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An analytical formulation with ill-conditioned numerical scheme and its remedy: scattering by two circular impedance cylinders

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Abstract: The regularization of the well-known analytical formulation of the monochromatic electromagnetic wave scattering problem from a syvstem with two neighbor impedance circular cylinders is presented. It is the improvement and extension of the work done for scattering from two perfectly conducting circular cylinders. Numerical results show that it is numerically much safer to solve the obtained infinite algebraic system at a lower truncation number also by ensuring the reliability of the solution.

Key words: Impedance cylinder, regularization, numerical stability, electromagnetic scattering

1. Introduction

During research for either basic sciences or engineering, capabilities of computers nowadays are alleviating many difficulties of numerical simulations. With proper software, obtaining the features of interest for an investigation has become a rather swift process regardless of its staying within certain limits. For a more general model, scientific computation requires generalization of the basic approaches and writing specified codes. Combined together, the ways to simulate a physical or an engineering process are quite various, owing to the present techniques of using the capabilities of modern hardware and software.

We can meet the concerns about the numerically stable simulation of an analytical model, starting at least five decades ago [1,2]. An end-of-millennium survey of efforts to improve numerical stability of the growing class of problems that can be modeled in mathematical rigor can be found in [3]. With the advent of the numerical techniques, the variety of attacked problems in engineering electromagnetics attained such a level that even the capacity of modern computers required appropriate usage with the same concerns mentioned above (e.g., [4]). Nevertheless, the examples among analytical formulations can show how the abundance of computational facilities may lead to overlooking the numerical implementation issues. Since many modern studies on nanotechnology, bioelectromagnetics, acoustics, and designs of power lines of transmission have to rely on their important problems being modeled analytically (e.g., [4–6]), reviving these issues is quite important.

Here we present an electromagnetic scattering problem that has a very simple geometrical configuration (Figure 1) and is important for nanotechnology and metamaterial science [6,7]. This formulation corresponds to the integral formulation in [1] and will be expressed as an infinite linear algebraic system of the first kind (LAES1). As stated in [3] and exemplified in [8], the verifications of the numerical solutions to a certain

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boundary value problem reduced to such a LAES1, being held with physical requirements such as reciprocity of the fields or boundary conditions, are circumstantial. This fact was taken into account in [2,3] and [8], in the first for two perfectly conducting circular cylinders and in the third for two layered eccentric circular cylinders. It is well known [9] that the only practical way to obtain a reliable and stable solution avoiding round-off errors is to make sure that the condition number of the truncated algebraic system does not reach (preferably) or exceed (meaning no significant digits are obtained) the word length (i.e. 53 bits or 16 decimal figures for a standard PC). In this work, the novel extension of the well-conditioning strategy is being presented for two neighbor circular impedance cylinders as seen in Figure 1. The resulting linear algebraic system is of the second kind (LAES2), i.e. (**I**+**H**)y=g, y,g$\in l_2$ (space of square-summable sequences), where **I** is the identity operator and **H** is a compact operator in l_2. The illumination is performed parallel to the longitudinal axes of the cylinders, and thus a two-dimensional problem is under consideration for both TE and TM polarizations. Nevertheless, all the formulations will be given for the TM mode, since TE mode formulation is very similar.

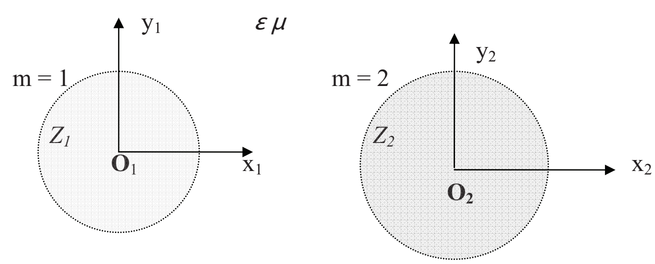

Figure 1. Geometry of the two neighbor circular impedance cylinders.

2. Formulation

2.1. Posing of the Problem

Let two parallel circular cylinders that are homogeneous in the longitudinal **Oz** direction be given (Figure 1). It is assumed that the cylinders are illuminated by an E-polarized (TM) wave with given value $E_z = E_z^{incident}$, and on the boundaries of the cylinders the boundary conditions of impedance type are given as:

$$\hat{\mathbf{n}} \times \mathbf{E} = Z_m \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{H}), \quad m = 1, 2, \tag{2.1}$$

where $\hat{\mathbf{n}}$ is the unit outward normal and Z_m is the corresponding impedance to surface m. If the conditions in Eq. (2.1) are proposed to model well-conductive media inside cylinders, we arrive at the classic form of Z_m [10–12], i.e. to the Leontovich–Schukin boundary condition. The application areas of the condition of the kind in Eq. (2.1) are much wider than the mentioned modeling. For example, they are useful in many physical applications during the simulation of the boundary of:

- a well or perfectly conductive medium,
- a dielectric medium with big values of material parameters ε and/or μ,
- surfaces with small inhomogeneities, i.e. metal gratings of small magnitude, and so on.

The calculation of the values of Z_m for such models is of great interest and the subject of numerous publications (see [13–18] and the references therein).

As is known, an electromagnetic field can be represented in a cylindrical coordinate system as the superposition of TE and TM polarized waves [19]. In particular, TM waves can be represented by means of the only component E_z.

$$E_\rho = E_\phi = H_z = 0; \quad H_\phi = \frac{i}{\omega\mu}\frac{\partial E_z}{\partial \rho}; \quad H_\rho = -\frac{i}{\omega\mu}\frac{1}{\rho}\frac{\partial E_z}{\partial \phi}, \tag{2.2}$$

where ω is the angular frequency and μ is the permeability of the media, and (ρ, ϕ, z) are the corresponding cylindrical coordinates.

