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Approximations of higher-order fractional differentiators and integrators using indirect discretization

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Abstract: This paper describes new approximations of fractional order integrators (FOIs) and fractional order differentiators (FODs) by using a continued fraction expansion-based indirect discretization scheme. Different tenth-order fractional blocks have been derived by applying three different s-to-z transforms described earlier by Al-Alaoui, namely new two-segment, four-segment, and new optimized four-segment operators. A new addition has been done in the new optimized four-segment operator by modifying it by the zero reflection method. All proposed half ($s^{\pm 1/2}$) and one-fourth ($s^{\pm 1/4}$) differentiator and integrator models fulfill the stability criterion. The tenth-order fractional differ-integrators ($s^{\pm\alpha}$) based on the modified new optimized four-segment rule show tremendously improved results with relative magnitude errors (dB) of $\leq$ –15 dB for $\alpha = 1/2$ and $\leq$ –20 dB for $\alpha = 1/4$ in the full range of Nyquist frequency so these have been further analyzed. The main contribution of this paper lies in the reduction of these tenth-order blocks into four new fifth-order blocks of half and one-fourth order models of FODs and FOIs. The analyses of magnitude and phase responses show that the proposed new fifth-order half and one-fourth differ-integrators closely approximate their ideal counterparts and outperform the existing ones.

Key words: Fractional order differentiators, fractional order integrators, half and one-fourth order differentiators and integrators, continued fraction expansion

1. Introduction

Fractional calculus is an old but still unusual topic that has drawn the interest of many researchers in all domains of science and engineering including electrical networks, automated control, image processing, radio engineering, and signal processing. Newton–Cotes integration rules [1–5] have been widely experimented on by Al-Alaoui using linear interpolation techniques to obtain differentiators and integrators with improved performances. Later these interpolated integer order models were adopted by different researchers for obtaining their fractional order counterparts by using different discretization schemes. Recently in [6,7] new improved differentiation and integration rules were developed by pole inversion and optimization (simulated annealing) techniques. The design procedure, by which a fractional operator in the Laplace domain ($s^{\pm\alpha}$) is discretized for its implementation in the time domain, is mainly classified in two different streams, namely direct discretization and indirect discretization. The direct discretization approach, as the name suggests, finds a generating function and expands it by direct application of any expansion technique. On the other hand, the indirect discretization technique discretizes an efficiently fitted s-domain rational approximation by using any existing s-to-z transformation.

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In the literature, several existing techniques vastly used for direct discretization are power series expansion (PSE) [8,9], continued fraction expansion (CFE) [10–18], Taylor series expansion (TSE) [19–23], and numerical integration formulae. The resulting rational approximations can be cut short to any arbitrary 'N' number of terms, which gives the order of resultant fractional order differentiators (FODs) and fractional order integrators (FOIs) after the truncation process. Recently in [7,8] some improved rational approximations were obtained by applying PSE truncation and signal modeling technique on Simpson integrator. The proposed FODs showed that the PSE-modeling technique performs better as compared to PSE-truncation. In [10–12], the CFE technique was used for direct discretization of linearly interpolated generating functions, whereas a combination of three different techniques, namely s-to-z transform, PSE, and signal modeling, was used on Pade, Prony, and Shanks' noniterative methods in [13] for obtaining the fractional order integrals and derivatives. The work of Chen and fellow researchers [14–17] is of noticeable significance in the implementation of different approximations. Major advancements in discretization schemes have opened new trends of practicing them for higher-order [18–22] approximations for more improved and accurate results. The study of [23] gives the effects of variation of fractional order 'α' for derivation and the integration part of fractional controllers on their overall performance.

To date, most of the work on higher-order approximations has been centered on the direct discretization approach and the scope of the indirect discretization scheme [24–28] in this field is not fully explored. This paper is an attempt to refocus on higher-order approximations by using indirect discretization, which is a more effective yet simpler technique as compared to direct discretization. In this paper the authors have adopted the latest improved versions of Al-Alaoui's differentiation rules, namely new two-segment [7], four-segment [7], and new optimized four-segment rules [7], for their present work for developing fractional order differ-integrators. The new optimized four-segment operator has been modified here by zero reflection method [29] for deriving new improved FODs and FOIs. The authors have concluded that any operator, when used as an s-to-z transform for discretizing rational approximations for finding stable resultant FODs and FOIs, should be stabilized first if it is unstable. All the proposed tenth-order blocks based on the modified new optimized four segment operator have been observed to present tremendous results with very smaller relative magnitude errors (dB) (RMEs) of ≤ -15 dB for half and ≤ -20 dB for one-fourth order differ-integrators. These models have been reduced from the tenth to the fifth order by canceling closely overlapping complex conjugate pairs of poles and zeros using pole-zero cancellation. The reason is that for a suitable fitting of transfer functions in the z-domain, complex poles/zeros are strictly undesirable. Later, the fifth-order half differentiator has been tested by pole/zero displacement for finding optimal results for minimum magnitude errors. The responses of all the proposed fifth-order FODs and FOIs have also surpassed the existing ones of the same order and have shown improved performances.

This paper is organized as follows: Section 2 deals with a brief introduction of the formulation of the indirect discretization technique that is used here and also describes all the three second-order digital differentiators. In Section 3, first the new optimized four-segment operator is slightly modified by the zero reflection method and then the rational approximations formulated in Section 2 are discretized using the differentiators mentioned in Sections 2 and 3. Transfer functions of tenth-order half and one-fourth differentiators and integrators are given and their responses are compared in the same section. In Section 4, some tenth-order half and one-fourth differentiators and integrators have been reduced to the fifth order. Simulation plots for comparisons of magnitude responses, phase responses, and relative magnitude errors of all the proposed fifth-order blocks of FODs and FOIs with existing models have been presented and their performances are discussed in Section 5. Section 6 concludes the paper.

