

Effect of nonuniform varying delay on the rate of convergence in averaging-based consensus

Onur CİHAN, Mehmet AKAR*

Department of Electrical and Electronic Engineering, Boğaziçi University, Bebek, İstanbul, Turkey

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Abstract: This paper discusses the effect of nonuniform varying communication delay on distributed consensus algorithms in discrete time. After introducing the delayed mathematical model, we first investigate the ergodicity of the delayed system using the properties of scrambling matrices. Subsequently, the effect of nonuniform varying delay on convergence is examined. It is shown theoretically that nonuniform delay is not detrimental to the convergence rate of the algorithm for directed acyclic graphs. The results are also illustrated with several numerical examples.

Key words: Consensus, distributed control, networks with time delays

1. Introduction

The problem of achieving a common value, referred to as consensus, agreement, or synchronization, in a distributed information sharing context has drawn significant research attention as of late [1–13]. The focus of this paper is to examine the performance of averaging-based discrete-time distributed consensus algorithms in communication networks where delay exists in data receptions. As packet losses and sampling are potential sources of delay in real systems, the possible adverse effect of delay in consensus algorithm performance has to be studied appropriately.

Although there has been a vast amount of research on distributed consensus (see, e.g., [1,2] and the references therein), only a few studies were focused on the performance of such algorithms under time delays. In [3], the authors proposed consensus protocols so that agents with integrator dynamics could achieve average consensus (i.e. nodes converge to the arithmetic mean of the initial values) under a constant amount of delay. Later, based on the results of [3], the effect of time-varying delays on convergence was examined in [4]. Some sufficient conditions for achieving consensus in directed networks in the presence of nonuniform delays were given in [5]. In [6], the authors used contraction theory and a simplified wave variable design in the stability analysis of interacting nonlinear systems with time delayed communications.

While the above work was for continuous-time networks, the focus of this paper is discrete-time consensus, which was addressed in [7–12]. Conditions for achieving consensus in discrete time were first given in [7] and were later reevaluated in [8] for bounded delays and conditions on the connectivity of the network. The results of [7,8] were extended in [9] by relaxing the convexity of the allowed regions for the state transition map of each agent. In [10], the authors studied the convergence properties of a discrete-time consensus algorithm under the assumptions of bounded delay and connectivity of the network using the concept of a spanning tree, which

*Correspondence: mehmet.akar@boun.edu.tr

was utilized by [11] in proposing a condition for achieving consensus. In [12], the author considered a network of interacting agents and showed that convergence to a common value would be achieved, provided that old information was uniformly purged from the system.

While all of the above works examined conditions for achieving consensus in delayed discrete-time systems, the main focus of this paper is to examine the effect of delay on the rate of convergence performance of the algorithm. To this end, the authors of [13] provided bounds on the time required to reach consensus. Naturally, these bounds turn out to be geometric for the conditions under which consensus is reached. However, their work did not provide enough insight into (non)adverse effects of delay on the performance of averaging-based consensus. While we also demonstrate that nonuniform delay does not adversely effect convergence of the consensus algorithm by using properties of scrambling matrices, the main contribution of this paper is to examine a class of network topologies for which the convergence rates of the delayed and nondelayed consensus algorithms are equal to each other (regardless of the weighting coefficients). More specifically, we will analytically show that nonuniform delay is not detrimental to the convergence rate for directed acyclic graphs that are of interest herein.

The organization of this paper is as follows. In Section 2, we introduce the delayed version of the consensus algorithm and examine its convergence properties using tools of scrambling matrices and graph theory. In Section 3, some terms related to the analysis of convergence speed are described and illustrated with examples. In Section 4, the effect of delay on the convergence rate of distributed consensus is discussed in detail, while Section 5 concludes the paper.