It can be shown that for a TM-polarized incident field propagating perpendicular to the **Oz** direction, the scattered field must be TM-polarized also. For cylinder m, we introduce cylindrical coordinate system $\mathbf{O}_m\mathbf{z}$ coinciding with the axis of this cylinder. As far as the incident and the scattered fields do not depend on z, we can choose $z = 0$ and introduce the polar coordinate systems (ρ_m, ϕ_m) for two neighbor circular impedance cylinders indexed via $m = 1{,}2$ respectively. Substituting Eq. (2.2) in Eq. (2.1) gives the relation

$$\left[E_z^t - Z_m \frac{i}{\omega\mu}\frac{\partial E_z^t}{\partial \rho_m}\right]_{\rho_m = a_m} = 0, \qquad m = 1,2 \tag{2.3}$$

on the boundary of cylinder m (BoC_m), where a_m is its radius and $E_z^t = E_z^{incident} + E_z^{scattered}$ is the z component of the total electric field. The wave number is $k = \omega(\varepsilon\mu)^{1/2}$ and the incident field on the BoC_m can be represented by the following series (see [19]):

$$E_z^{incident}(\rho_m, \phi_m) = \sum_{n=-\infty}^{\infty} T_n J_n(k\rho_m) e^{in\phi_m}; \quad T_n = \left\{ \begin{array}{c} i^{-n};\ plane\, wave \\ H_n^{(1)}(k\rho_m^{(0)});\ line\, source\ at\ \rho_m^{(0)} \end{array} \right. . \tag{2.4}$$

It is convenient for us to formulate a slightly more general boundary value problem (BVP), where we denote $E_z^{scattered}(\rho_m, \phi_m)$ as function $u^s(\rho_m, \phi_m)$. It is necessary to find the function u^s as the solution of the following BVP:

$$u^s \in C^{2,\alpha}(\Omega) \cap C^{1,\alpha}\left(\bar{\Omega}\right), \tag{2.5}$$

where the domain Ω is the space R^2 without the cylindrical regions being considered as closed domains, i.e. they contain their BoC_m centered at $\mathbf{O}_m$ with radius a_m, $m = 1{,}2$. Taking the time dependence of the electromagnetic waves $\mathrm{e}^{-i\omega t}$ to be omitted everywhere below, very standard posing of the scattering problem (see [20]) requires the scattered field u^s to satisfy the homogeneous Helmholtz equation:

$$\left(\Delta + k^2\right) u^s(q) = 0, \qquad q \in \Omega; \tag{2.6}$$

Sommerfeld radiation condition:

$$\sqrt{|q|}\left(\frac{\partial}{\partial |q|} - ik\right) u^s(q) = \mathrm{O}(1) \quad for \quad |q| \to \infty; \tag{2.7}$$

and boundary condition of the following kind on BoC_m:

$$\lim_{h\to+0} D_m^{\alpha,\beta}\left[u^s(\rho_m + h, \phi_m)\right] = f^{(m)}(\phi_m); \quad m = 1,2; \qquad \left(D_m^{\alpha,\beta} = \alpha + \frac{\beta}{k}\frac{\partial}{\partial \rho_m}\right), \tag{2.8}$$

where $D_m^{\alpha,\beta}$ is the differential operator for the impedance boundary condition, α and β are some given constants such that $|\alpha|+|\beta|>0$, $u^s(\rho_m+h,\ \phi_m)$ is $u^s(q)$ written in local coordinate system (ρ_m,ϕ_m), and $f^{(m)}(\phi_m)$ are some (infinitely) smooth functions of ϕ_m.

Functions $f^{(m)}(\phi_m)$ are nothing else but the result of the acting of the differential operator in Eq. (2.8) to the incident field. Typically, the incident field satisfies the Helmholtz equation (Eq. (2.6)) in at least a small vicinity of each BoC_m, and that is why the incident field and the functions $f^{(m)}(\phi_m)$ are representable by their Fourier series

$$f^{(m)}(\phi_m)=\sum_{n=-\infty}^{\infty} f_n^{(m)} e^{in\phi_m};\quad m=1,2, \tag{2.9}$$

with Fourier coefficients $f_n^{(m)}$ decaying like superalgebraically factor inverse power $Const\cdot n^{-t}$ for any $t>0$ ($Const$ here depends of course on t).

As is known [21], the solution $u^s(q)$ corresponding to the infinitely smooth function $f^{(m)}(\phi_m)$ is infinitely smooth itself until the BoC_m, m = 1,2. This means that not only Eq. (2.5) is valid but also

$$u^s(q)\in C^\infty\left(\bar{\Omega}\right). \tag{2.10}$$

We will construct the solution of the BVP of Eqs. (2.5)–(2.8) on the basis of the separation of variables method (SVM). This corresponds to the "series expansion" context of [1] and in many texts this method is explained as the sequence of the following steps:

- Find all eigenmodes of the homogeneous Helmholtz equation,
- Remove those eigenmodes that do not satisfy radiation and finite energy conditions from consideration,
- Make the superpositions of the rest of the eigenmodes and find the superposition coefficients from boundary conditions of the BVP.

Such a naive algorithm works sometimes, but not always, because it misses the most important requirement: *any* solution of the BVP must be representable by the mentioned superposition in a *suitable* metric – at least, in a continuous metric. Roughly speaking, such an approach is similar to the well-known Rayleigh hypothesis; see [22–24]. The Rayleigh hypothesis is found to work under certain restrictions, but is wrong in general.

In order to give an example of improper usage of the SVM, let us consider the case $\alpha=1$, $\beta=0$, i.e. the Dirichlet BVP. Let us search for the solution of the BVP in the following form:

$$u^s(q)=\sum_{n=-\infty}^{\infty} R_n^{(1)} H_n^{(1)}(k\rho_1)\, e^{in\phi_1}, \tag{2.11}$$

i.e. as a superposition of harmonics connected with the origin of coordinate $\mathbf{O}_1$ only. As far as the number of unknowns $\left\{R_n^{(1)}\right\}_{n=-\infty}^{\infty}$ is infinite, one can try to satisfy N conditions (by point matching or moment methods, etc.) for both cylinders. Unfortunately, such an approach is based on the strong violation of restrictions of the geometrical structure, for which the Rayleigh hypothesis is correct [22–24]. That is why there will be divergent results when N is growing. The corresponding algebraic system will be extremely ill-conditioned.

