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Some properties of digital H-spaces

Özgür EGE[1,*], **İsmet KARACA**[2]
[1]Department of Mathematics, Celal Bayar University, Manisa, Turkey
[2]Department of Mathematics, Ege University, İzmir, Turkey

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Abstract: In this paper, we study certain properties of digital H-spaces. We prove that a digital image that has the same digital homotopy type with any digital H-space is also a digital H-space. We show that the digital fundamental group of a digital H-space is abelian. We give examples that are related to a digital homotopy associative H-space and a κ-contractible digital H-space. Several important applications of digital H-spaces are given in computer vision and image processing. Finally, we deal with the importance of digital H-space in digital topology and image processing. We conclude that any κ-contractible digital image is a digital H-space.

Key words: Digital image, digital H-space, digital H-group, digital homotopy, image processing

1. Introduction

Although algebraic topology is an area of pure mathematics, there are several applications of algebraic topology in engineering and science. Interesting techniques have been applied, especially in computer vision and image processing. In the last decade, homotopy and homology groups, which are significant tools in algebraic topology, play an important role for some problems of computer vision and image processing with computational properties.

Digital image processing and pattern recognition are developing fields with various applications in medicine, geology, biology, industry, etc. The principal aim of digital topology is to determine the topological properties of discrete objects. Researchers have determined significant properties of two- and three-dimensional digital images with tools from topology and algebraic topology. In homotopy theory, a variety of properties of topological spaces that are invariant under continuous deformations are studied. Computer graphics, geometric modeling, computer vision, and digital image analysis are important usage areas of digital topology.

H-spaces play an important role in the basic properties in homotopy theory. For example, a fundamental group of an H-space is abelian. Additionally, Adams [1] proved that the spheres S^0, S^1, S^3, S^7 are H-spaces. Digital homotopy groups that can be related to digital H-spaces are used to classify digital images. A digital homotopy group is a significant invariant for image analysis. This generalizes the digital fundamental group, which gives information about loops and holes of a space. A general method that is related to deciding whether two digital images have isomorphic homotopy groups could be a powerful tool for image processing. We think that digital H-spaces will be a very useful tool in determining digital homotopy groups.

Researchers in this area wish to characterize the properties of digital images. Since we aim to get important results for digital H-spaces, we benefit from [2]. Ayala et al. [3] showed that the digital fundamental

*Correspondence: ozgur.ege@cbu.edu.tr

group of a digital object is isomorphic to the fundamental group of its continuous analog and give a digital version of the Seifert–Van Kampen theorem. Boxer [4] dealt with the digital versions of several notions from topology, including homeomorphism, retraction and homotopy. He [5] introduced the digital fundamental group by using classical methods of algebraic topology. He also [6] studied various digital continuous functions that preserve homotopy types and gave [7] several theorems about covering spaces. Boxer and Karaca [8] investigated the conditions under which the fundamental group of a Cartesian product of digital images is isomorphic to the product of the fundamental groups of the factors. Ege and Karaca [9] studied the fixed point properties of digital images. They [10] introduced a digital H-space, a digital H-group, and a digital H-map between digital H-spaces, and also obtained some results about digital H-spaces. Ege and Karaca [11] constructed a cohomology theory of digital images and defined a digital simplicial cup product. The digital fundamental group was discussed in [12,13]. Herman [14] gave several adjacency relations. He [15] presented an algorithm to find any finite boundary between two components in any binary picture in any finitary 1-simply connected digital space. Kong [16] defined a digital fundamental group of discrete objects in the 3D digital Euclidean space, which gives us information about the structure of the digital image. Kopperman [17] gave an approximation on finite spaces and digital topology. Kovalevsky [18] introduced a notion of a cell list and data structure for encoding segmented images. He also gave some applications of these data to image analysis. Malgouyres [19] gave a complete algebraic presentation of the digital fundamental group of any object in a 2D digital image. He concluded that two 2D connected objects have isomorphic fundamental groups if and only if they have the same number of holes. Mazo et al. [20] proved that the fundamental group of a digital space is isomorphic to the fundamental-like group, which is generally considered in digital image analysis. They also used finite spaces for image analysis and processing. Rosenfeld and Kak [21] presented some concepts and mathematical techniques of digital image processing and analysis. Rosenfeld [22] addressed continuous functions between two digital images. Since H-spaces were widely characterized in [23,24], we also benefit from them. In this paper, we prove that a digital image that has the same digital homotopy type with any digital H-space is also a digital H-space, and we show that the digital fundamental group of a digital H-space is abelian.