2. Model and convergence analysis of delayed distributed consensus

2.1. Delayed consensus model and mathematical preliminaries

In communication networks where communication delay is unavoidable, the distributed consensus algorithm can be mathematically expressed as

$$x_i(t+1) = a_{ii}(t)x_i(t) + \sum_{j=1, j \neq i}^n a_{ij}(t)x_j(t - \tau_{ij}), \tag{1}$$

where $a_{ij}(t) \geq 0$ is the averaging coefficient and τ_{ij} is the amount of delay in data transmission between nodes j and i . The following conditions are assumed throughout the paper [8]:

Assumption 1

- (i) $a_{ii}(t) > \xi, \forall i = 1, 2, \dots, n, \forall t \geq 0$ for some constant $\xi > 0$.
- (ii) $a_{ij}(t) \in [\xi, 1] \cup \{0\}, \forall i, j = 1, 2, \dots, n, \forall t \geq 0$.
- (iii) $\sum_{j=1}^n a_{ij}(t) = 1, \forall i = 1, 2, \dots, n, \forall t \geq 0$.
- (iv) $\tau_{ij} \leq \tau_{max}, \forall i, j = 1, 2, \dots, n$.

Assumption (i) guarantees that nodes use their own data in their updates, whereas the second condition requires that the received data from a neighbor should be used with strictly positive weighting. Assumption

(iii) ensures that the sum of the weighting coefficients for each node is equal to one. Finally, (iv) excludes unbounded delay in the context of this paper by restricting the maximum delay amount to τ_{max} .

When there is no communication delay, i.e. $\tau_{ij} = 0$, the matrix representation of Eq. (1) is given by

$$x(t+1) = A(t)x(t), \quad (2)$$

where the system matrix $A(t) \in \mathbb{A}$ belongs to a special class of so called row-stochastic averaging matrices under the conditions of Assumptions 1(i)–(iii) that are assumed to hold throughout the paper.

Definition 1 We say that a nonnegative matrix $A \in R^{n \times n}$ is row-stochastic if $\sum_{j=1}^n a_{ij} = 1, \forall i \in \{1, 2, \dots, n\}$.

For row-stochastic matrices, we have the following well-known result due to Markov.

Lemma 1 [14] Let A be a row-stochastic matrix. Let x be a nonnegative vector and $y = Ax$. Then we have:

$$\max_{i=1,2,\dots,n} y_i - \min_{i=1,2,\dots,n} y_i \leq \tau(A) \left(\max_{i=1,2,\dots,n} x_i - \min_{i=1,2,\dots,n} x_i \right) \quad (3)$$

where

$$\tau(A) = \frac{1}{2} \max_{i,j} \sum_k |a_{ik} - a_{jk}| \quad (4)$$

is defined to be the coefficient of ergodicity.

Definition 2 Let $A \in R^{n \times n}$ be a row-stochastic matrix. (i) We say that A is scrambling if $\tau(A) < 1$.

(ii) We say that A is ergodic if $\lim_{t \rightarrow \infty} A^t = ec^\top$ for some $c \in \mathbb{R}^n$ where $e = [1 \dots 1]^\top$.

For a row-stochastic matrix A , its coefficient of ergodicity satisfies $0 \leq \tau(A) \leq 1$. Although scrambling matrices are ergodic, not every ergodic system matrix is scrambling [15].

Delay is well known to deteriorate the performance of a system in general; furthermore, it can even lead to instability if not compensated properly. The focus in this paper is to study the effect of nonuniform delay on the convergence rate of the distributed consensus algorithm in Eq. (1) using mathematical tools of stochastic matrices, ergodicity, and graph theory. In the rest of this section, we examine the convergence properties, while Section 4 is devoted to the effect of delay on convergence rate.

2.2. Convergence analysis for uniform delay and fixed topology

Suppose that there exists a uniform and fixed amount of delay between all nodes, i.e. $\tau_{ij}(t) = \tau, \forall i, j, t$. Then Eq. (1) is reexpressed as

$$x(t+1) = A_D x(t) + (A - A_D)x(t-\tau) \quad (5)$$

where $A_D = \text{diag}(a_{ii})$. Let the augmented state vector \hat{x} be defined as

$$\hat{x}(t) = [x(t+\tau)^\top, x(t+\tau-1)^\top, \dots, x(t)^\top]^\top.$$

Then Eq. (5) can be rewritten in the following form:

$$\hat{x}(t+1) = \hat{A}(t)\hat{x}(t), \quad (6)$$

where \hat{A} is given by

$$\hat{A} = \begin{bmatrix} A_D & 0 & \dots & 0 & A - A_D \\ I & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \dots & I & 0 \end{bmatrix}. \tag{7}$$