2.2. Reducing the BVP to algebraic systems

Due to the explanation above, we will derive the corresponding superposition from the solid mathematical background, namely from Green's formulae technique and the relevant integral representation of the scattered field (see also [1] for integral formulation). Indeed, as is well known [20], any solution of the Helmholtz equation in domain Ω can be represented as:

$$u^s(q) = -\int\limits_S \left[u^{s(+)}(p)\frac{\partial}{\partial n_p}G_2(k|q-p|) - G_2(k|q-p|)\,v^{s(+)}(p)\right]dS_p, \qquad q \in \Omega, \tag{2.12}$$

where $S = BoC_1$?[U+F8F3]?[U+F8FE]BoC_2, and $G_2(k|q--p|)$ is the Green function of the two dimensional free space, i.e.

$$G_2(k|q-p|) = -\frac{i}{4}H_0^{(1)}(k|q-p|), \tag{2.13}$$

$$u^{s(+)}(a_m) = \lim_{h\to+0} u^s(a_m+h,\phi_m), \tag{2.14}$$

$$v^{s(+)}(a_m) = \lim_{h\to+0}\frac{\partial}{\partial n}u^s(a_m+h,\phi_m). \tag{2.15}$$

Using the following expansion for Eq. (2.13) by virtue of Graf's addition theorem for Bessel functions [25,26],

$$-\frac{i}{4}H_0^{(1)}(k|q-p|) = -\frac{i}{4}\sum_{s=-\infty}^{\infty} J_s(k\rho_<)\,H_s^{(1)}(k\rho_>)\,e^{is(\phi_q-\phi_p)}, \tag{2.16}$$

where $\rho_< = \min(\rho_q, \rho_p)$, $\rho_> = \max(\rho_q, \rho_p)$, and expansion of Eqs. (2.14) and (2.15) into their Fourier series followed by substitution of all the above obtained series representations into Eq. (2.12) and their term-by-term integration there gives the following expression for the scattered field:

$$u^s(\rho_\nu,\phi_\nu) = \sum_{m=1}^{2}\sum_{n=-\infty}^{\infty} R_n^{(m)}H_n^{(1)}(k\rho_m)\,e^{in\phi_m}, \quad \rho_\nu > a_\nu, \qquad \nu = 1,2. \tag{2.17}$$

If $R_n^{(m)}$ are fast decaying for n$\to\infty$, which is the case here, as we know, Eq. (2.17) is valid on S, too. Eq. (2.17) gives the necessary superposition of the harmonics $H_n^{(1)}(k\rho_m)\,e^{in\phi_m}$, $m=$ 1,2, and as is shown, any solution of the Helmholtz equation in Ω is representable as Eq. (2.17).

Before going further, one should find a suitable way of applying the boundary conditions of Eq. (2.8). Such a way is provided by virtue of Graf's addition theorem for Bessel functions once again [25,26], having the special case considered here in the following view:

$$H_n^{(1)}(k\rho_m)e^{in\phi_m} = \sum_{s=-\infty}^{\infty} J_s(k\rho_\nu e^{(i\ s\phi_\nu)})H_{n-s}^{(1)}(kd_{m\upsilon})e^{i(n-s)(\theta_{m\upsilon})}, \;\; n = 0, \pm1, \pm2, ... \tag{2.18}$$

Here the vector for the coordinate differences in polar coordinates is defined as $\mathbf{d_{m\upsilon}} = (d_{m\upsilon}, \theta_{m\upsilon}) = \mathbf{O}_\upsilon - \mathbf{O_m}$, with neighbor index υ and $(m,\upsilon) \in \{1,2\}$, $m \neq \upsilon$. The expansion of Eq. (2.18) is valid for any point in the small vicinity of both BoC_m until $\rho_m < d$. Such a vicinity always exists for nonovercrossing BoC_m (as in the considered case here), i.e. $a_1 + a_2 < d_{12}$. Thus, the scattered field in the mentioned small vicinity is:

$$u^s(\rho_m, \phi_m) = \sum_{n=-\infty}^{\infty} e^{in\phi_m} \left\{ \begin{array}{l} R_n^{(m)} H_n^{(1)}(k\rho_m) \\ + \sum\limits_{s=-\infty}^{\infty} R_s^{(v)} H_{s-n}^{(1)}(kd_{mv}) J_n(k\rho_m) e^{i(s-n)(\theta_{mv})} \end{array} \right\}. \tag{2.19}$$

Substituting Eq. (2.19) into the boundary conditions of Eq. (2.8) and taking into account that the equality of functions means the equality of their Fourier coefficients, we arrive at:

$$\begin{array}{l} R_n^{(m)} D_m^{\alpha,\beta}\left[H_n^{(1)}(ka_m)\right] + \sum\limits_{s=-\infty}^{\infty} R_s^{(v)} H_{s-n}^{(1)}(kd_{mv}) D_m^{\alpha,\beta}\left[J_n(ka_m)\right] e^{i(s-n)(\theta_{mv})} \\ = f_n^{(m)}, \qquad m = 1,2, \;\; n = 0, \pm 1, \pm 2, ... \end{array} \tag{2.20}$$

Following Ivanov in [2] (and also [28] later), one can transform the LAES1 arrived at in Eq. (2.20) into a form looking like LAES2:

$$\begin{array}{l} R_n^{(m)} + \sum\limits_{s=-\infty}^{\infty} R_s^{(v)} H_{s-n}^{(1)}(kd_{mv}) \frac{D_m^{\alpha,\beta}[J_n(ka_m)]}{D_m^{\alpha,\beta}\left[H_n^{(1)}(ka_m)\right]} e^{i(s-n)(\theta_{mv})} \\ = \frac{f_n^{(m)}}{D_m^{\alpha,\beta}\left[H_n^{(1)}(ka_m)\right]}, \;\; m = 1,2, \;\; n = 0, \pm 1, \pm 2, ... \end{array} \tag{2.21}$$

which is still a LAES1 as shown through the numerical experiments performed below.

3. Regularization

3.1. On the analytical regularization method

The infinite algebraic system

$$\mathbf{A}x = b \tag{3.1}$$

is called a LAES2 if

$$\mathbf{A} = \mathbf{I} + \mathbf{H}, \tag{3.2}$$

where $\mathbf{I}$ is the identical operator and $\mathbf{H}$ is a compact operator in space l_2. Otherwise, Eq. (3.1) is a LAES1. In spite of the widely used definition of $\mathbf{H}$ as a compact, and not necessarily in space l_2, operator in functional analysis, we restrict this space just to space l_2. This is due to the restriction from the numerical point of view (see below). We refer our reader to [9,31] for further details on numerical stability and theoretical and practical applications of the truncation method. In particular, the condition number of the operator in Eq. (3.2) can be defined as:

$$\nu = \lim_{N\to\infty} \nu_N; \qquad \nu_N = \|\mathbf{I} + \mathbf{H}_N\| \left\|(\mathbf{I} + \mathbf{H}_N)^{-1}\right\|, \tag{3.3}$$

where $\mathbf{H}_N$ are any truncated $N \times N$ matrices under restriction $||\mathbf{H}\text{-}\mathbf{H}_N|| \to 0$ for $N \to \infty$. If $(\mathbf{I}+\mathbf{H})^{-1}$ exists then ν exists and is finite as well as ν_N being uniformly bounded for big enough N. On the contrary, condition numbers ($\nu = ||\mathbf{A}|| \cdot ||\mathbf{A}^{-1}||$) of a LAES1 are typically growing until infinity for $N \to \infty$.