This paper is organized as follows. In the preliminaries section, we give some basic definitions related to digital topology such as digital κ-adjacencies, a digital (κ_1, κ_2)-continuous function, digital (κ_1, κ_2)-isomorphism, and digital (κ_1, κ_2)-homotopy. In the next section we introduce a digital H-space and a digital H-group. Moreover, we give examples of digital homotopy-associative H-space and κ-contractible digital H-spaces. In Section 4, we deal with several applications of digital H-spaces in image processing and computer vision. Finally, we arrive to certain important conclusions about digital H-spaces.

2. Preliminaries

In this work, we denote the set of integers by $\mathbb{Z}$. A finite subset of $\mathbb{Z}^n$ with an adjacency relation is called a digital image, which is denoted by (X, κ), where $\mathbb{Z}^n$ represents the set of lattice points in Euclidean n-dimensional space and κ is an adjacency relation for the members of X. Various adjacency relations are used in the study of digital images.

Definition 2.1. [16]. *Consider the following statements:*

(1) Two points p and q in $\mathbb{Z}$ are 2-adjacent if $|p-q| = 1$ (Figure 1).

(2) Two points p and q in $\mathbb{Z}^2$ are 8-adjacent if they are distinct and differ by at most 1 each coordinate.

(3) Two points p and q in $\mathbb{Z}^2$ are 4-adjacent if they are 8-adjacent and differ by exactly one coordinate (see Figure 2).

(4) Two points p and q in $\mathbb{Z}^3$ are 26-adjacent if they are distinct and differ by at most 1 each coordinate.

(5) Two points p and q in $\mathbb{Z}^3$ are 18-adjacent if they are 26-adjacent and differ by at most two coordinates.

(6) Two points p and q in $\mathbb{Z}^3$ are 6-adjacent if they are 18-adjacent and differ by exactly one coordinate (Figure 3).

Figure 1. The 2-adjacency.

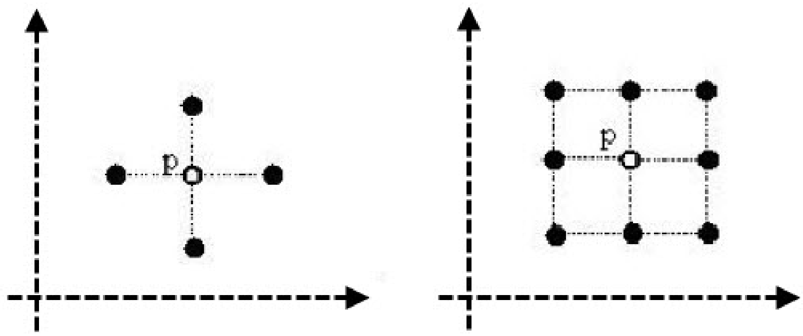

Figure 2. The 4-adjacency and 8-adjacency.

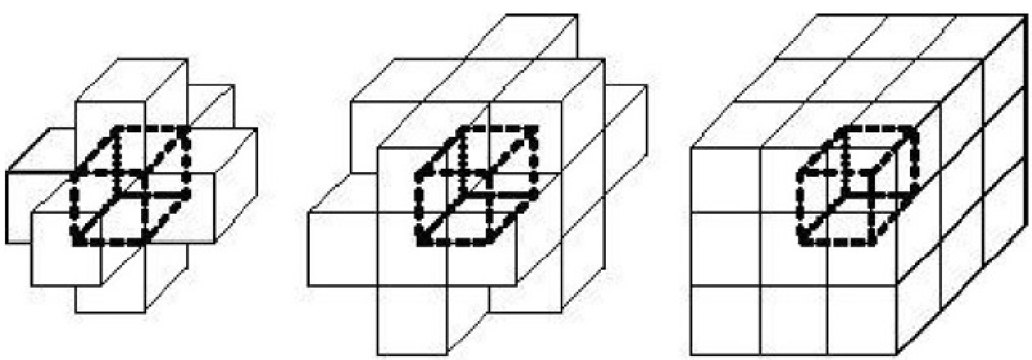

Figure 3. The 6-adjacency, 18-adjacency, and 26-adjacency.

Let κ be an adjacency relation defined on $\mathbb{Z}^n$. A κ-neighbor of $p \in \mathbb{Z}^n$ is a point of $\mathbb{Z}^n$ that is κ-adjacent to p. A digital image $X \subset \mathbb{Z}^n$ is κ-connected [14] if and only if for every pair of different points $x, y \in X$, there is a set $\{x_0, x_1, \ldots, x_r\}$ of points of a digital image X, such that $x = x_0$, $y = x_r$ and x_i and x_{i+1} are κ-neighbors where $i \in \{0, 1, \ldots, r-1\}$. A κ-component of a digital image X is a maximal κ-connected subset of X. Let $a, b \in Z$ with $a < b$. A digital interval [4] is a set of the form

$$[a, b]_{\mathbb{Z}} = \{z \in \mathbb{Z} | a \leq z \leq b\}.$$

For the Cartesian product of two digital images, the adjacency is defined as follows:

Definition 2.2 [12]. *Given two points* $x_i, y_i \in (X_i, \kappa_i)$, $i \in \{0,1\}$, (x_0, x_1) *and* (y_0, y_1) *are adjacent in* $X_0 \times X_1$ *if and only if one of the following is satisfied:*

a) $x_0 = y_0$ and x_1 and y_1 are κ_1-adjacent; or

b) x_0 and y_0 are κ_0-adjacent and $x_1 = y_1$; or

c) x_0 and y_0 are κ_0-adjacent and x_1 and y_1 are κ_1-adjacent.

We will denote κ_* for the adjacency of the Cartesian product of digital images (X_0, κ_0) and (X_1, κ_1).

Definition 2.3 [5]. *Let* $(X, \kappa_1) \subset \mathbb{Z}^{n_1}$ *and* $(Y, \kappa_2) \subset \mathbb{Z}^{n_2}$ *be digital images. A function* $f : X \to Y$ *is called* (κ_1, κ_2)*-continuous if the image under* f *of every* κ_1*-connected subset of* X *is a* κ_2*-connected subset of* Y.

A function $f : X \to Y$ is (κ_1, κ_2)-continuous [5,22] if and only if for every κ_1-adjacent points $\{x_1, x_2\}$ of X, either $f(x_1) = f(x_2)$ or $f(x_1)$ and $f(x_2)$ are κ_2-adjacent in Y.

Let $(X, \kappa_1) \subset \mathbb{Z}^{n_1}$ and $(Y, \kappa_2) \subset \mathbb{Z}^{n_2}$ be digital images. A function $f : X \to Y$ is a (κ_1, κ_2)-isomorphism [7] if f is (κ_1, κ_2)-continuous and bijective and $f^{-1} : Y \to X$ is (κ_2, κ_1)-continuous.

Definition 2.4 [5]. *Let* $X \in \mathbb{Z}^{n_1}$ *and* $Y \in \mathbb{Z}^{n_2}$ *be digital images with* κ_1*-adjacency and* κ_2*-adjacency, respectively. Two* (κ_1, κ_2)*-continuous functions* $f, g : X \to Y$ *are said to be digitally* (κ_1, κ_2)*-homotopic in* Y *if there is a positive integer* m *and a function* $H : X \times [0, m]_{\mathbb{Z}} \to Y$, *such that:*

- for all $x \in X$, $H(x, 0) = f(x)$ and $H(x, m) = g(x)$;
- for all $x \in X$, the induced function $H_x : [0, m]_{\mathbb{Z}} \to Y$, defined by

$$H_x(t) = H(x, t) \quad forall\ t \in [0, m]_{\mathbb{Z}},$$

  is $(2, \kappa_2)$-continuous; and
- for all $t \in [0, m]_{\mathbb{Z}}$, the induced function $H_t : X \to Y$, defined by

$$H_t(x) = H(x, t) \quad forall\ x \in X,$$

  is (κ_1, κ_2)-continuous.

The function H is called a digital (κ_1, κ_2)-homotopy between f and g. The notation $f \simeq_{(\kappa_1, \kappa_2)} g$ is used to indicate that functions f and g are digitally (κ_1, κ_2)-homotopic in Y. The digital (κ_1, κ_2)-homotopy relation [5] is one of equivalence among digitally continuous functions $f : (X, \kappa_1) \to (Y, \kappa_2)$.

Definition 2.5 [5]. *Let* $f : X \to Y$ *be a* (κ_1, κ_2)*-continuous function and let* $g : Y \to X$ *be a* (κ_2, κ_1)*-continuous function, such that* $f \circ g \simeq_{(\kappa_2, \kappa_2)} 1_Y$ *and* $g \circ f \simeq_{(\kappa_1, \kappa_1)} 1_X$. *Then we say that* X *and* Y *have the same* (κ_1, κ_2)*-homotopy type and that* X *and* Y *are* (κ_1, κ_2)*-homotopy equivalent.*

Definition 2.6 [4]. *(i) A digital image* (X, κ) *is said to be* κ*-contractible if its identity map is* (κ, κ)*-homotopic to a constant function* $\bar{c}$ *for some* $c \in X$, *where the constant function* $\bar{c} : X \to X$ *is defined by* $\bar{c}(x) = c$ *for all* $x \in X$.