2. Indirect discretization based on different operators

2.1. Rational approximations for FODs and FOIs

In [30], Lagrange's method was used for finding the CFE of binomial functions by using generalized solutions of the Riccati differential equation. In this paper the authors have adopted these solutions by truncating them up for deriving rational approximations for fractional order differentiation and integration operators. The formulae for different rational approximations of FODs and FOIs are as follows.

$$(s)^{\pm\alpha} = \frac{1}{1-}\,\frac{\alpha s}{(1+s)-}\,\frac{(1-\alpha)s}{2-}\,\frac{(1+\alpha)s}{3(1+s)-}\,\frac{(2-\alpha)s}{2-}\,\frac{(2+\alpha)s}{5(1+s)-}\,\frac{(3-\alpha)s}{2-}\,\frac{(3+\alpha)s}{7(1+s)-}\,\frac{(4-\alpha)s}{2-}\,\frac{(4+\alpha)s}{9(1+s)} \tag{1}$$

$$(s)^{\pm\alpha} = \frac{1}{1-}\,\frac{\alpha s}{1+}\,\frac{(1+\alpha)s}{2+}\,\frac{(1-\alpha)s}{3+}\,\frac{(2+\alpha)s}{2+}\,\frac{(2-\alpha)s}{5+}\,\frac{(3+\alpha)s}{2+}\,\frac{(3-\alpha)s}{7+}\,\frac{(4+\alpha)s}{2+}\,\frac{(4-\alpha)s}{9+} \tag{2}$$

$$(s)^{\pm\alpha} = \frac{1}{1-}\,\frac{\alpha s}{1+(1+\alpha)s-}\,\frac{(1+\alpha)s(1+s)}{2+(3+\alpha)s-}\,\frac{2(2+\alpha)s(1+s)}{3+(5+\alpha)s-}\,\frac{3(3+\alpha)s(1+s)}{4+(9+\alpha)s-}\,\frac{4(4+\alpha)s(1+s)}{5+(9+\alpha)s-}\,\frac{5(5+\alpha)s(1+s)}{6+(11+\alpha)s-}\,\frac{6(6+\alpha)s(1+s)}{7+(13+\alpha)s-} \tag{3}$$

$$(s)^{\pm\alpha} = \frac{1}{1-}\,\frac{\alpha s}{1+\alpha s+}\,\frac{(1-\alpha)s}{2-(1-\alpha)s+}\,\frac{2(2-\alpha)s}{3-(2-\alpha)s+}\,\frac{3(3-\alpha)s}{4-(3-\alpha)s+}\,\frac{4(4-\alpha)s}{5-(4-\alpha)s+}\,\frac{5(5-\alpha)s}{6-(5-\alpha)s+}\,\frac{6(6-\alpha)s}{7-(6-\alpha)s+} \tag{4}$$

CFE-based indirect discretization techniques have been previously used in [24,25] for deriving half and one-fourth order differentiators and integrators based on first-order operators only, namely the Al-Alaoui operator and bilinear operator. The authors of this paper have used higher-order s-to-z transforms [7] for finding more efficient and improved approximations. Among the above-mentioned rational approximations, the one described by Eq. (3) only fulfils the condition of convergence of CFE by uniformly converging in the finite s-plane. Pole-zero recognition is easier using this convergence when this is later converted into discretized form by s-to-z transformations. After simplification it can be written in generalized form as follows.

$$s^{\pm\alpha} = \frac{B_0 s^5 + B_1 s^4 + B_2 s^3 + B_3 s^2 + B_4 s + B_5}{A_0 s^5 + A_1 s^4 + A_2 s^3 + A_3 s^2 + A_4 s + A_5} \tag{5}$$

Coefficients for numerator and denominator polynomials used in all the rational approximations for half and one-fourth order differ-integrators are listed in Table 1.

Table 1. Coefficients for numerator and denominator polynomials in the rational approximations.

Coefficients for numerator and denominator polynomials in the rational approximations				
	$\alpha = 1/2$	$\alpha = -1/2$	$\alpha = 1/4$	$\alpha = -1/4$
B0 = A5	–324.8438	–29.5313	–203.9502	–64.2920
B1 = A4	–4.8727e+003	–1.6242e+003	–3.8751e+003	–2.2502e+003
B2 = A3	–1.3643e+004	–9.7453e+003	–1.2917e+004	–1.0930e+004
B3 = A2	–9.7453e+003	–1.3643e+004	–1.0930e+004	–1.2917e+004
B4 = A1	–1.6242e+003	–4.8727e+003	–2.2502e+003	–3.8751e+003
B5 = A0	–29.5313	–324.8438	–64.2920	–203.9502

2.2. Digital differentiators adopted for rational approximations

Three different differentiation rules given in [7] are considered for s-to-z mapping of rational approximations of half and one-fourth differentiators and integrators. The differentiator operators adopted for discretization are given below with the value of sampling period T as 1 s.

- Transfer function of the new two-segment operator [7] is:

$$H_{two\_seg}(z) = \left(\frac{8.8438z^2 - 8.8438}{7z^2 + 8.2543z + 2.4333}\right). \tag{6}$$

- Transfer function of the four-segment operator [7] is:

$$H_{four\_seg}(z) = \left(\frac{21z^2 - 21}{17.4509z^2 + 19.2696z + 5.3195}\right). \tag{7}$$

- Transfer function of the new optimized four-segment operator [7] is:

$$H_{optim\_four}(z) = \left(\frac{238z^2 + 9.401z - 241.3227}{217z^2 + 259.9009z + 46.8232}\right). \tag{8}$$

3. New modified s-to-z transformation and discretized tenth-order blocks of FODs and FOIs

3.1. New modified optimized four segment differentiator

The differentiator operators mentioned above have been checked for the stability criterion before fitting them in the continuous time domain, according to which these differentiators should have all their poles and zeros inside the unit circle because a stable s-domain fitted transfer function can only ensure a stable and minimum phase rational transfer function in discretized form. Rational approximations derived in Section 1 have been discretized by using the above-mentioned second-order differentiator operators as s-to-z transformations, by substituting the transfer function of differentiator operators in place of ‘s’ in Eq. (5) one by one. As an example, the derivation process is explained here for the new two-segment operator.

$$s^{\pm\alpha} = \frac{\left(\begin{array}{l} B_0\left[\frac{8.8438z^2-8.8438}{7z^2+8.2543z+2.4333}\right]^5 + B_1\left[\frac{8.8438z^2-8.8438}{7z^2+8.2543z+2.4333}\right]^4 + B_2\left[\frac{8.8438z^2-8.8438}{7z^2+8.2543z+2.4333}\right]^3 \\ +B_3\left[\frac{8.8438z^2-8.8438}{7z^2+8.2543z+2.4333}\right]^2 + B_4\left[\frac{8.8438z^2-8.8438}{7z^2+8.2543z+2.4333}\right] + B_5 \end{array}\right)}{\left(\begin{array}{l} A_0\left[\frac{8.8438z^2-8.8438}{7z^2+8.2543z+2.4333}\right]^5 + A_1\left[\frac{8.8438z^2-8.8438}{7z^2+8.2543z+2.4333}\right]^4 + A_2\left[\frac{8.8438z^2-8.8438}{7z^2+8.2543z+2.4333}\right]^3 \\ +A_3\left[\frac{8.8438z^2-8.8438}{7z^2+8.2543z+2.4333}\right]^2 + A_4\left[\frac{8.8438z^2-8.8438}{7z^2+8.2543z+2.4333}\right] + A_5 \end{array}\right)} \tag{9}$$