According to [9], a small or not critically big condition number guarantees stability and good accuracy of the solution of the LAES2

$$(\mathbf{I} + \mathbf{H})\, x = b, \quad x, b \in l_2, \tag{3.4}$$

as well as the numerical convergence of the truncated system solutions to the solution of the infinite system. Nothing of this kind is true, in general, for the LAES1. The solutions of the truncated LAES1 are, as a rule, unstable and divergent for $N \to \infty$.

Let us suppose that the LAES1 of Eq. (3.1) is given, where $\mathbf{A}$ is acting on a pair of functional spaces, say $\boldsymbol{U}$ and $\boldsymbol{V}$, namely $\mathbf{A}: \boldsymbol{U} \to \boldsymbol{V}$, and let a pair of operators ($\mathbf{L}$,$\mathbf{R}$) be given so that $\mathbf{L}: \boldsymbol{V} \to l_2$, $\mathbf{R}: l_2 \to \boldsymbol{U}$, and $\mathbf{LAR} = \mathbf{I} + \mathbf{H}$ with $\mathbf{H}$ compact in l_2. We call such pair a "two-sided regularizer"; if $\mathbf{R} = \mathbf{I}$ or $\mathbf{L} = \mathbf{I}$, we call $\mathbf{L}$ or $\mathbf{R}$ respectively a left or right one-sided regularizer. Having a regularizer, one can easily reduce the LAES1 to a LAES2. Indeed, let $\mathbf{L}$ act on Eq. (3.1) from the left side and a new unknown $y \in l_2$ be introduced as $x = \mathbf{R}y$. Eq. (3.1) becomes a LAES2:

$$(\mathbf{I} + \mathbf{H})\, y = \mathbf{L}b, \quad y, \mathbf{L}b \in l_2. \tag{3.5}$$

The purpose of the analytical regularization method (ARM) is just to construct such a regularizing pair analytically, which is sometimes a rather nontrivial task. For further details and principle ideas of the ARM we refer our reader to [3,29,31] to obtain a clear initial explanation. As good examples of the ARM implementation that are not covered by the aforementioned papers, [32–34] can also be cited.

3.2. Regularization of the algebraic system

The infinite LAES1 in Eq. (2.20) can be rewritten in the following block matrix notation using the definition of the incident field in Eq. (2.4):

$$\begin{bmatrix} H_n^{(1)m}(a_m) & J_{n,s}^m(a_m) \\ J_{n,s}^v(a_v) & H_n^{(1)v}(a_v) \end{bmatrix} \begin{bmatrix} R_n^{(m)} \\ R_n^{(v)} \end{bmatrix} = \begin{bmatrix} -T_n J_n^m(a_m) \\ -T_n J_n^v(a_v) \end{bmatrix}; \quad n, s \in (-\infty, \infty), \ \ m, v \in \{1, 2\}. \tag{3.6}$$

Applying truncation, indices $n, s \in$[–N,N] for each block in the matrix of Eq. (3.6), and thus it is a (2×(2N+1)) × (2×(2N+1)) square algebraic system. Here, the single subindex entries are vector blocks or matrix entries that form diagonal matrix blocks. Double index matrix entries obey the following form:

$$J_{n,s}^{(m,v)} = J_n^{(m,v)} H_{s-n}^{(1)}\left(kd_{(mv,vm)}\right) e^{i(s-n)\left(\theta_{(mv,vm)}\right)}. \tag{3.7}$$

The dielectric permittivity ε and the magnetic permeability μ of the host medium, as well as the surface impedances of the cylinders $Z_{1,2}$, are complex valued quantities in general. As a result of the action of the IBC operator in Eq. (2.8), for all the entries of the matrix and the right-hand-side vector in Eq. (3.6), the relation $B_n^m(\rho_m) = B_n(k\rho_m)$ $-i\zeta B_n$'$(k\rho_m)$ is valid with B being any of the considered Bessel and Hankel functions (' is derivative with respect to argument). Here $\zeta = Z_m/Z$ with $Z = (\mu/\varepsilon)^{1/2}$ the intrinsic impedance of the host medium.

The properties in Table 1 are crucial to understand the infinite LAES1 in Eq. (3.6), which is in the form of $\mathbf{A}$x = b. The passage from the expressions there in column 2 to 3 is performed using the asymptotic Stirling formula $\mathrm{n}! \sim (2\pi \mathrm{n})^{1/2}(\mathrm{n/e})^n$ [26]. Consequently, the truncation procedure of the LAES1 in Eq. (3.6) for numerical evaluation is ill-conditioned since the Euclidian (Sturm–Liouville) norm [30] of infinite matrix A is unbounded. Therefore, solutions of such a LAES1 are unprotected from round-off errors. In [2] it was recognized that the proper scaling for the unknown coefficients regularizes the LAES1 to a LAES2 for two perfectly conducting neighbor cylinders. Our aim here is to implement the same for the two neighbor cylinders with IBC. Understanding the structure of the reflection coefficients in Eq. (2.17) of an integral formulation

[1] can be useful for accomplishing this. Also, referring to the parallel explanations in Section 2.2 above, one can conclude that they possess rapidly decreasing behavior proportional to $J_n(t)$ and its derivative as n$\to\infty$ (Table 1). Therefore, their scaling in the form of $x=\mathbf{R}y$ is formulated with the diagonal matrix $\mathbf{R}$, which bears asymptotically similar functions to the ones presented in Table 1 on its diagonal to regularize the entries in $\mathbf{A}$:

$$x=\mathbf{R}y\to\begin{bmatrix} R_n^{(m)} \\ R_n^{(\upsilon)} \end{bmatrix}=\begin{bmatrix} F_n^m(a_m) & 0 \\ 0 & F_n^{\upsilon}(a_\upsilon) \end{bmatrix}\begin{bmatrix} \tilde{R}_n^{(m)} \\ \tilde{R}_n^{(\upsilon)} \end{bmatrix}. \tag{3.8}$$

The two choices of $F_n^m(\rho_m)$ and the corresponding left operator $\mathbf{L}$ to transform the LAES1 $\mathbf{A}x=b$ into a LAES2 $(\mathbf{I}+\mathbf{H})y=g$ where $y,g\in l_2$, as $(\mathbf{I}+\mathbf{H})=\mathbf{LAR}$, $y=\mathbf{R}^{-1}x$, and $g=\mathbf{L}b$ with $\mathbf{I}$ the unit operator are tabulated in Table 2.