(ii) We say that a (κ_1, κ_2)*-continuous function* $f : X \to Y$ *is* κ_2*-nullhomotopic if* f *is* κ_2*-homotopic to a constant function* $\bar{c}$ *in* Y*.*

(iii) Let (X, A) *be a digital image pair with* κ*-adjacency. Let* $i : A \to X$ *be the inclusion function.* A *is called a* κ*-retract of* X *if and only if there is a* κ*-continuous function* $r : X \to A$ *such that* $r(a) = a$ *for all* $a \in A$*. Then the function* r *is called a* κ*-retraction of* X *onto* A*.*

Definition 2.7 [6]. *A digital homotopy* $H : X \times [0, m]_{\mathbb{Z}} \to X$ *is a* κ*-deformation retract if the induced map* $H(-, 0)$ *is the identity map* 1_X *and the induced map* $H(-, m)$ *is* κ*-retraction of* X *onto* $H(X \times \{m\}) \subset X$*. The set* $H(X \times \{m\})$ *is called a* κ*-deformation retract of* X*.*

For a digital image (X, κ) and its subset (A, κ), we call (X, A) a digital image pair with κ-adjacency. Moreover, if A is a singleton set $\{x_0\}$, then (X, x_0) is called a pointed digital image.

3. Digital H-spaces

In this section we explore digital H-spaces. We generally benefit from [2,23,24].

Definition 3.1. *Let* (X, p, κ) *be a pointed digital image. For a digital continuous multiplication* $\mu : X \times X \to X$ *and a digital constant map* $c : X \to X$*, if we have*

$$\mu \circ (c, 1_X) \simeq_{(\kappa,\kappa)} 1_X \simeq_{(\kappa,\kappa)} \mu \circ (1_X, c),$$

then (X, p, κ) *is called a digital H-space.*

Example 3.2. *Let* $X = \{p\}$ *be a single point digital image with* κ*-adjacency. We now show that* (X, p, κ) *is a digital H-space. Since*

$$\mu \circ (c, 1_X)(p) = \mu(c(p), 1_X(p)) = \mu(p, p) = p = \mu(1_X(p), c(p)) = \mu \circ (1_X, c)(p)$$

where $c : X \to X$ *is a digital constant map, i.e.* $c(p) = p$*, we have*

$$\mu \circ (c, 1_X) \simeq_{(\kappa,\kappa)} \mu \circ (1_X, c) \simeq_{(\kappa,\kappa)} 1_X.$$

As a result, we find that (X, p, κ) *is a digital H-space.*

Definition 3.3. *A digital H-space* (X, p, κ) *is called digital homotopy associative if we have* $\mu \circ (1_X \times \mu) \simeq_{(\kappa^*,\kappa)} \mu \circ (\mu \times 1_X)$*, where* $\mu : X \times X \to X.$

Example 3.4. *Let* $X = [0, 1]_{\mathbb{Z}}$*. Then* $(X, p, 2)$ *is a digital H-space where* $p = 0$ *or* 1*. We first determine images of* $X \times X \times X$ *and* $X \times X$*.*

$$X \times X \times X = MSS_6^{'} = [0, 1]_{\mathbb{Z}} \times [0, 1]_{\mathbb{Z}} \times [0, 1]_{\mathbb{Z}} \subset \mathbb{Z}^3$$

is a digital image with 6*-adjacency [13] (Figure 4).*

On the other hand, $X \times X = [0, 1]_{\mathbb{Z}} \times [0, 1]_{\mathbb{Z}} \subset \mathbb{Z}^2$ is a digital image with 4-adjacency (Figure 5).

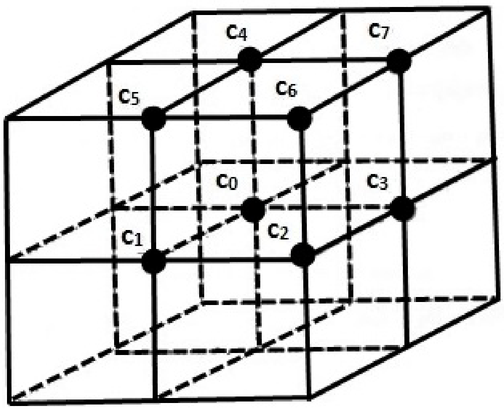

Figure 4. Digital image MSS_6'.

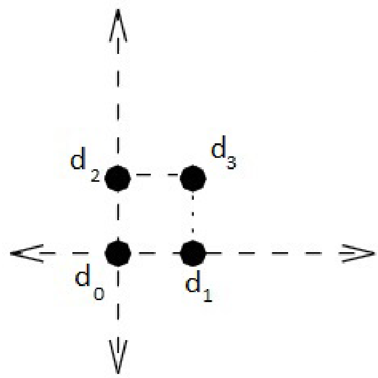

Figure 5. Digital image $X \times X$.

