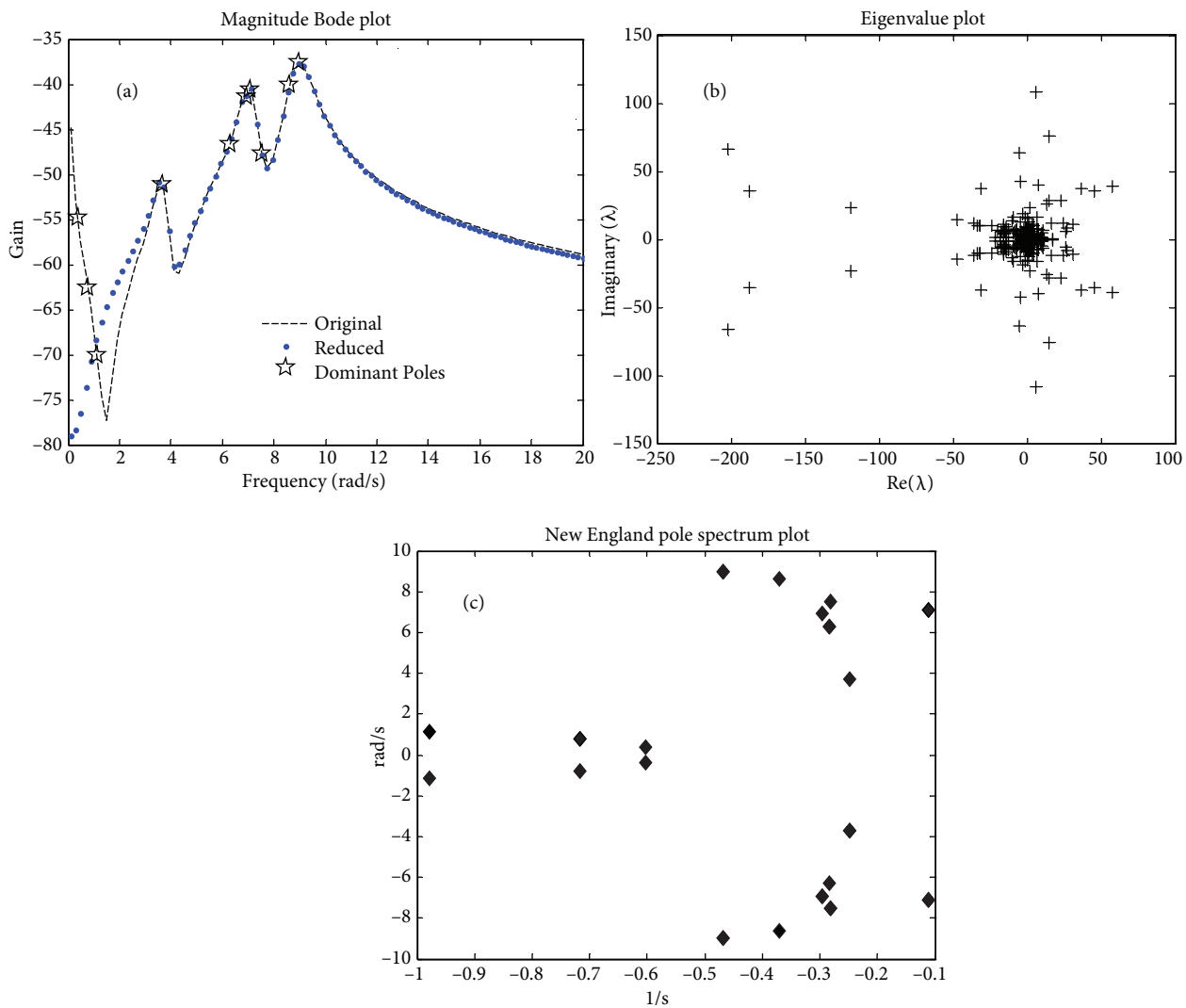


Figure 2. a) Magnitude plot of the proposed model, b) eigenvalue plot showing left and right eigenvalues, and c) pole spectrum of the New England data.

follows a linear behavior compared to that of EVD/PCA. This confirms the effectiveness of the proposed method for high-dimensional applications. The New England 39-bus system with 10 generators is used to test the continuation of invariant subspaces, where 9 state variables are considered for each generator with 2 algebraic equations for each bus. In the experiments, a convergence tolerance of $\epsilon = 10^{-10}$ is used. The tests are carried out in MATLAB 7.0 on an AMD Turion-X2 dual core 64-bit 2.10-GHz PC with a 4.0-GB RAM.

The proposed method is applied to the transfer functions of the New England data to find the number of dominant poles. Starting with one shift, after having the first pole converged, the next most dominant approximate pole is the new shift afterwards. Results are given in the Table for the 10 most dominant eigenvalues with corresponding damping and frequency for the New England 39-bus system. The system Bode plot is depicted on Figure 2a, where it is observed that the proposed algorithm successfully computes the complex eigenvalues of the system under study. Figure 2c shows the pole spectrum of the New England data. The ICPCA method enables the computational cost to come down to an optimal level, where the growth of the dominant poles does not affect the system's dimensionality reduction, without any human interaction during the process.

5. Conclusions

The proposed algorithm presented in this paper is relatively fast in computing the eigenvalues and corresponding eigenvectors. When compared to existing methods, it has several advantages. First, it is more robust as it uses the inverse covariance factorization. Second, it converges at a lower number of iterations. Third, the possibility of missing a dominant pole is lower. Moreover, it is unsupervised, i.e. many dominant poles can be computed with one shift, without human intervention. The proposed ICPCA method could be applied to any other related engineering problems.

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