Eq. (9) is simplified after substituting values of coefficients in the numerator and denominator from Table 1 for any value of fractional order. The resultant approximations for $\alpha = 1/2$ is:

$$H_{twoseg\_1/2\_diff}(z) = \left(\frac{\left(\begin{array}{c} 1.124(z-0.9796)(z-0.8198)(z-0.5191)(z+0.9869)(z+0.9986) \\ (z+0.9587)(z+0.9018)(z+0.7914)(z+0.272)(z-0.1269) \end{array}\right)}{\left(\begin{array}{c} (z-0.9191)(z-0.6849)(z-0.3299)(z+0.935)(z+0.9754) \\ (z+0.9944)(z+0.8554)(z+0.7043)(z+0.4448)(z+0.07783) \end{array}\right)}\right). \tag{10}$$

The resultant transfer functions for half ($s^{\pm 1/2}$) and one-fourth ($s^{\pm 1/4}$) differ-integrators obtained by different operators were observed to be tenth-order blocks. These blocks were again checked for the criterion for a better fit in continuous frequency response. In this analysis, only the discretized transfer functions based on the new optimized four-segment rule were found to be unstable. Instead of applying the pole reflection method to these unstable tenth-order resultant fractional operators, we have done some empirical modification in their base operator itself by applying the zero reflection [29] method on a nonunity zero (1.027) before deriving its rational approximations. The new optimized four-segment operator after modification has been named NOFSAM and the original operator before modification is named NOFSBM [7].

$$H_{NOFSBM}(z) = \left(\frac{238z^2 + 9.401z - 241.3227}{217z^2 + 259.9009z + 46.8232}\right) = \left(\frac{1.0968(z-0.9874)(z+1.027)}{(z+0.9768)(z+0.2209)}\right) \tag{11}$$

$$H_{NOFSAM}(z) = \left(\frac{1.0968}{1.027}\right)\left(\frac{(z-0.9874)\left(z+\frac{1}{1.027}\right)}{(z+0.9768)(z+0.2209)}\right) = \left(\frac{1.1264z^2 - 0.01542z - 1.083}{z^2 + 1.198z + 0.2158}\right) \tag{12}$$

Mathematical models of the all the discretized half and one-fourth differentiators and integrators are summarized in Tables 2 and 3 respectively with the help of values of gain, poles, and zeros.

Table 2. Details of gain, poles, and zeros for mathematical model of tenth-order half and one-fourth differentiators based on different operators.

Symbols	New two segment operator		Four segment operator		NOFSBM operator		NOFSAM operator	
	$\alpha = 1/2$	$\alpha = 1/4$	$\alpha = 1/2$	$\alpha = 1/4$	$\alpha = 1/2$	$\alpha = 1/4$	$\alpha = 1/2$	$\alpha = 1/4$
Gain	1.124	1.0602	1.097	1.0474	1.0473	1.0234	1.0611	1.0301
Zeros of the mathematical models of half and one-fourth order differentiators								
Z1	0.9796	0.9669	0.9796	0.9669	0.9656	0.952	0.9658	0.9522
Z2	0.8198	0.7883	0.8206	0.7894	0.7988	0.7668	0.7984	0.7663
Z3	0.5191	0.4735	0.5252	0.4808	0.506	0.464	0.5035	0.4612
Z4	–0.9869	–0.9535	–0.9837	–0.9806	–0.9866	–0.9847	–0.9771	–0.9733
Z5	–0.9986	–0.9843	–0.9983	–0.9972	–1.004	–1.002	0.162	0.1224
Z6	–0.9587	–0.9978	–0.9495	–0.9433	–1.016	–1.015	0.1143	–0.1369
Z7	–0.9018	–0.8918	–0.883	–0.8717	–1.023	–1.023	(–0.9730 + 0.0165i)	(–0.9675 + 0.1317i)
Z8	–0.7914	–0.7735	–0.7611	–0.7419	–1.027	–1.026	(–0.9730 –0.0165i)	(–0.9675 –0.1317i)
Z9	–0.272	–0.3142	–0.2394	–0.2804	0.167	0.1275	–0.9760	– 0.9903
Z10	0.1269	0.07661	0.1454	0.09692	0.111	–0.1341	–0.9750	–0.9587
Poles of the mathematical models of half and one-fourth order differentiators								
P1	0.9191	0.9366	0.9192	0.9367	0.9015	0.9199	0.9015	–0.9731
P2	0.6849	0.7209	0.6875	0.7229	0.664	0.6995	0.6628	0.92
P3	0.3299	0.379	0.3413	0.389	0.336	0.3791	0.332	0.6985
P4	–0.935	–0.9417	–0.9214	–0.9293	–0.9796	–0.9812	–0.9771	0.3755
P5	–0.9754	–0.9786	–0.9696	–0.9736	–0.9953	–0.9975	–0.1935	–0.1765
P6	–0.9944	–0.9956	–0.993	–0.9946	–1.011	–1.012	0.008357	0.04558
P7	–0.8554	–0.8689	–0.8309	–0.8458	–1.02	–1.021	–0.9906	–0.9951
P8	–0.7043	–0.7309	–0.6693	–0.697	–1.025	–1.026	–0.9574	–0.9589
P9	–0.4448	–0.3998	–0.4081	–0.3639	–0.1924	–0.1749	(–0.9755 + 0.0173i)	–0.9816
P10	–0.0778	–0.0255	–0.0518	–0.0014	0.01322	0.05062	(–0.9755 –0.0173i)	–0.9674

It was observed that a minor modification in the operator before substituting it in rational approximations for discretization has resulted in stable mathematical models of tenth-order FODs and FOIs with all their poles and zeros inside the unit circle. The reason for this behavior is that any minute shifting in position of poles or zeros by delta value ($\pm\Delta$) in the original operator always leads to entirely different resultant fractional models. Rational approximations for fractional operator $s^{\pm\alpha}$ cumulatively converge towards a discretized (z-domain) model form by CFE, which repeatedly tries to correlate between the two domains.

Table 3. Details of gain, poles, and zeros for mathematical model of tenth-order half and one-fourth integrators based on different operators.