Lemma 2 *Suppose that at least one node exists in the network from which the rest of the nodes are accessible. Then consensus in the network is achieved despite the existence of bounded uniform delay τ .*

Proof Let there be a node in the network from which the rest of the nodes are accessible. From [15], A^{n-1} is known to be scrambling. We now show that the network achieves consensus by proving that the $((\tau + 1)n - 1)$ th power of \hat{A} is scrambling. For mathematical convenience, denote the (i,j) th block of \hat{A}^τ by $\hat{A}_{\tau,ij}$ that can be computed as

$$\hat{A}_{\tau,ij} = \begin{cases} \bar{f}_{i,j}(A) & i < j \\ f_{i,1}(A_D) & j = 1, i \leq \tau \\ 0 & j \leq i, j \neq 1 \\ I & j = 1, i = \tau + 1 \end{cases} \tag{8}$$

where $f_{i,j}(A)$ and A have equal adjacency matrices and $\bar{f}_{i,j}(A)$ and A have equal adjacency matrices, except that $\bar{f}_{i,j}(A)$ has zero diagonal elements. In the sequel, $g_{i,j}(A)$, $\bar{g}_{i,j}(A)$, $h_{i,j}(A)$, and $\bar{h}_{i,j}(A)$ are defined similarly.

Using Eq. (8), $\hat{A}^{2\tau+1} = [\hat{A}_{2\tau+1,ij}]$ is computed as

$$\hat{A}_{2\tau+1,ij} = \begin{cases} \bar{g}_{i,j}(A^2) & i < j \\ g_{i,1}(A) & j = 1 \\ \bar{g}_{i,j}(A) & j \neq 1, j \leq i. \end{cases} \tag{9}$$

From Eq. (9), one can deduce that $\hat{A}^{2\tau+1}$ is scrambling given that A is.

Suppose that A is non-scrambling. Then the subblocks of the $((\tau + 1)n - 1)$ th power of \hat{A} are calculated as

$$\hat{A}_{(\tau+1)n-1,ij} = \begin{cases} \bar{h}_{i,j}^{(n)}(A) & i < j \\ h_{i,1}(A^{n-1}) & j = 1 \\ \bar{h}_{i,j}^{(n-1)}(A) & j \leq i, j \neq 1 \end{cases} \tag{10}$$

where $\bar{h}_{i,j}^{(k)}(\cdot)$ is a polynomial of order k . One can note that each of the first column components of Eq. (10) has the same adjacency as A^{n-1} . If at least one node exists in the network from which the rest of the nodes are accessible, the matrix A^{n-1} is scrambling [15]. Hence, from Eq. (10), the $((\tau + 1)n - 1)$ th power of \hat{A} is also scrambling. Therefore, we can conclude that the network achieves consensus under bounded uniform delay. \square

2.3. Convergence analysis for uniform delay and varying topologies

Consider a network with N different topologies characterized by averaging matrices $\mathbb{A} = \{A_1, A_2, \dots, A_N\}$. Suppose that a uniform and fixed amount of delay exists between nodes. Let $\hat{\mathbb{A}} = \{\hat{A}_1, \dots, \hat{A}_N\}$ consist of the corresponding system matrices for the delayed systems.

Lemma 3 *Given the set of averaging matrices, $\mathbb{A} = \{A_1, \dots, A_N\}$, suppose that the network consists of a node (possibly different for each topology) in the network from which the rest of the nodes are accessible. Then consensus is achieved in the delayed network for $\tau_{ij} = \tau$ and arbitrary switching of the topologies.*

Proof Let $\pi_k, k \geq 1$ be sets that consist of all possible k matrix products from the set $\hat{\mathbb{A}}$. It can be seen that all of the elements of π_K are scrambling for some $K \leq (\tau + 1)n - 1$, and therefore the network achieves consensus. \square

2.4. Nonuniform delay varying topology networks

Each node may receive data from other nodes with different and varying amounts of delay. Convergence analysis of the consensus algorithm for nonuniform delay networks is crucial since uniform delay does not exist in practical systems.