Table 1. Asymptotes and upper bound estimates of radial functions.

Function	Asymptotic form while n$\to\infty$	Upper bound estimate while n$\to\infty$
$\lvert J_n(t)\rvert$	$\sim\frac{1}{\sqrt{2\pi n}}\left\lvert\left(\frac{et}{2n}\right)^n\right\rvert$	$\le\frac{e^{\lvert Imt\rvert}}{n!}\left\lvert\left(\frac{t}{2}\right)^n\right\rvert$
$\left\lvert H_n^{(1)}(t)\right\rvert$	$\sim\sqrt{\frac{2}{\pi n}}\left\lvert\left(\frac{2n}{et}\right)^n\right\rvert$	$\le\frac{(n-1)!}{e^{\lvert Imt\rvert}}\left\lvert\left(\frac{2}{t}\right)^n\right\rvert$

Table 2. Regularizing functions of right operator R.

$F_n^m(t)$	Upper bound estimate while n$\to\infty$	Vector Λ for left operator L = diagΛ
$J_n^m(t)$	$\le\frac{e^{\lvert Imt\rvert}}{(n-1)!}\left\lvert\left(\frac{t}{2}\right)^{n-1}\right\rvert$	$\begin{bmatrix} (J_n^m(a_m))^{-1}\big/H_n^{(1)m}(a_m) \\ (J_n^{\upsilon}(a_\upsilon))^{-1}\big/H_n^{(1)\upsilon}(a_\upsilon) \end{bmatrix}^T$
$\frac{1}{H_n^{(1)m}(t)}$	$\le\frac{e^{\lvert Imt\rvert}}{(n-1)!}\left\lvert\left(\frac{t}{2}\right)^{n}\right\rvert$	$\begin{bmatrix} I \\ I \end{bmatrix}^T$

Letting $\Lambda_{1,2}$ be some real-valued constants of asymptotic analysis, both functions in Table 2 lead to the following inequalities for upper bounds of the nonzero entries of $\mathbf{H}$ while n,s$\to\infty$,

$$\mathbf{H}=\begin{bmatrix} 0 & K_I \\ K_{II} & 0 \end{bmatrix};\qquad K_I=\left\{k_{ns}^{(1)}\right\}_{n,s=-\infty}^{\infty};\qquad K_{II}=\left\{k_{ns}^{(2)}\right\}_{n,s=-\infty}^{\infty}$$
$$\left\lvert k_{ns}^{(1)}\right\rvert<\Lambda_1\times\left[\frac{(\lvert n\rvert+\lvert s\rvert)!}{\lvert n-1\rvert!\lvert s-1\rvert!}\right]\left(\frac{a_m}{d_{m\upsilon}}\right)^{\lvert n\rvert}\left(\frac{a_\upsilon}{d_{m\upsilon}}\right)^{\lvert s\rvert};\quad\left\lvert k_{ns}^{(2)}\right\rvert<\Lambda_2\times\left[\frac{(\lvert n\rvert+\lvert s\rvert)!}{\lvert n-1\rvert!\lvert s-1\rvert!}\right]\left(\frac{a_\upsilon}{d_{m\upsilon}}\right)^{\lvert n\rvert}\left(\frac{a_m}{d_{m\upsilon}}\right)^{\lvert s\rvert} \tag{3.9}$$

and prove that $\mathbf{H}$ is a compact operator in l_2 as long as $d_{m\upsilon}>\ a_m+a_\upsilon$, which is naturally satisfied.

4. Numerical results

The numerical results below are calculated on a couple of impedance cylinders with radius $ka_{1,2}=1$ and identical surface impedances $Z_{1,2}=Z$ when their centers are aligned on the x-axis and $kd=5$ unless otherwise indicated.

In Figure 2, the bistatic radar cross-section (RCS) is plotted for several real and imaginary impedance values to make a comparison of the obtained results with the ones provided by another solver having the other time dependence, $e^{+i\omega t}$ (Ansoft HFSS). The incident field is a TM-polarized plane wave approaching perpendicularly (i.e. $\phi^{incidence} = \pi/2$) to the common axis of both cylinders. Results are verified with perfect agreement, as can be seen in Figure 2.

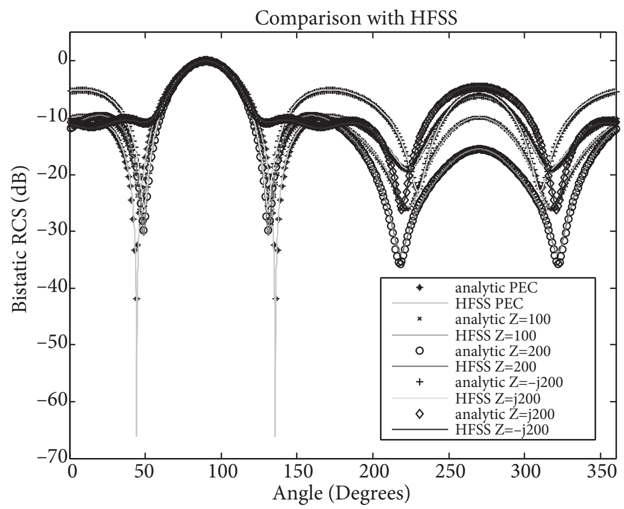

Figure 2. Comparison of bistatic RCS results of suggested technique and an EM software package for several impedance values. Imaginary impedance values vary due to different time dependences of the solvers.

In the following two figures, $Z_{1,2} = 100+100i$ and the solutions obtained after regularization as explained in Section 3.2, before regularization as in Eq. (2.20), and before regularization according to Ivanov [2] (and [28]) as in Eq. (2.21) are being compared. Among the three mentioned solutions, only the first is well-conditioned and Figures 3 and 4 are both meant to highlight this via several mathematical and physical properties.

In Figure 3, the abscissa is the truncation number N while representing an infinite system index $n \in (-\infty, \infty)$ via $n \in [-N, N]$ per-block for each boundary. Thus, the algebraic system size for the two cylinder system is obtained through doubling the block size, i.e. $(2\times(2\mathrm{N}+1)) \times (2\times(2\mathrm{N}+1))$ for the considered cases below. The Euclidean norm [30] condition number $\nu = ||\mathrm{A}||_2||\mathrm{A}^{-1}||_2$, and the rank of the matrices involved and the maximum deviation from satisfaction of the boundary conditions on $BoC_{1,2}$ are the ordinates in the subplots. With increasing N, the well-conditioned system

1. has a bounded ν while the others have quickly growing ν,

2. has a linearly increasing rank equal to the number of equations while the others are rank-deficient, and

3. satisfies the boundary condition perfectly with machine precision while the boundary conditions for others remain nonzero.