Symbols	New two segment operator		Four segment operator		NOFSBM operator		NOFSAM operator	
	$\alpha = -1/2$	$\alpha = -1/4$	$\alpha = -1/2$	$\alpha = -1/4$	$\alpha = -1/2$	$\alpha = -1/4$	$\alpha = -1/2$	$\alpha = -1/4$
Gain	0.88967	0.94322	0.91159	0.95477	0.95486	0.97717	0.94239	0.97077
Zeros of the mathematical models of half and one-fourth order integrators								
Z1	0.9191	0.9366	0.9192	0.9367	0.9015	0.9199	0.9015	–0.9731
Z2	0.6849	0.7209	0.6875	0.7229	0.664	0.6995	0.6628	0.92
Z3	0.3299	0.379	0.3413	0.389	0.336	0.3791	0.332	0.6985
Z4	–0.935	–0.9417	–0.9214	–0.9293	–0.9796	–0.9812	–0.9771	0.3755
Z5	–0.9754	–0.9786	–0.9696	–0.9736	–0.9953	–0.9975	–0.1935	–0.1765
Z6	–0.9944	–0.9956	–0.993	–0.9946	–1.011	–1.012	0.008357	0.04558
Z7	–0.8554	–0.8689	–0.8309	–0.8458	–1.02	–1.021	–0.9906	–0.9951
Z8	–0.7043	–0.7309	–0.6693	–0.697	–1.025	–1.026	–0.9574	–0.9589
Z9	–0.4448	–0.3998	–0.4081	–0.3639	–0.1924	–0.1749	(–0.9755 + 0.0173i)	–0.9816
Z10	–0.07783	–0.0255	–0.0518	–0.0014	0.01322	0.05062	(–0.9755 – 0.0173i)	–0.9674
Poles of the mathematical models of half and one-fourth order integrators								
P1	0.9796	0.9669	0.9796	0.9669	0.9656	0.952	0.9658	0.9522
P2	0.8198	0.7883	0.8206	0.7894	0.7988	0.7668	0.7984	0.7663
P3	0.5191	0.4735	0.5252	0.4808	0.506	0.464	0.5035	0.4612
P4	–0.9869	–0.9535	–0.9837	–0.9806	–0.9866	–0.9847	–0.9771	–0.9733
P5	–0.9986	–0.9843	–0.9983	–0.9972	–1.004	–1.002	0.162	0.1224
P6	–0.9587	–0.9978	–0.9495	–0.9433	–1.016	–1.015	–0.1143	–0.1369
P7	–0.9018	–0.8918	–0.883	–0.8717	–1.023	–1.023	(–0.9730 + 0.0165i)	–0.9888
P8	–0.7914	–0.7735	–0.7611	–0.7419	–1.027	–1.026	(–0.9730 – 0.0165i)	–0.9642
P9	–0.272	–0.3142	–0.2394	–0.2804	0.167	0.1275	–0.9760	–0.9903
P10	0.1269	0.07661	0.1454	0.09692	–0.111	–0.1341	–0.9750	–0.9587

3.2. Performance discussion of FODs and FOIs based on new two-segment, four-segment, NOFSBM, and NOFSAM operators

The RME value is the decibel (dB) equivalent of the relative magnitude error and is given by

$$RME = 20\log_{10}\left|\frac{\varepsilon}{k}\right|, \tag{13}$$

where $\varepsilon = |k - k_{approx}|$ is the difference between the magnitude of the ideal value of the operator (k) and that of the operator we are approximating (k_{approx}).

Figure 1 shows the responses of RME values of half and one-fourth differentiators, whereas Figure 2 presents the same for their integrator counterparts. Tenth-order blocks based on NOFSAM have improved

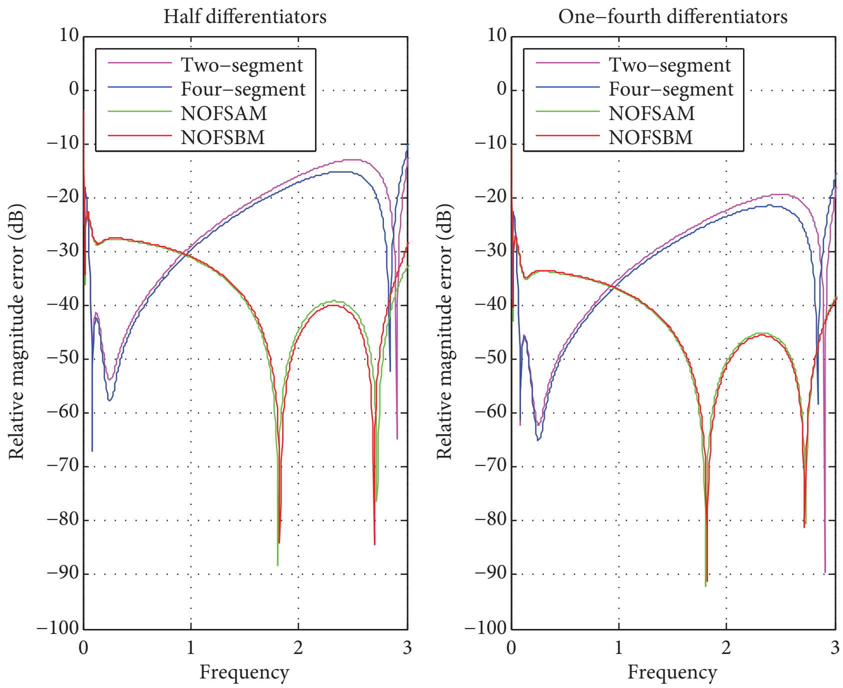

Figure 1. Comparison of relative magnitude errors (dB) of new tenth-order blocks of half and one-fourth differentiators based on different operators.