Theorem 1 *Given the set of averaging matrices, $\mathbb{A} = \{A_1, \dots, A_N\}$, suppose that the network consists of a node (possibly different for each topology) from which the rest of the nodes are accessible. Then consensus is achieved for arbitrarily varying bounded delay $\tau_{ij}, i, j = 1, 2, \dots, n, i \neq j$ and arbitrary switching of the topologies.*

Proof First let us consider the case of a nonuniform fixed amount of delay, which will later be extended to varying delay and network topologies. Under this setting, the consensus algorithm of Eq. (1) can be put into the following matrix form for networks with nonuniform delay [16]:

$$\hat{A} = \begin{bmatrix} A_D + A_{\bar{D},0} & A_{\bar{D},1} & \dots & A_{\bar{D},\tau_{max}} \\ I & 0 & \dots & 0 \\ & \ddots & & \\ 0 & \dots & I & 0 \end{bmatrix} \tag{11}$$

where the (i, j) th element of $A_{\bar{D},l}, l = 0, 1, \dots, \tau_{max}$ is given by

$$[A_{\bar{D},l}]_{ij} = \begin{cases} a_{ij} & \text{if } \tau_{ij} = l \text{ and } i \neq j \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

so that $A_{\bar{D},0} + \dots + A_{\bar{D},\tau_{max}} = A - A_D$ is satisfied. The minimum power that makes the system matrix scrambling is no more than $(\tau_{max} + 1)n - 1$ with the equality if delay is uniform and $\tau_{ij} = \tau_{max}$. Since $\hat{A}^{(\tau_{max}+1)n-1}$ is guaranteed to be scrambling under Assumption 1, consensus follows for a nonuniform fixed amount of delay and the given network topology.

In order to complete the proof for arbitrarily varying topology networks and varying delays, let $\tilde{\pi}_k, k \geq 1$ be the sets that consist of all possible matrix products of length k from the set of all matrices of the form of Eq. (11) for all delay combinations and different topologies corresponding to $\mathbb{A} = \{A_1, \dots, A_N\}$. Note that $\tilde{\pi}_1$ consists of $N(\tau_{max} + 1)^{n^2-n}$ matrices. Analogous to the proof of Lemma 3, it can be seen that all of the elements of $\tilde{\pi}_K$ are scrambling for some $K \leq (\tau_{max} + 1)n - 1$. \square

Remark: Under the conditions of Theorem 1, we note that consensus is achieved despite the existence of bounded nonuniform and varying delay. While we rely on properties of scrambling matrices in proving the end result, the stated conditions in the theorem are relatively easy to check, i.e. one needs to ensure that the maximum amount of delay is bounded and that the network consists of a node (possibly different for each topology) from which the rest of the nodes are accessible.

3. Terms related to convergence speed of consensus algorithms

Given

$$d_0 = \max_{i=1,2,\dots,n} x_i(0) - \min_{i=1,2,\dots,n} x_i(0), \tag{13}$$

where $x(0)$ is the initial state, we are interested in determining the number of steps, k , required to achieve a chosen consensus accuracy level, ϵ . In this section, we will discuss two terms related to the study of convergence speed of consensus algorithms: the coefficient of ergodicity and the second largest eigenvalue of the update matrix.

3.1. The coefficient of ergodicity

Recall that the coefficient of ergodicity, denoted by $\tau(A)$, for a row-stochastic matrix A is computed from Eq. (3). As already discussed in Section 2, the coefficient of ergodicity satisfies $\tau(A) \leq 1$ for a row-stochastic matrix A . Furthermore, from Lemma 1, we know that the maximum state difference is contracting when the matrix is scrambling.