For all $x \in X$, let $c : X \to X$ be defined by $c(x) = p$ where $p \in \{0,1\}$. It is clear that $\mu \circ (c, 1_X) \simeq_{(2,2)} \mu \circ (1_X, c)$. Similarly, for all $(x,y,z) \in X \times X \times X$, we have

$$\mu \circ (1_X, \mu) \simeq_{(6,2)} \mu \circ (\mu, 1_X).$$

As a result, $(X, p, 2)$ is a digital $(6,2)$-homotopy associative H-space.

Definition 3.5. *Let (X,p,κ) be a digital H-space. A map $\eta : (X,p,\kappa) \to (X,p,\kappa)$ is called a digital homotopy inverse for (X,p,κ) if we have*

$$\mu \circ (\eta, 1_X) \simeq_{(\kappa,\kappa)} \mu \circ (1_X, \eta) \simeq_{(\kappa,\kappa)} p$$

where $p(x) = p$, $\mu : X \times X \to X$.

Definition 3.6. *For a digital H-space (X,p,κ), if $\mu \circ T \simeq_{(\kappa^*,\kappa)} \mu$, where T is defined by $T(x,y) = (y,x)$, then we say μ is digital homotopy-commutative.*

Definition 3.7. *A digital H-group is a digital H-space (X,p,κ) with the digital homotopy associative multiplication μ and digital homotopy inverse η.*

Definition 3.8. *Let (X,p,κ) and (Y,q,κ') be digital H-spaces. A map $f : X \to Y$ is called a digital H-map if we have $f \circ \mu \simeq_{(\kappa,\kappa')} \mu' \circ (f \times f)$ where $\mu' : Y \times Y \to Y$.*

Theorem 3.9. *Let (X,p,κ) be a digital H-space. If (X,p,κ) and (Y,q,κ') have the same (κ,κ')-homotopy type, then (Y,q,κ') is a digital H-space.*

Proof. Let $g : Y \to X$ be the digital homotopy inverse of $f : X \to Y$. Since (X,p,κ) is a digital H-space, it has a digital continuous multiplication $\mu : X \times X \to X$. If we define $\mu' : Y \times Y \to Y$ by $\mu' = f \circ \mu \circ (g \times g)$, then μ' is a digital continuous multiplication in Y and

$$\mu' \circ (1, c') = f \circ \mu \circ (1, c) \circ g \simeq_{(\kappa',\kappa')} f \circ g$$

where c and c' are digital constant maps. Because $f \circ g \simeq_{(\kappa',\kappa')} 1_Y$, we get

$$\mu' \circ (1, c') \simeq_{(\kappa',\kappa')} 1_Y.$$

Similarly, we have $\mu' \circ (c', 1) \simeq_{(\kappa', \kappa')} 1_Y$. Thus, have we obtained the required result. □

Corollary 3.10. *Any κ-contractible digital image (X, κ) is a digital H-space.*

Example 3.11. *Let 0_n be the origin of $\mathbb{Z}^n$. Boxer [7] defines a sphere-like digital image as follows: $S_n = [-1,1]_{\mathbb{Z}}^{n+1} \setminus \{0_{n+1}\} \subset \mathbb{Z}^{n+1}$. We define a digital version of the real projective line $\mathbb{Z}P^1$ via quotient map from S_1 with antipodal points, and we denote the digital projective line by $\mathbb{Z}P^1$ (Figure 6).*

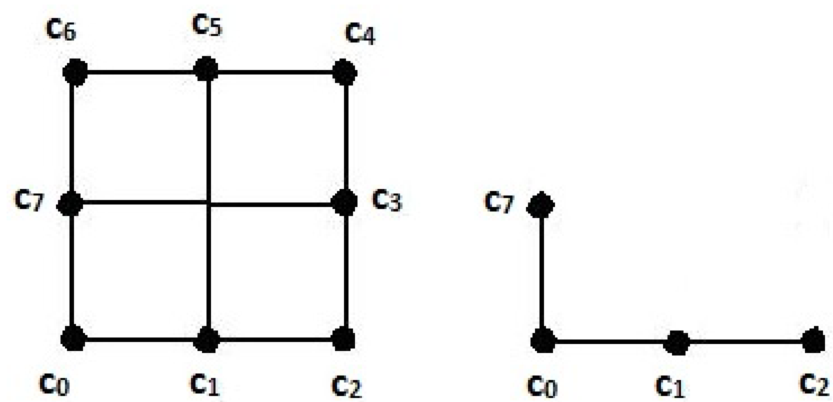

Figure 6. S_1 and the digital projective line $\mathbb{Z}P^1$.