In Figure 4, the algebraic system size is fixed to 162×162 (i.e. $\mathrm{N} = 40$). The abscissa is the whole period samples of the polar angle. While the ordinate is the current density on the cylinder surfaces, the abscissa represents the points on circular surfaces in its own polar coordinates, and while the ordinate is the bistatic RCS, the abscissa represents the direction in the far zone.

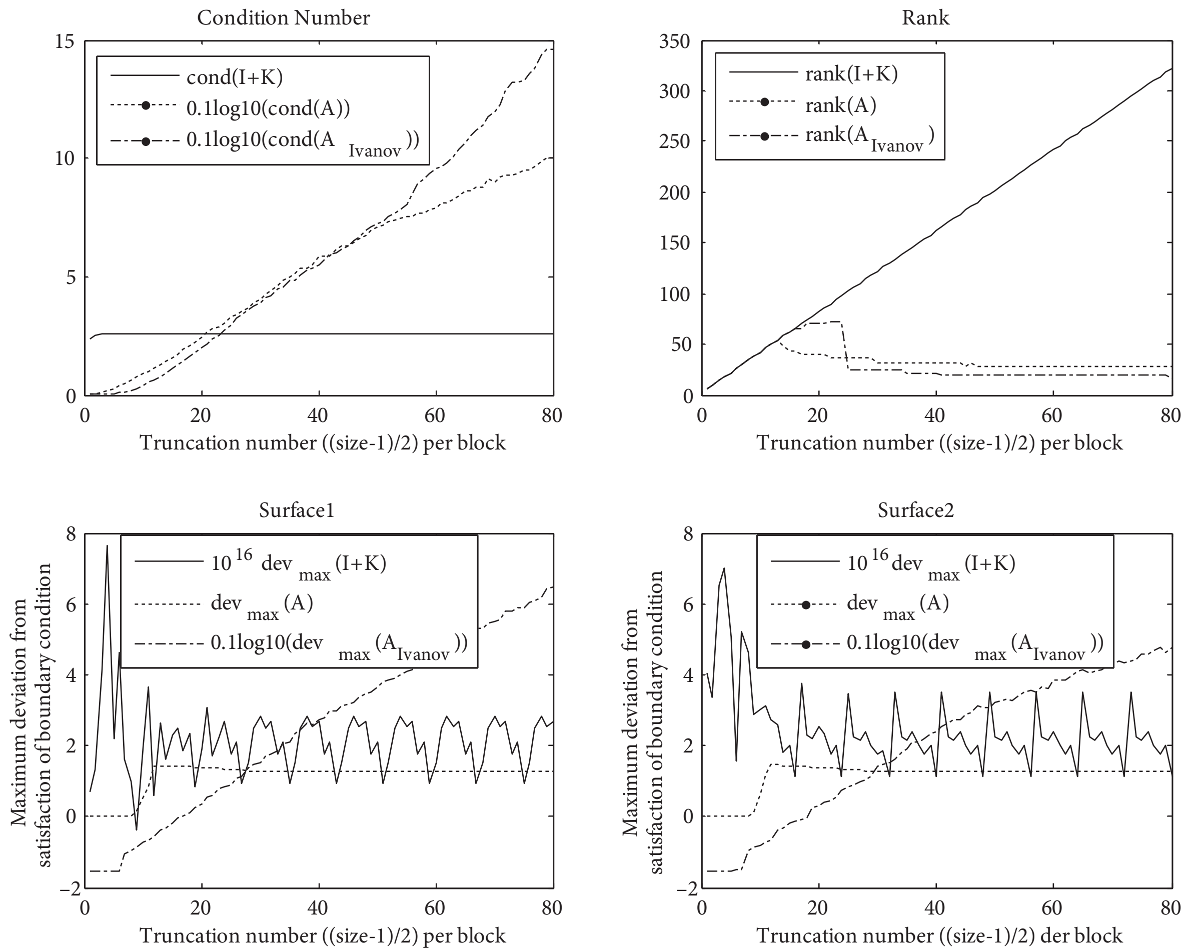

Figure 3. The condition number, the rank, and the performance of satisfying the boundary conditions for the considered algebraic systems with increasing system dimensions are depicted. Their values are conveniently scaled when necessary as indicated in legends to fit the plot frame for a good perception of their comparison.

In all the subplots in Figure 4, the effect of the deviation from boundary conditions on the surface currents and the bistatic RCS are clearly observed, the correct one being obtained by the well-conditioned system, i.e. after regularization.

The LAES1, i.e. the algebraic system before regularization, is without any doubt extremely ill-conditioned according to the depictions in Figures 3 and 4, especially after having a look at the indicators stated above in points a and 5. The plots and calculations here are performed via MATLAB, making such analysis extremely easy with the handy tools provided there for matrices and their solutions either being via LU decomposition or iterative methods. Despite being extremely ill-conditioned and rank-deficient for N = 40, the MATLAB LU solver kept on working perfectly well for the LAES1 of Eqs. (2.20) and (3.6), while it expectedly failed for the LAES1 of Eq. (2.21) according to Ivanov [2]. That is why we switched to the "quasi-minimal residual" (qmr) method as one of many options of MATLAB provided as iterative solvers. The LAES2 explained in Section 3.2, i.e. the well-conditioned algebraic system, quickly converged to its solution with the qmr, but LAES1 never did, as expected. Figures 3 and 4 are depicted via such failing solutions of the LAES1 reflecting its true mathematical nature. It seemed that the inner measures that MATLAB takes to enhance its performance masked this poor property of the LAES1 in Eq. (3.6). Therefore, we performed the LU decomposition solution in C++ with