In order to derive an estimate of the convergence speed of a system with a scrambling matrix A , let d_k be the difference between the maximum and minimum components of the vector $x(k)$ at time k , $k \geq 0$. From Eq. (3), we have $d_{k+1} \leq \tau(A)d_k$, $k \geq 0$, which can be iterated to obtain $d_k \leq \tau(A)^k d_0$. Since it is desired that $d_k \leq \epsilon$, one can impose $d_k \leq \tau(A)^k d_0 \leq \epsilon$, i.e. $k \log \tau(A) \leq \log(\epsilon/d_0)$, which leads to $k \geq \log_{\tau(A)}(\epsilon/d_0)$ (since $\tau(A) < 1$ for a scrambling matrix). Therefore, in at most

$$k_{max} = \lceil \log_{\tau(A)}(\epsilon/d_0) \rceil \tag{14}$$

number of iterations where $\lceil k \rceil$ denotes the smallest integer greater than or equal to k , the specified accuracy level ϵ is achieved for the given initial difference d_0 . It can be deduced from Eq. (14) that k_{max} is smaller for a smaller coefficient of ergodicity $\tau(A) < 1$; however, this does not necessarily ensure faster convergence as illustrated in the following example.

Example 1 Consider two systems with system matrices A_1 and A_2 given as

$$A_1 = \begin{bmatrix} 2/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 3/4 & 1/4 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 3/5 & 2/5 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 \end{bmatrix}. \tag{15}$$

The state evolutions of these two systems are depicted in Figure 1 for the initial state $x(0) = [10, 2, 0]^T$, whereas the maximum difference of the components of the state vector is shown in Figure 2 for ease of comparison. We note from Figures 1 and 2 that the first system achieves consensus in fewer steps than the second one.

In Figures 1 and 2 and the rest of the simulation results presented in this paper, the selected numbers do not immediately correspond to actual physical values, but rather scaled versions of physical quantities, such as heading angle that is encountered in formation control, or clock frequencies of the nodes in, e.g., sensor networks.

The below table summarizes the theoretical maximum and actual number of steps required for a range of ϵ/d_0 . Although the theoretical k_{max} value computed via Eq. (14) for System 1 is greater than that of System 2 for all ϵ/d_0 , the actual required number of steps is always less or equal. Moreover, it is also noted from the

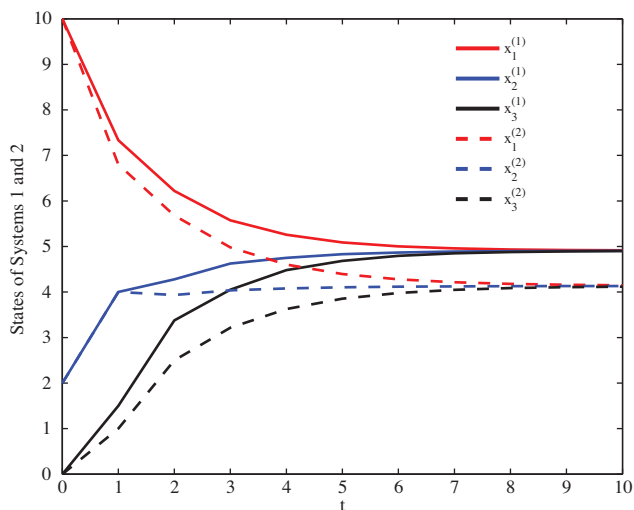


Figure 1. State values for System 1 and System 2 ($x_i^{(j)}$ denotes the i th state of the j th system).

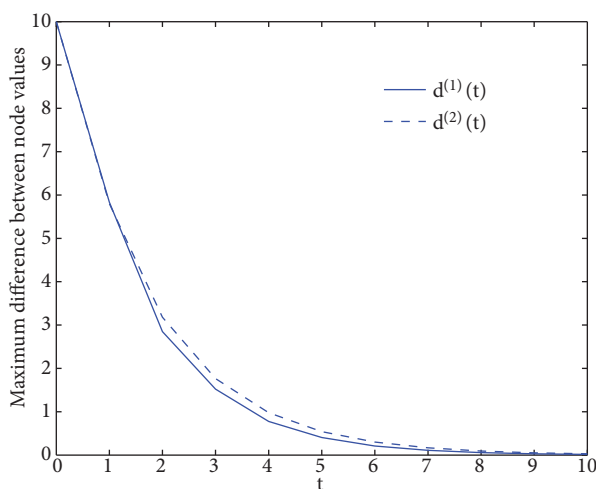


Figure 2. Maximum differences, $d^{(1)}$ and $d^{(2)}$, for System 1 and System 2.

same table that k_{actual} increases as ϵ/d_0 decreases, which is expected since higher accuracy requires a larger number of iterations.