It is easy to show that S_1 is an 8-deformation retract of $\mathbb{Z}P^1$. Therefore, $\mathbb{Z}P^1$ is an 8-contractible image and $\mathbb{Z}P^1$ has the same $(8,2)$-homotopy type as a single point image. As shown by Example 3.2 and Theorem 3.9, $(\mathbb{Z}P^1, 8)$ is a digital H-space.

Example 3.12. *Ege and Karaca [9] defined a digital version of the real projective plane $\mathbb{Z}P^2$ via a quotient map from S_2 with antipodal points. They denoted the digital projective plane by P^2 (Figure 7).*

It is easy to show that S_2 is a 6-deformation retract of P^2. As a result, P^2 is a 6-contractible image and has the same $(6,2)$-homotopy type as a single-point image. As shown by Example 3.2 and Theorem 3.9, $(P^2, 6)$ is a digital H-space.

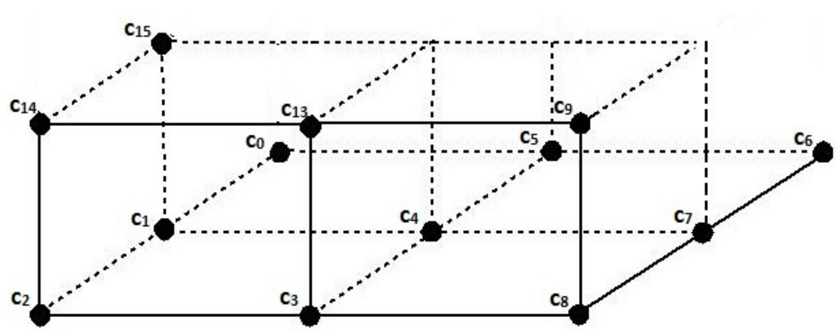

Figure 7. Digital projective plane P^2.

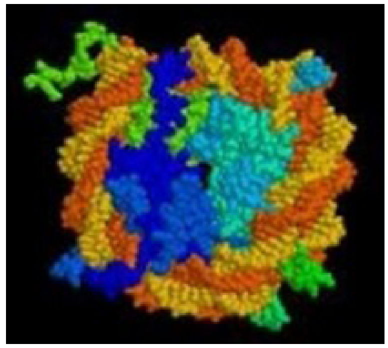

Figure 8. Complex molecule of a protein.

Theorem 3.13. *If (X, p, κ) and (Y, q, κ') are digital H-spaces, then the product space $(X \times Y, p \times q, \kappa^*)$ is a digital H-space, where κ^* is the product adjacency.*

Proof. Let μ and ν be digital continuous multiplication maps for (X, p, κ) and (Y, q, κ'), respectively. Now let θ be a digital multiplication map for $(X \times Y, p \times q, \kappa^*)$. We define θ by $\theta = (\mu \times \nu)(1 \times T \times 1)$, where $T : X \times Y \to Y \times X$ is defined by $T(x, y) = (y, x)$. Since

$$\theta \circ (c, 1)(x, y) = (x, y) \text{ and } \theta \circ (1, c)(x, y) = (x, y),$$

where c is digital constant map and $1_{X \times Y}$ is the identity map on $X \times Y$, we have $\theta \circ (c, 1) \simeq_{(\kappa^*, \kappa^*)} 1_{X \times Y}$ and $\theta \circ (1, c) \simeq_{(\kappa^*, \kappa^*)} 1_{X \times Y}$. As a result, θ is the digital continuous multiplication for $X \times Y$, and $(X \times Y, p \times q, \kappa^*)$ is a digital H-space. □

Denote $[X, Y]$ by the set of pointed digital (κ, κ')-homotopy classes of pointed maps from X to Y. If (Y, q, κ') is a digital H-space and (X, p, κ) is any digital image, then the set $[X, Y]$ can be given binary operation, which is defined as follows:

Let $f, g : X \to Y$ be digital maps and define

$$f + g = \mu \circ (f \times g) \circ d = \mu \circ (f, g).$$

Here, $d : X \to X \times X$ is the diagonal map. If $\alpha = [f]$, $\beta = [g] \in [X, Y]$, then we set $\alpha + \beta = [f + g]$.