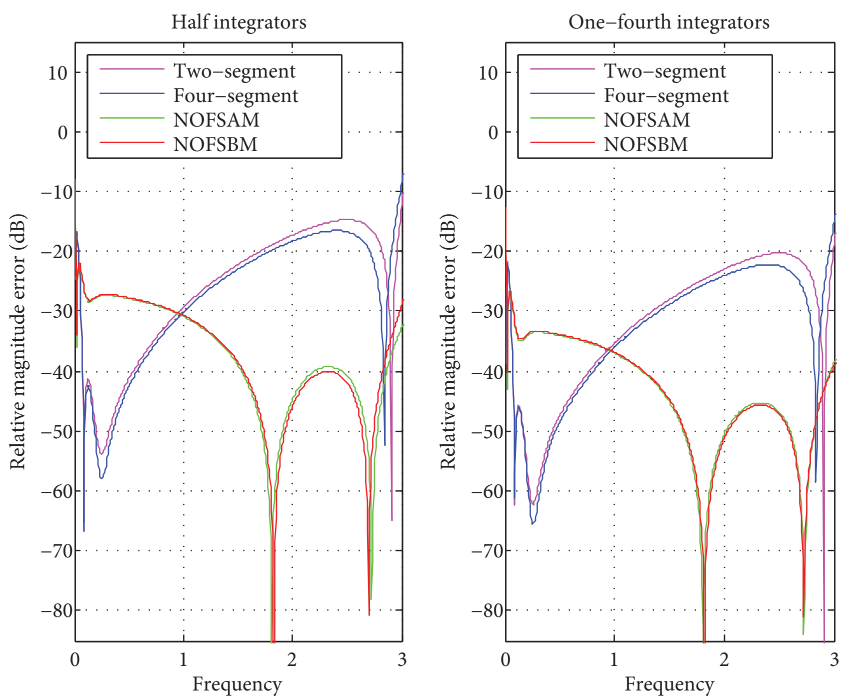

Figure 2. Comparison of relative magnitude errors (dB) of new tenth-order blocks of half and one-fourth integrators based on different operators.

responses of RME as compared to fractional operators based on NOFSBM, mainly in the high frequency regions. Half and one-fourth differentiator models based on new two-segment [7], four-segment [7], NOFSBM [7], and NOFSAM [7] operators have been observed to have perceptible performances. It can be clearly observed that NOFSAM based discretizations for FODs present negligible RME values of ≤ -15 dB for $\alpha = 1/2$ that decrease to ≤ -20 dB for $\alpha = 1/4$ for both the half and one-fourth orders for the complete Nyquist frequency range and outperform all the other proposed as well as the existing tenth-order differentiators. Among all the proposed half and one-fourth order integrators, those obtained by NOFSAM-based discretizations surpass the others by closely following the ideal response, with RME ≤ -30 dB in the range of $0.2 \leq \omega \leq 1.5$ rad and ≤ -40 dB in the range $1.5 \leq \omega \leq 3$ rad for the complete Nyquist range, but these have a single disadvantage of large order, i.e. 10.

4. Proposed fifth-order one-half and one-fourth differentiators and integrators

In spite of the efficient performance of tenth-order blocks of FODs and FOIs based on NOFSAM with negligible errors, these have been further analyzed for removing their only disadvantage of large order (N = 10), which has camouflaged their performance (see Tables 2 and 3). In this section, four new fifth-order blocks of fractional operators have been proposed by reducing their higher order by pole-zero cancellation technique. Application of this technique on these four tenth-order blocks is clearly shown in Eqs. (14) through (17).

$$H_{NOFSAM\_1/2\_diff}(z) = \left(\frac{\left(\begin{array}{c} (1.0611)(z-0.9658)(z-0.7984)(z-0.5035)(\cancel{z+0.9771})(z-0.162) \\ (z+0.1143)(z^2+\cancel{1.946z}+0.947)(z^2+\cancel{1.951z}+0.9516) \end{array} \right)}{\left(\begin{array}{c} (z-0.9015)(z-0.6628)(z-0.332)(\cancel{z+0.9771})(z+0.1935) \\ (z-0.008357)(z^2+1.948z+\cancel{0.9484})(z^2+\cancel{1.951z}+0.9519) \end{array} \right)} \right) \quad (14)$$

$$H_{NOFSAM\_1/2\_int}(z) = \left(\frac{\left(\begin{array}{c} (0.94239)(z-0.9015)(z-0.6628)(z-0.332)(\cancel{z+0.9771})(z+0.1935) \\ (z-0.008357)(z^2+\cancel{1.948z}+0.9484)(z^2+\cancel{1.951z}+0.9519) \end{array} \right)}{\left(\begin{array}{c} (z-0.9658)(z-0.7984)(z-0.5035)(\cancel{z+0.9771})(z-0.162) \\ (z+0.1143)(z^2+\cancel{1.946z}+0.947)(z^2+\cancel{1.951z}+0.9516) \end{array} \right)} \right) \quad (15)$$

$$H_{NOFSAM\_1/4\_diff}(z) = \left(\frac{\left(\begin{array}{c} (1.0301)(z-0.9522)(z-0.7663)(z-0.4612)(\cancel{z+0.9733})(z-0.1224) \\ (z+0.1369)(z^2+\cancel{1.953z}+0.9534)(z^2+\cancel{1.949z}+0.9494) \end{array} \right)}{\left(\begin{array}{c} (\cancel{z+0.9731})(z-0.92)(z-0.6985)(z-0.3755)(z+0.1765) \\ (z-0.04558)(z^2+\cancel{1.954z}+0.9542)(z^2+\cancel{1.949z}+0.9496) \end{array} \right)} \right) \quad (16)$$

$$H_{NOFSAM\_1/4\_int}(z) = \left(\frac{\left(\begin{array}{c} (0.97077(\cancel{z+0.9731})(z-0.92)(z-0.6985)(z-0.3755)(z+0.1765) \\ (z-0.04558)(z^2+\cancel{1.954z}+0.9542)(z^2+\cancel{1.949z}+0.9496) \end{array} \right)}{\left(\begin{array}{c} (z-0.9522)(z-0.7663)(z-0.4612)(\cancel{z+0.9733})(z-0.1224) \\ (z+0.1369)(z^2+\cancel{1.953z}+0.9534)(z^2+\cancel{1.949z}+0.9494) \end{array} \right)} \right) \quad (17)$$

The transfer functions of all the four tenth-order half and one-fourth differentiators and integrators based on the NOFSAM operator had one real pole and two pairs of complex conjugate poles and zeros, exactly overlapping over each other in pole-zero plots. This pole-zero cancellation technique requires a necessary and sufficient condition that states that frequency responses of higher-order FODs or FOIs (which we are degenerating) and their lower-order counterparts (obtained after reduction) should exactly match in their bode plots, leaving no effect of the degeneration of order on their magnitude and phase responses. Following the same bottom line, we have reduced the above-mentioned half and one-fourth order differentiators and integrators into their corresponding fifth-order counterparts after application of the pole-zero cancellation technique. The transfer functions of reduced fifth-order half and one-fourth differentiators and integrators are listed in Table 4.

Table 4. Mathematical models of fifth-order proposed half and one-fourth differentiators and integrators.