Table. Theoretical maximum and actual number of steps required for varying ϵ/d_0 .

		$\epsilon/d_0 = 10^{-2}$		$\epsilon/d_0 = 10^{-3}$		$\epsilon/d_0 = 10^{-4}$		$\epsilon/d_0 = 10^{-5}$	
System	τ	k_{max}	k_{actual}	k_{max}	k_{actual}	k_{max}	k_{actual}	k_{max}	k_{actual}
1	0.667	12	8	18	11	23	15	29	18
2	0.6	10	8	14	12	19	16	23	20

Since the coefficient of ergodicity is not a good means to compare convergence speeds as illustrated by the above example, we have to use alternative properties of the system matrices to make better comparisons.

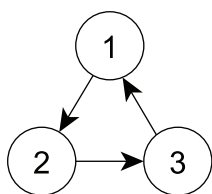


Figure 3. A simple network of $n = 3$ nodes.

3.2. The second largest eigenvalue

The largest eigenvalue of a row-stochastic matrix is equal to one. Moreover, all other eigenvalues are strictly less than one (in magnitude) when consensus is achieved. Let λ_2 be the second largest eigenvalue in this case, i.e.

$$\lambda_1 = 1 > |\lambda_2| \geq \dots \geq |\lambda_n|. \tag{16}$$

Suppose that the system matrix A is related to its Jordan form as $A = TJT^{-1}$. Then the k th power of the system matrix can be computed as

$$A^k = TJ^kT^{-1} = W_{11}1^k + \sum_{i=2}^q \sum_{l=0}^{m_i-1} W_{il}k^l \lambda_i^k \tag{17}$$

where q is the number of distinct eigenvalues of A , m_i is the multiplicity of i th eigenvalue λ_i , and W_{ij} are appropriate scaling matrices. Then the state vector at the k th step is equal to

$$x(k) = A^k x_0 = W_{11}x_0 + \sum_{i=2}^q \sum_{l=0}^{m_i-1} (W_{il}x_0)k^l \lambda_i^k. \tag{18}$$

Note that the sum term in the above expression vanishes faster when the second largest eigenvalue is smaller. Therefore, the convergence speed is directly related to λ_2 , i.e. a smaller λ_2 (in magnitude) results in faster convergence.

Example 2 Consider the system matrices given in Example 1. For A_1 and A_2 we have $\lambda_2(A_1) = 0.5181$ and $\lambda_2(A_2) = 0.5537$, respectively. Since $|\lambda_2(A_1)| < |\lambda_2(A_2)|$, System 1 converges faster.

4. The delay effect on the rate of convergence

In mathematical system theory, delay is well known to deteriorate system performance in general and may even lead to an unstable system if not compensated properly. Although it is noted that algorithm convergence is not adversely affected from bounded delay by Theorem 1, its effect on the convergence rate must be examined. To this end, we first consider a simple example to demonstrate that delay does not necessarily deteriorate the rate of convergence.

Example 3

Consider the system matrix

$$A = \frac{1}{1000} \begin{bmatrix} 924 & 0 & 76 \\ 52 & 948 & 0 \\ 0 & 507 & 493 \end{bmatrix} \tag{19}$$

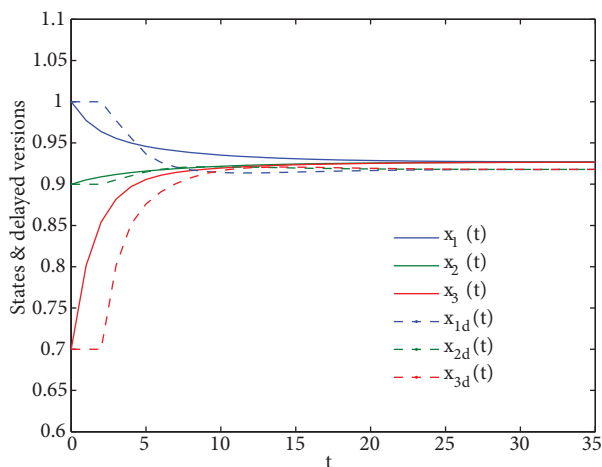


Figure 4. State evolution for the nominal and the delayed system in Example 3 ($x_i(t)$: nominal system states, $x_{id}(t)$: delayed system states).