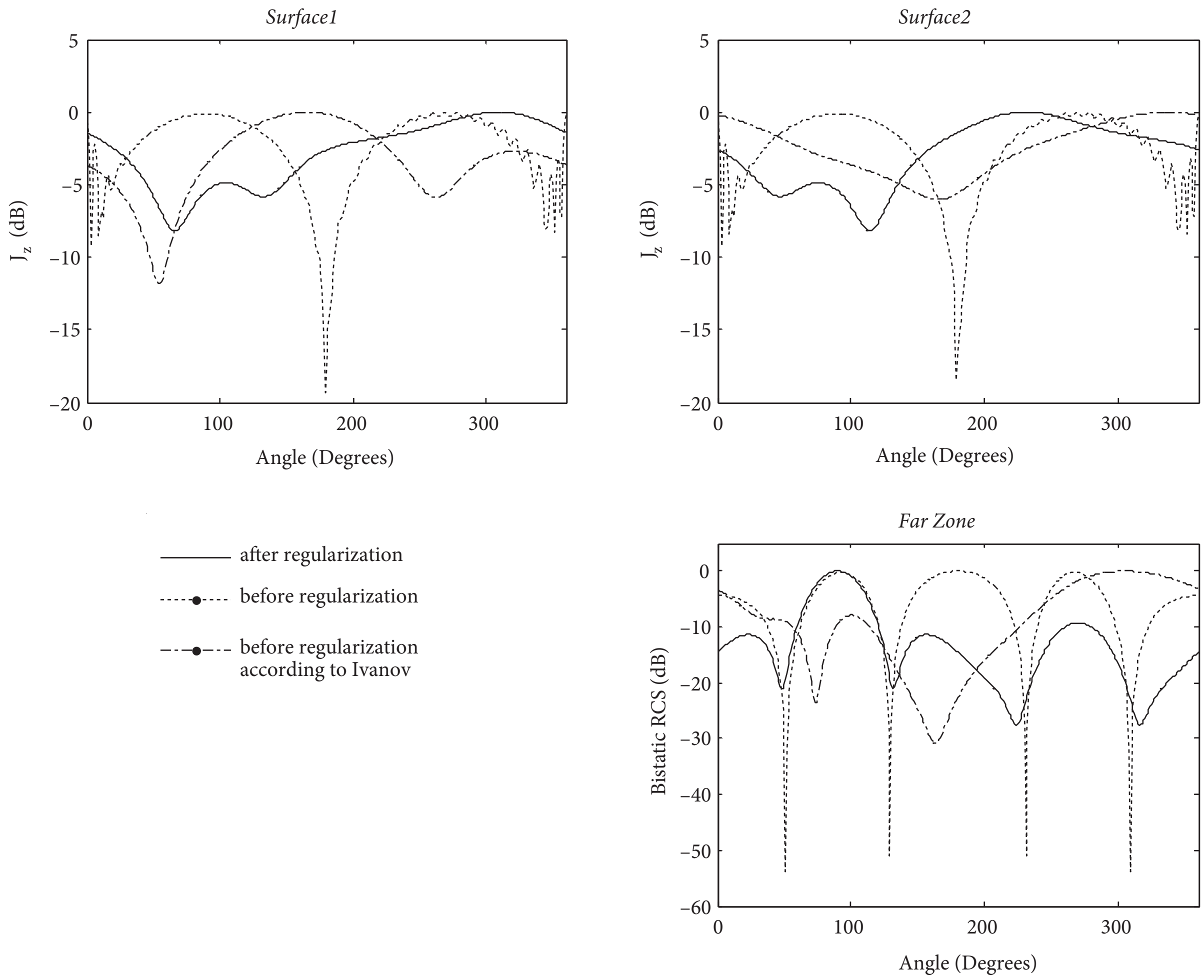

Figure 4. For a fixed truncation number per block (i.e. N = 40) the currents on the impedance surfaces and the bistatic RCS of the scattered field from them are depicted via all the considered algebraic systems.

its double precision set to 16 significant digits. The maximum deviation from the satisfaction of the boundary condition calculated via this solution unveiled the mask over LU solutions of LAES1 in Eq. (3.6) that MATLAB kept. With increasing N, the boundary condition is far from being satisfied via the several truncated LAES1 under consideration according to the tabulation in Table 3.

Table 3. Maximum deviation from satisfaction of the boundary condition by a LU solver of mantissa length 10^{16} solving via matrix A.

Truncation number ((size-1)/2) per block	Maximum deviation from satisfaction of the boundary condition	
	Surface 1	Surface 2
10	2.107708154721642e-08	2.110741476084647e-08
20	4.199639234666350e-02	2.110740774266010e-08
40	9.852222875038121e+11	2.110740707965443e-08
80	5.767939322592583e+39	1.592155168152142e+40
100	3.172408189417764e+64	5.632395455446972e+65

5. Conclusion

In this study, an example of an analytical formulation with an ill-conditioned numerical scheme and its remedy by means of two-sided regularization is presented to emphasize the pitfalls of a numerical treatment during a scientific computation with analytical formulations.

For both TE and TM polarizations, scattering of the electromagnetic waves by two circular impedance cylinders is represented based on the integral formulation [1]. Its essence is delineated thorough the formulation of the problem for the TM case. The TE polarization case solutions can easily be obtained from the resulting TM polarization expressions only by noticing that H_z is taken instead of E_z as the scattered field in Eq. (2.5) and, $\zeta = -Z/Z_m$ in Eq. (3.6) where the IBC $H_z = (1/Z_m)E_\phi$ is suppressed instead of Eq. (2.3) due to Eq. (2.1). The corresponding boundary value problem set via the IBC resulted in an infinite linear algebraic equation of the first kind that is subject to severe round-off errors during its solution. We revived an old regularization approach with its new version with alternatives for the considered problem. It transformed the ill-conditioned equation system into an infinite linear algebraic equation of the second kind in l_2, thus well-conditioning it for solving it by means of a numerical truncation procedure. This well-conditioning is demonstrated not only by the plots of the mathematical indicators such as the condition numbers and ranks of both algebraic systems, but also the physical quantities such as the satisfaction of the boundary condition and surface currents on the boundaries and the bistatic radar cross-section in the far field. They also proved that the numerical solutions obtained via the well-conditioned system do not necessitate verification with physical requirements such as reciprocity of the fields or boundary conditions in the first place. The regularization technique itself is based on the analytical formulation of the considered problem and obtained results are tested via Ansoft HFSS with perfect agreement between them. For such EM solver packages and other numerical approaches (e.g., [35]) it can be used as a comparison and benchmarking tool. This can help test the validity and capability of the methods in both ways and lead to an extension of the considered simple problem to a more general one.

Future perspectives of the analytical model involved in this paper include formulation for more than two impedance circular cylinders, which can be inside a circular cavity (or a waveguide) with impedance walls and in free space under oblique incident plane wave excitation.