Fifth-order half differentiator and integrator	Fifth-order one-fourth differentiator and integrator
$H_{5th-diff-1/2}(z)=\left(\frac{1.061z^5-2.457z^4+1.856z^3-0.4514z^2-0.01295z+0.007628}{z^5-1.711z^4+0.7642z^3+0.01147z^2-0.03853z+0.0003208}\right)$	$H_{5th-diff-1/4}(z)=\left(\frac{1.03z^5-2.23z^4+1.518z^3-0.2863z^2-0.0313z+0.005809}{z^5-1.863z^4+0.9813z^3-0.06156z^2-0.04165z+0.001941}\right)$
$H_{5th-int-1/2}(z)=\left(\frac{0.9428z^5-1.629z^4+0.7336z^3+0.01479z^2-0.03878z+0.0003281}{z^5-2.314z^4+1.745z^3-0.422z^2-0.013z+0.0071483}\right)$	$H_{5th-int-1/4}(z)=\left(\frac{0.9708z^5-1.809z^4+0.9526z^3-0.05976z^2-0.04043z+0.001885}{z^5-2.165z^4+1.474z^3-0.2779z^2-0.03039z+0.005639}\right)$

5. Simulation results of comparisons of proposed fifth-order differ-integrators with the existing models of same orders

5.1. Simulation results of comparisons of proposed fifth-order half differentiator model with the existing ones

The mathematical model of the fifth-order half differentiator, i.e. $H_{NOFSAM\_1/2\_diff}$ (z), mentioned above, has been modified by pole-zero displacement method for obtaining lower magnitude errors with improved response. Transfer function in pole-zero form with delta (0.1 to 0.9) values added and subtracted in values of gain, poles, and zeros is given below.

$$H_{5th\_diff\_1/2}(z)=\left(\frac{(1.0611\pm\delta_1)(z-0.9658\pm\delta_2)(z-0.7984\pm\delta_3)(z-0.5035\pm\delta_4)(z-0.162\pm\delta_5)(z+0.1143\pm\delta_6)}{(z-0.9015\pm\delta_7)(z-0.6628\pm\delta_8)(z-0.332\pm\delta_9)(z+0.1935\pm\delta_{10})(z-0.008357\pm\delta_{11})}\right) \tag{18}$$

The transfer function of the proposed fifth-order half differentiator obtained after pole-zero displacement is

$$H_{5th\_diff\_1/2}(z)=\left(\frac{1.061z^5-2.454z^4+1.851z^3-0.4476z^2-0.01379z+0.007582}{z^5-1.728z^4+0.7781z^3+0.01569z^2-0.04113z+0.000348}\right). \tag{19}$$

The fifth-order half differentiator ($H_{5th-diff-1/2}$(z)) has been compared with the existing benchmark half differentiator of same order given by Leulmi and Ferdi by using the PSE modeling technique. It is named here as ($H_{LF-diff-1/2}$(Z)).

The transfer function of the half differentiator given in [8] is

$$H_{LF-diff-1/2}(z)=\left(\frac{1.076z^5-3.068z^4+2.86z^3-0.7134z^2-0.2548z+0.0999}{z^5-2.261z^4+1.375z^3+0.1088z^2-0.2484z+0.02629}\right). \tag{20}$$

Plots for comparisons of magnitude and phase responses of proposed half differentiators are presented in Figure 3 and their RME responses are given in Figure 4.

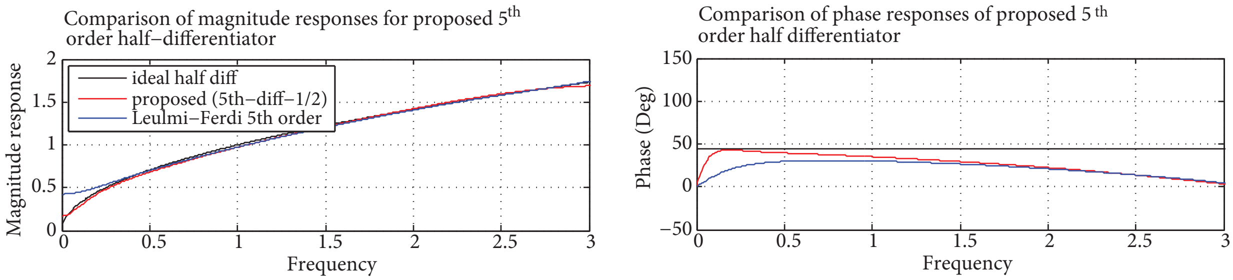

Figure 3. Comparison of magnitude and phase responses of proposed fifth-order half differentiator with the existing [8] half differentiator and ideal half differentiator.

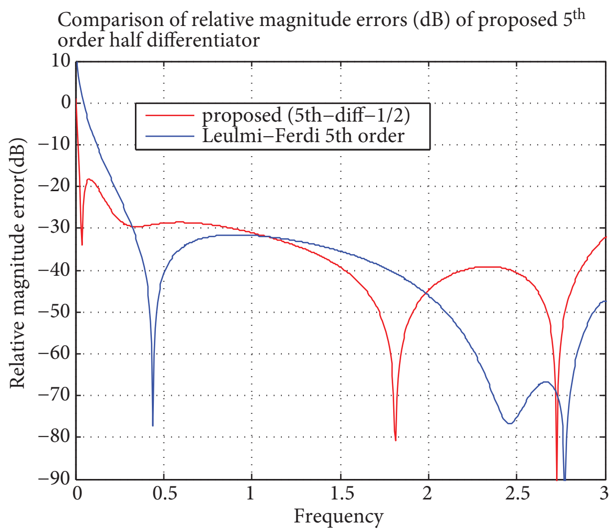

Figure 4. Comparison of relative magnitude errors (dB) of proposed fifth-order half differentiator with the existing [8] half differentiator and ideal half differentiator.

Response of the proposed fifth-order half differentiator $H_{5th-diff-1/2}(z)$ is observed in close vicinity to that of $H_{LF-diff-1/2}(z)$ and it outperforms the LF-half-differentiator in the ranges of $0 \leq \omega \leq 0.35$ rad and $1 \leq \omega \leq 1.98$ rad with RME –20 dB and –31.78 dB, respectively. Phase responses of both the proposed fifth-order and $H_{LF-diff-1/2}(z)$ half differentiators exactly overlap and follow linear curves over the entire Nyquist frequency range. The LF-half-differentiator performs better than the proposed one in the range of $0.32 \leq \omega \leq 1$ rad and $1.98 \leq \omega \leq 3$ rad, but the phase response of the proposed operator surpasses it in the complete spectrum of frequencies.