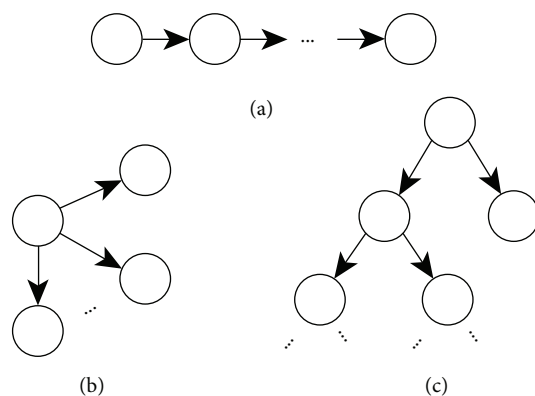


Figure 5. Some example network topologies under consideration: chain topology (a), master-slave topology (b), hierarchical topology (c).

for the network given in Figure 3. For the two-step delayed and the nominal systems, we have $|\lambda_2(\hat{A})| = 0.7722 < \lambda_2(A) = 0.8613$, which implies that the delayed system converges faster. The simulation results for the nominal and the delayed system are depicted in Figure 4 for the initial state values $x(0) = [1, 0.9, 0.7]^T$.

Although Example 3 illustrates that delay does not necessarily deteriorate the rate of convergence for certain choices of the weighting coefficients, we know that this is not true in general. In the sequel, we examine a class of network topologies for which the convergence rates of the delayed and nondelayed consensus algorithms are equal to each other (regardless of the weighting coefficients). Some examples of the topologies under consideration are depicted in Figure 5. The topology in Figure 5a arises in the problem of vehicle platooning, where each vehicle is trying to follow the preceding one at a safe distance. Figure 5b depicts the *master-slave topology*, in which the slaves receive data from the master. This topology is widely used in clock synchronization problems, where the master clock dictates its local clock frequency with its neighbors, namely slave clocks. Figure 5c depicts the *hierarchical topology*, where the nodes receive the leading node’s data directly or indirectly. This

can be thought of as a multilevel master-slave network where slaves of the n th level are the masters of the $(n + 1)$ th level. The following theorem states that nonuniform delay is not detrimental to the convergence rate for the topologies of interest in this paper. Before we state the result, we recall that two matrices A and B are said to be permutation-similar if there exists a permutation matrix P such that $B = P^{-1}AP$.

Theorem 2 *For directed acyclic graphs (i.e., the graphs whose adjacency matrix is permutation-similar to an upper (or equivalently lower) triangular matrix), the nominal and delayed consensus algorithms of Eq. (1) have the same nonzero spectra at each iteration step for arbitrarily varying nonuniform bounded delay.*

Proof We first consider the nonuniform fixed delay among nodes. In this case, the system matrix can be expressed in the form of Eq. (11), whose characteristic polynomial is given by

$$|\lambda I - \hat{A}| = \begin{vmatrix} \lambda I - (A_D + A_{\bar{D},0}) & -A_{\bar{D},1} & \dots & -A_{\bar{D},\tau_{max}} \\ -I & \lambda I & 0 & 0 \\ & \ddots & & \\ 0 & \dots & -I & \lambda I \end{vmatrix}. \tag{20}$$

By using the properties of the determinant for block matrices, Eq. (20) can be simplified to a lower-dimensional matrix determinant:

$$|\lambda I - \hat{A}| = \lambda^n \begin{vmatrix} \lambda I - (A_D + A_{\bar{D},0}) & -A_{\bar{D},1} & \dots & -A_{\bar{D},\tau_{max}-1} - \frac{1}{\lambda}A_{\bar{D},\tau_{max}} \\ -I & \lambda I & 0 & 0 \\ & \ddots & & \\ 0 & \dots & -I & \lambda I \end{vmatrix}. \tag{21}$$