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References

[1] Bates RHT. Analytic constraints on electromagnetic field computations. IEEE T Microw Theory 1975; 23: 605–623.

[2] Ivanov EA. Diffraction of Waves from Two Bodies. Minsk, Belarus: Nauka i Tekhnika, 1968 (in Russian).

[3] Poyedinchuk AY, Tuchkin YA, Shestopalov VP. New numerical-analytical methods in diffraction theory. Math Comput Model 2000; 32: 1029–1046.

[4] Ergül Ö. Analysis of composite nanoparticles with surface integral equations and the multilevel fast multipole algorithm. J Opt 2012; 14: 062701.

[5] Anastassiu HT. Error estimation of the method of auxiliary sources (MAS) for scattering from an impedance circular cylinder. Prog Electromagn Res 2005; 52: 109–128.

[6] Ramm AG. Scattering of electromagnetic waves by many nano-wires. Mathematics 2013; 1: 89–99.

[7] Salem MA, Kamel AH. Scattering by an impedance cylinder immersed halfway between dielectric-metamaterial half-spaces. In: EMTS 2013 International Symposium on Electromagnetic Theory; 20–24 May 2013; Hiroshima, Japan. p. 287.

[8] Dikmen F, Vatansever S, Tuchkin YA. A well-conditioned algebraic system for scattering from eccentrically layered circular cylinders. In: EMTS 2013 International Symposium on Electromagnetic Theory; 20-24 May 2013; Hiroshima, Japan. p. 1047.

[9] Wilkinson JH. The Algebraic Eigenvalue Problem. Oxford, UK: Clarendon Press, 1975.

[10] Schukin AN. Radio-Waves Propagation. Moscow, Russia: Sviaz'izdat, 1940 (in Russian).

[11] Leontovich MA. On approximate boundary conditions for electromagnetic field on the surface of well-conductive bodies. Investigation on Wave Propagation 1948; 2: 5–12 (in Russian).

[12] Rytov CM. Calculation of skin-effect by perturbation method, J Exp Theor Phys 1940; 10: 180–190 (in Russian).

[13] Slepyan GY, Maksimenko SA, Lakhtakia A, Yevtushenko O, Gusakov AV. Electrodynamics of carbon nanotubes: dynamic conductivity, impedance boundary conditions, and surface wave propagation. Phys Rev B 1999; 60: 17136–17149.

[14] Dykhnea AM, Kaganova IM. The Leontovich boundary conditions and calculation of effective impedance of inhomogeneous metal. Opt Commun 2002; 206: 39–56.

[15] Ammari H, Latiri-Grouz C, Nedelec JC. Scattering of Maxwell's equations with a Leontovich boundary condition in an inhomogeneous medium: a singular perturbation problem. SIAM J Appl Math 1999; 59: 1322–1334.

[16] Theron IP, Cloete JH. On the use of the Leontovich boundary condition in the analysis of microstrip antennas. In: Antennas and Propagation Society International Symposium; 18–25 June 1992; Chicago, IL, USA. pp. 620–623.

[17] Hoppe DJ, Rahmat-Samii Y. Impedance Boundary Conditions in Electromagnetics. Washington, DC, USA: Taylor & Francis, 1995.

[18] Yuferev SV, Ida N. Surface Impedance Boundary Conditions. A Comprehensive Approach. Boca Raton, FL, USA: CRC Press, 2009.

[19] Balanis CA. Advanced Engineering Electromagnetics. New York, NY, USA: John Wiley & Sons Inc., 1989.

[20] Colton DL, Kress R. Integral Equations Methods in Scattering Theory. Malabar, FL, USA: Krieger Publishing Company, 1992.

[21] Bers L, John F, Schechter M. Partial Differential Equations. New York, NY, USA: Interscience, 1964.

[22] van den Berg PM, Fokkema JT. The Rayleigh hypothesis in the theory of reflection by a grating. J Opt Soc Am 1979; 69: 27–31.

[23] van den Berg PM, Fokkema JT. The Rayleigh hypothesis in the theory of diffraction by a perturbation in a plane surface. Radio Sci 1980; 15: 723–732.

[24] Apelt'cin VF, Kurkchan AG. Analytical Properties of Wave's Fields. Moscow, Russia: Moscow University Press, 1989 (in Russian).

[25] Chew WC. Waves and Fields in Inhomogeneous Media. New York, NY, USA: Wiley-IEEE Press, 1999.

[26] Abramowitz M, Stegun IA. Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. New York, NY, USA: Dover Publications, 1972.

[27] Il'inski AS, Smirnov YG. Electromagnetic Wave Diffraction by Conducting Screens: Pseudodifferential Operators in Diffraction Problems. Boca Raton, FL, USA: CRC Press, 1998.

[28] Scharstein RW. Acoustic scattering from two parallel soft cylinders. In: Southeastcon '92 Proceedings; 12–15 April 1992; Birmingham, AL, USA. New York, NY, USA: IEEE. pp. 534–537.

[29] Poyedinchuk AE, Tuchkin YA, Shestopalov VP. On regularization of the spectral problems of wave scattering by unclosed screens. Doklady AN SSSR 1987; 295: 1328–1342.

[30] Kantorovich LV, Akilov GP. Functional Analysis in Normed Spaces (Translated from Russian). Oxford, UK: Pergamon Press, 1982.

[31] Shestopalov VP, Tuchkin YA, Poyednichuk AY, Sirenko YK. Novel Methods for Solving Direct and Inverse Problems of Diffraction Theory, Vol. 1: Analytical Regularization of Electromagnetic Boundary Value Problems. Kharkov, Ukraine: Osnova, 1997 (in Russian).

[32] Dikmen F, Tuchkin YA. Analytical regularization method for electromagnetic wave diffraction by axially symmetrical thin annular strips. Turk J Electr Eng Co 2009; 17: 107–124.

[33] Özkan E, Dikmen F, Tuchkin YA. Scalar wave diffraction by perfectly soft thin circular cylinder of finite length; analytical regularization method. Turk J Electr Eng Co 2002; 10: 459–472.

[34] Dikmen F, Karaçuha E, Tuchkin YA. Scalar wave diffraction by a perfectly soft infinitely thin circular ring. Turk J Electr Eng Co 2001; 9: 199–220.

[35] Fostier J, Michiels B, Bogaert I, De Zutter D. A fast 2-D parallel multilevel fast multipole algorithm solver for oblique plane wave incidence. Radio Sci 2011; 46: RS6006.