5.2. Simulation results of comparisons of proposed fifth-order half integrator model with the existing ones

A new fifth-order half integrator has been proposed by inverting the transfer function of the fifth-order half differentiator discussed above and this is given as follows.

$$H_{5th\_int\_1/2}(z) = \left(\frac{0.9428z^5-1.629z^4+0.7336z^3+0.01479z^2-0.03878z+0.0003281}{z^5-2.314z^4+1.745z^3-0.422z^2-0.013z+0.007148}\right) \qquad (21)$$

This proposed fifth-order half integrator has been compared with the existing half integrators of the same order and their transfer functions are given below.

Transfer function of the fifth-order half integrator given in [11] is

$$H_{gupta}(z) = \left(\frac{0.9309z^5 - 0.9040z^4 - 0.4901z^3 + 0.4421z^2 + 0.0726z - 0.0366}{z^5 - 1.5644z^4 + 0.0194z^3 + 0.7243z^2 - 0.11744z - 0.0608} \right). \tag{22}$$

Transfer function of the fifth-order trapezoidal half integrator (H_T) [11] is

$$H_{trap}(z) = \left(\frac{(0.707)(z^5 - 0.443z^4 - 1.067z^3 + 0.43z^2 + 0.187z - 0.045)}{z^5 - 1.443z^4 - 0.123z^3 + 0.776z^2 - 0.181z - 0.024} \right). \tag{23}$$

Transfer function of the fifth order Al-Alaoui half integrator [26] is

$$H_{Al-Alaoui}(z) = \left(\frac{(0.935)(z^5 - 1.962z^4 + 1.175z^3 - 0.179z^2 - 0.021z + 0.002)}{z^5 - 2.533z^4 + 2.214z^3 - 0.748z^2 + 0.067z - 0.004} \right). \tag{24}$$

The plots for comparisons of magnitude responses and phase responses of the proposed half integrator ($H_{5th\_int\_1/2}$ (z)) with the existing ones [11,26] are shown in Figure 5. Figure 6 presents comparisons of RME responses of proposed half integrators and the existing ones. The proposed fifth-order half integrator has RME error values of $\leq$ –40 dB for 0.2 $\leq \omega \leq$ 2.5 rad of the complete Nyquist frequency range. H_{gupta}(z) has RME values $\leq$ –33.56 dB in the full spectrum. $H_{Al-Alaoui}$(z) shows good response with RME of –37.15 dB in the range of 0.74 $\leq \omega \leq$ 2.46 rad but does not perform well in low- and high-frequency regions. The proposed half integrator has a linear phase curve and it performs better than the existing half integrators.

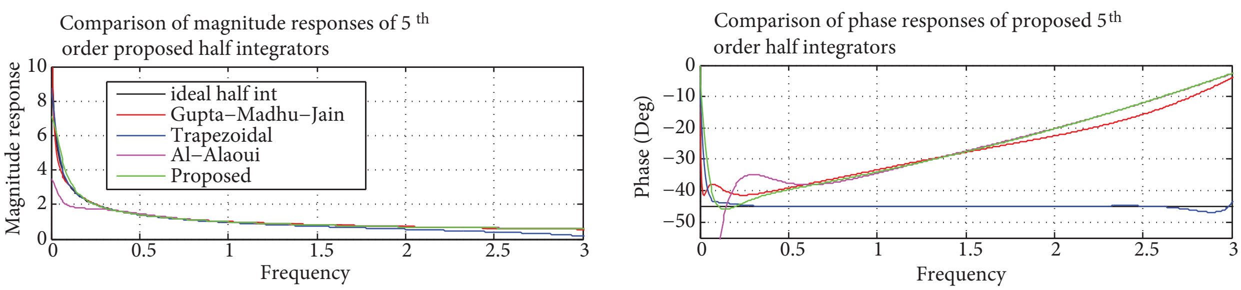

Figure 5. Comparison of magnitude and phase responses of proposed fifth-order half integrator with the existing [11,26] half integrators and ideal half integrator.

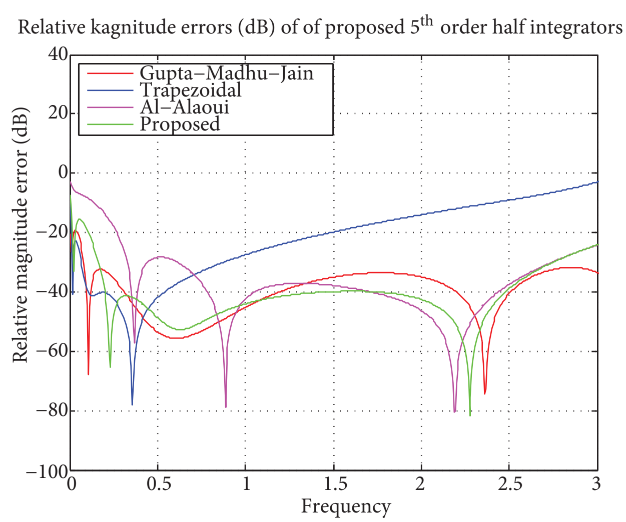

Figure 6. Comparison of relative magnitude errors (dB) of proposed fifth-order half integrator with the existing [11,26] half integrators and ideal half integrator.

5.3. Simulation results of the proposed fifth-order one-fourth differentiator and integrator models

The already existing fifth-order one-fourth differentiators and integrators are only those developed in [21] using TSE and CFE techniques on the Schneider operator and Al-Alaoui-SKG rule. Responses of the proposed fifth-order half and one-fourth differentiators and integrators, namely $H_{5th-diff-1/4}(z)$, $H_{5th-int-1/4}(z)$, were compared with those obtained in [21] for the same order and the ideal response in the full spectrum of normalized frequency. Figures 7 and 8 show comparisons of magnitude, phase, and RME responses of the proposed one-fourth differentiator with those of the ideal one-fourth differentiator and existing models [21]. Figure 9 compares responses of magnitude and phase of the fifth-order one-fourth integrator with the ideal one-fourth integrator, whereas Figure 10 presents plots of their corresponding RME values.