Theorem 3.14. *Let (Y, q, κ') be a digital H-group and its digital multiplication map be $\mu : Y \times Y \to Y$. For all digital images(X, p, κ), $[X, Y]$ has a natural group structure given by $[f][g] = [\mu \circ (f, g)]$. If μ is digital homotopy commutative, then $[X, Y]$ is abelian.*

Proof. Since $\mu \circ (1_Y \times \mu) \simeq_{(\kappa^*, \kappa')} \mu \circ (\mu \times 1_Y)$, we have associativity. For every $[f], [g], [h] \in [X, Y]$, we get $[f]([g][h]) = ([f][g])[h]$. We claim that $[X, Y]$ has a two-sided identity element. Let $c : Y \to Y$ and $e : X \to Y$ be digital constant maps. We know that $\mu \circ (1, c) \simeq_{(\kappa', \kappa')} 1_Y$ and $\mu \circ (c, 1) \simeq_{(\kappa', \kappa')} 1_Y$. Since we get $[f][e] = [f]$ and $[e][f] = [f]$, then we have a two-sided identity element $[e]$ for $[X, Y]$.

$[\phi \circ f]$ is a two-sided inverse of f, because $[f][\phi \circ f] = [e]$ where $\phi \circ f$ represents $[f]^{-1}$ and $\phi : Y \to Y$ is a digital homotopy inverse for μ and Y. Thus, $[X, Y]$ is a group. If μ is digital homotopy commutative, then

$$[f][g] = [\mu \circ (f, g)] = [\mu \circ T \circ (f, g)] = [g][f].$$

Thus, we obtain the result. □

Theorem 3.15. *Let (X, p, κ) be a digital H-space, (Y, q, κ') a digital image, and $f : Y \to X$ a pointed map. If f has a left-pointed digital homotopy inverse, then (Y, q, κ') is a digital H-space.*

Proof. Let $r : X \to Y$ be a left-pointed digital homotopy inverse of f. Then $r \circ f \simeq_{(\kappa',\kappa')} 1_Y$. We define a digital multiplication map $\mu_Y : Y \times Y \to Y$ by $\mu_Y = r \circ \mu_X \circ (f \times f)$. It is clear that $\mu_Y \circ (1_Y \times \mu) \simeq_{(\kappa *,\kappa')} \mu_Y \circ (\mu_Y \times 1_Y)$. As a result, Y is a digital H-space. □

We immediately obtain the following.

Proposition 3.16. *If (X, p, κ_1), (Y, q, κ_2), and (Z, r, κ_3) are digital H-spaces and $f : Y \to Z$ is a digital H-map, then $f_* : [X, Y] \to [X, Z]$ is a homomorphism.*

Lemma 3.17. *A κ-retract of a digital H-space (X, p, κ) is a digital H-space.*

Proof. Let (X, p, κ) be a digital H-space and $A \subset X$ be a κ-retract with a κ-retraction $r : X \to A$. We know that X has a digital multiplication map $\mu : X \times X \to X$. Let $i : A \to X$ be an inclusion map. Because $\mu_A = r \circ \mu \circ (i \times i)$, we conclude that μ_A is a digital multiplication map. As a result, (A, q, κ) is a digital H-space where $r(p) = q \in A$. □

Example 3.18. *Let $X = [0, 1]_{\mathbb{Z}}$. By Example 3.4, we know that $(X, p, 2)$ is a digital H-space where $p = 0$ or 1. Consider the subset $A = [0]_{\mathbb{Z}} \subset X$. Since A is a 2-retract of $(X, p, 2)$, from Lemma 3.17 we conclude that one-pointed digital image A is a digital H-space.*

Using Lemma 3.17, we can give the following theorem.

Theorem 3.19. *Let (X, p, κ) and (Y, q, κ') be any two digital images. If $(X \times Y, p \times q, \kappa^*)$ is a digital H-space, then (X, p, κ) and (Y, q, κ') are both digital H-spaces, where κ^* is the product adjacency.*

Proof. •X is a digital κ-retract of $X \times Y$, since there is a κ-retraction map $r_1 : X \times Y \to X$ that can be defined by $r_1(x, y) = x$, such that $r_1 \circ i(x) = x \quad \Rightarrow \quad r_1 \circ i \simeq_{(\kappa,\kappa)} 1_X$ where $i : X \to X \times Y$ is an inclusion map.

•Y is a digital κ'-retract of $X \times Y$ because there is a κ'-retraction map $r_2 : X \times Y \to Y$ that can be defined by $r_2(x, y) = y$, such that $r_2 \circ j(y) = y \quad \Rightarrow \quad r_2 \circ j \simeq_{(\kappa',\kappa')} 1_Y$ where $j : Y \to X \times Y$ is an inclusion map.

From Lemma 3.17, we conclude that (X, p, κ) and (Y, q, κ') are digital H-spaces. □

If we associate with Theorem 3.13 and Theorem 3.19, we have the following corollary:

Corollary 3.20. *(X, p, κ) and (Y, q, κ') are both digital H-spaces if and only if $(X \times Y, p \times q, \kappa^*)$ is a digital H-space.*

4. Applications

In this section, we deal with certain applications of this theory in computer vision and image processing. Several problems show the important role played by digital topology in the analysis of digital images and computer vision. One problem concerns counting objects in an image, such as the number of people in a market, which is important for security. Another problem is distinguishing objects due to their properties. Computer scientists use homotopy invariants, such as the fundamental group, to compare the forms of objects. Other problems are reduction of data, image segmentation, and data compression.