By applying the above procedure recursively, we obtain

$$|\lambda I - \hat{A}| = \lambda^{n\tau_{max}} \left| \lambda I - \left(A_D + A_{\bar{D},0} + \frac{1}{\lambda}A_{\bar{D},1} + \dots + \frac{1}{\lambda^{\tau_{max}}}A_{\bar{D},\tau_{max}} \right) \right|. \tag{22}$$

For the network topologies that are covered in the statement of the Theorem, we note that $A_D + A_{\bar{D},0} + \frac{1}{\lambda}A_{\bar{D},1} + \dots + \frac{1}{\lambda^{\tau_{max}}}A_{\bar{D},\tau_{max}}$ and A (the system matrix without delay) have the same set of eigenvalues. Since the delayed system has the zero eigenvalue with multiplicity $n\tau_{max}$, the nominal and the delayed systems have the same nonzero spectra.

In order to complete the proof for arbitrarily varying delays, let $\tilde{\pi}_k, k \geq 1$ be the sets that consist of all possible matrix products of length k from the set of all matrices of the form of Eq. (11) for all delay combinations. Note that $\tilde{\pi}_1$ consists of $(\tau_{max} + 1)^{n^2-n}$ matrices. We now consider two subcases: $1 \leq k < \tau_{max}$ and $k \geq \tau_{max}$. For the latter case, any matrix $p_k(\hat{A})$ in $\tilde{\pi}_k$ can be expressed in the following form:

$$p_k(\hat{A}) = \begin{bmatrix} A_D^k + * & * & \dots & * \\ A_D^{k-1} + * & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ A_D^{k-\tau_{max}} + * & * & \dots & * \end{bmatrix}, \tag{23}$$

where $*$ above denotes an upper-triangular matrix with zero diagonal elements. By applying the same recursive procedure as in the fixed delay case, we obtain

$$|\lambda I - p_k(\hat{A})| = \lambda^{n\tau_{max}} |\lambda I - (A_D^k + *)|, \tag{24}$$

which implies that the nonzero spectra of $p_k(\hat{A})$ and A^k are equal to each other. For $1 \leq k < \tau_{max}$, the first k subrows of $p_k(\hat{A})$ have the form

$$\begin{bmatrix} A_D^k + * & * & \dots & * \\ A_D^{k-1} + * & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ A_D + * & * & \dots & * \end{bmatrix}, \tag{25}$$

whereas the last $\tau_{max} + 1 - k$ subrows are equal to $[I_{(\tau_{max}+1-k)n} \ 0]$, where $I_{(\tau_{max}+1-k)n}$ is the identity matrix of length $(\tau_{max} + 1 - k)n$. The same recursive procedure as above can be applied to $p_k(\hat{A})$ in this case as well to obtain the desired result. \square

Remark: From Theorem 2, we note that the nonzero spectra of the nominal system and the version for arbitrarily varying nonuniform bounded delay are equal to each other at each iteration. Hence, we conclude that bounded nonuniform varying delay does not adversely affect convergence rate for the network topologies under consideration in this paper regardless of the amount of delay and the choice of the weighting coefficients. As can be expected, this is not true for all topologies under which consensus is achieved. To this end, consider the fully connected network where each node uses equal averaging coefficients: $1/n$. It can be easily noted that the system matrix has the zero eigenvalue with multiplicity $n - 1$; therefore, consensus is achieved in a single iteration (dead-beat system). On the other hand, the delayed system has a nonzero second largest eigenvalue.

5. Concluding remarks

In this paper, the performance of averaging-based deterministic consensus under delayed information has been studied. It is proven mathematically that the rate of convergence of the distributed consensus algorithm is not reduced despite delay for directed acyclic graphs as discussed in Section 4. This result holds regardless of the values of the averaging coefficients so long as they satisfy Assumption 1. Although this paper has focused on the case where delay does not adversely effect the convergence rate, networks topologies where the convergence rate is always degraded are to be explored.

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