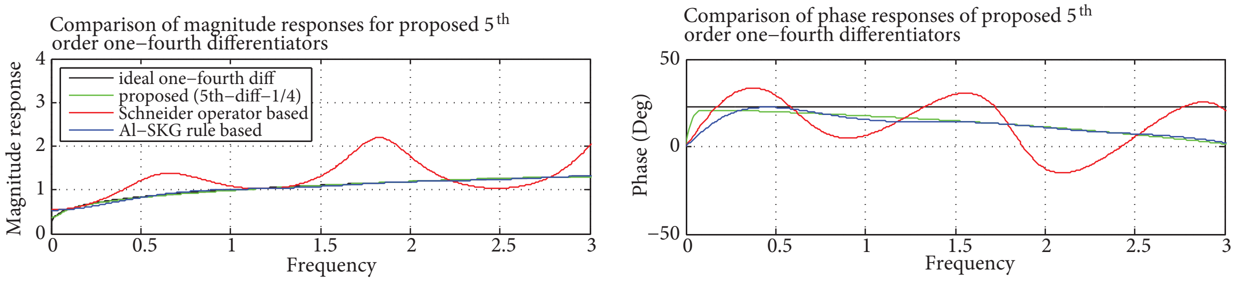

Figure 7. Comparison of magnitude and phase responses of proposed fifth-order one-fourth differentiator $H_{5th-diff-1/4}(z)$ with the existing models [21] and ideal one-fourth differentiator.

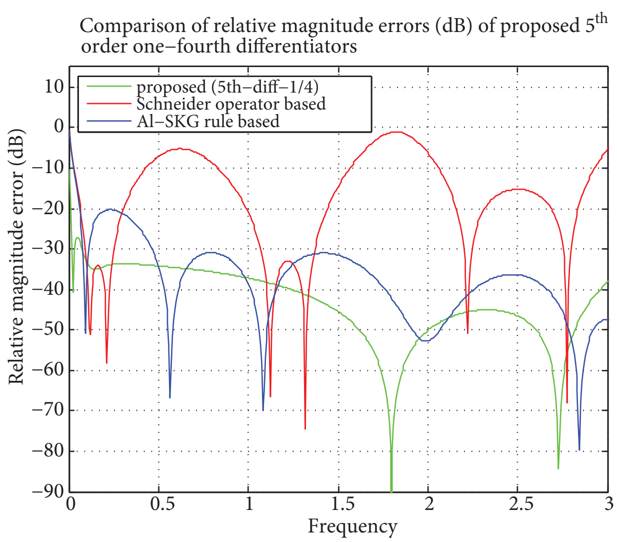

Figure 8. Comparison of relative magnitude errors (dB) of proposed fifth-order one-fourth differentiator $H_{5th-diff-1/4}(z)$ with the existing models [21] and ideal one-fourth differentiator.

We observed that the proposed fifth-order models clearly outperform the existing ones by a large margin with very smaller RME of ≤ –34.58 dB for one-fourth differentiators and ≤ –40 dB for one-fourth integrators in the complete range of Nyquist frequency. Al-SKG rule-based FODs [21] show comparable results with RME ≤ –30 dB in the full spectrum. Phase responses of proposed fifth-order blocks follow linear curves and are better than phase responses of the existing models in the complete frequency range.

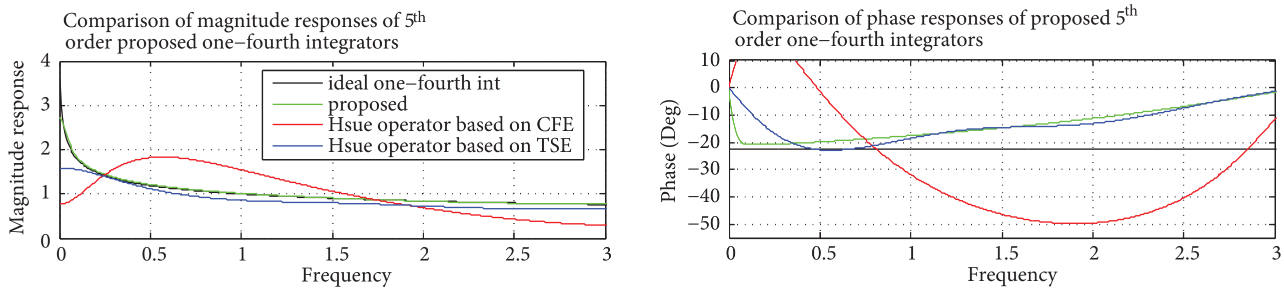

Figure 9. Comparison of magnitude and phase responses of proposed fifth-order one-fourth integrator $H_{5th-diff-1/4}(z)$ with the existing models [21] and ideal one-fourth integrator.

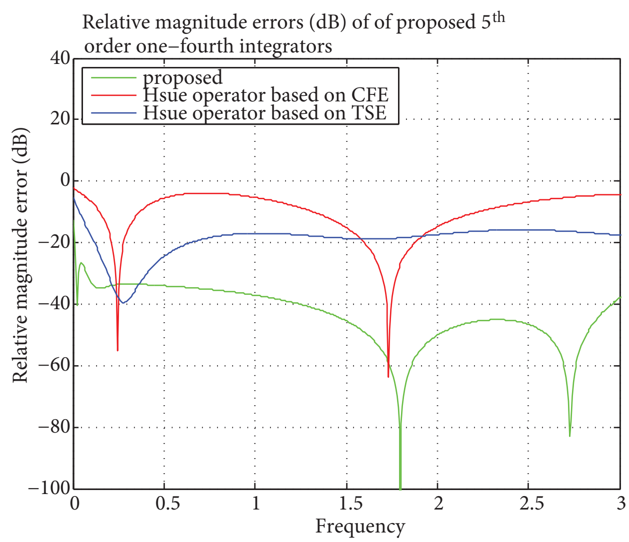

Figure 10. Comparison of relative magnitude errors (dB) of proposed fifth-order one-fourth integrator $H_{5th-diff-1/4}(z)$ with the existing models [21] and ideal one-fourth integrator.

6. Conclusion

This paper describes several new approximations of fractional order integrators and differentiators. Tenth-order blocks are derived by applying three different s-to-z transforms based on new two-segment [7], optimized four-segment [7], and new optimized four-segment operators [7] by indirect discretization scheme. A new modified s-to-z transformation (NOFSAM) has been proposed, which gives improved responses for all the resultant tenth-order blocks. Thus, the new modified operator clearly validates its effectiveness as responses are observed to accurately approximate corresponding ideal curves with very low RME values. The later part of the paper has shown four new interesting blocks of the fifth-order half and one-fourth differentiators and integrators obtained by reduction of FODs and FOIs based on the NOFSAM operator. The proposed fifth-order half differentiators have shown remarkably low magnitude errors and outperform the fifth-order benchmark Leulmi–Ferdi half differentiator. The proposed half and one-fourth integrators also outperform their existing counterparts. Phase responses of all these proposed differentiators and integrators follow the linear curve in the complete Nyquist range.

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