We now give three applications of this theory.

1) The following figure is a complex molecule of a protein.

   The 3D structure of the protein can be given as follows.

   If we obtain the digital simplicial complex structure of Figure 9, then we can calculate its digital fundamental group and digital homotopy groups. Our purpose is to gain information about the shapes, holes, and sizes of the components of digital images. This information is one of the important problems of image analysis. Since the digital homotopy groups are invariant from digital images, we can classify them. If we determine whether a digital image is a digital H-space or not, certain results are achieved about the image.

Figure 9. 3D structure of the protein.

2) If a digital image (X, κ) is κ-contractible, then we say that it is a digital H-space and has the same properties as the one-point digital image. Homotopy deals with topological invariants such as the number of connected components, holes, etc. of a topological space. Homotopy groups codify and shape properties. The following figure shows the machinery of this theory.

   There are no adjacent points in Figure 10, so its zero and first digital homotopy groups are $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. The other homotopy groups are trivial. We conclude that a digital H-space that has three components has the same homotopy-type as the image. This knowledge contains the topological properties of the digital image. Using these properties, we arrive to certain conclusions about the digital image, including the classification of images.

3) Thinning, an important operation in image analysis, aims to reduce data without altering crucial topological properties such as connectivity and homotopy type. If the computer data are reduced by digital

homotopy, the computer program of the digital image can be run quickly and easily. The following example is given to explain this procedure. An 8-deformation retract of the digital image $\mathbb{Z}^2 - \{(0,0)\}$ is the digital image MSC_8', which is defined by $MSC_8' = \{(1,0),(0,1),(-1,0),(0,-1)\}$ in $\mathbb{Z}^2$.

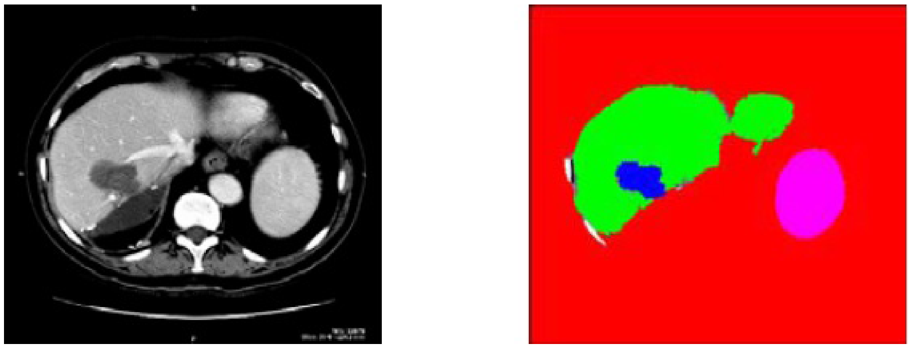

Figure 10. Original CAT mediastinum image and image with different connected components.

Since MSC_8' is an 8-contractible digital image [5], i.e. it has the same digital homotopy type with a one-pointed digital image, it is a digital H-space. Therefore, we conclude that this image has all the properties of a digital H-space. This knowledge will be useful for solving certain problems of digital image processing.

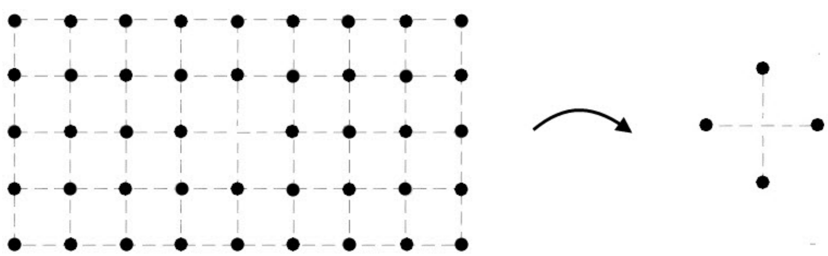

Figure 11. 8-deformation retract of the digital image $\mathbb{Z}^2 - \{(0,0)\}$ to MSC_8'.

5. Conclusion

The aim of this paper is to present several important properties of digital H-spaces. We arrived to useful conclusions about digital H-spaces. Since the digital homotopy groups related to digital H-spaces are powerful invariants that carry much of the topological information about a digital image, we can arrive to some useful conclusions about image processing and computer graphics algorithms. In the future, we aim to construct a homology and cohomology theory for digital H-spaces, because they are powerful tools with computational properties.

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