

Stability criterion for uncertain 2-*D* discrete systems with interval-like time-varying delay employing quantization/overflow nonlinearities

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Abstract: This paper considers the problem of global asymptotic stability of a class of two-dimensional (2-*D*) uncertain discrete systems described by the Fornasini–Marchesini second local state-space (FMSLSS) model under the influence of various combinations of quantization/overflow nonlinearities and interval-like time-varying delay in the state. The systems under consideration involve parameter uncertainties that are assumed to be deterministic and norm-bounded. A delay-dependent stability criterion is established by bounding the forward difference of the 2-*D* Lyapunov functional using the reciprocally convex approach. The criterion is compared with a recently reported criterion.

Key words: Delay-dependent stability criterion, finite wordlength effect, Lyapunov stability, state-delayed system, two-dimensional system, uncertain system

1. Introduction

During the past two decades, significant research has been done on two-dimensional (2-*D*) systems and their practical applications in the field of digital image processing, digital control systems, thermal processes, chemical reactors, river pollution modeling, seismographic data processing, gas filtration process [1–4], etc. Therefore, 2-*D* system analysis and design has received a substantial amount of interest and is considered a challenging task.

Finite wordlength nonlinearities such as quantization and overflow are inherently present in discrete systems implemented using fixed-point arithmetic. Such nonlinearities may lead to instability in the designed system [5]. The problem of global asymptotic stability of 2-*D* discrete systems with quantization or overflow nonlinearities has received considerable attention (see, for instance, [6–11] and the references cited therein). Since the practical 2-*D* discrete systems operate under the influence of the combination of quantization and overflow nonlinearities, the study of stability properties of discrete systems involving both types of nonlinearities appears to be more realistic [12–16].

Physical systems may suffer from parameter uncertainties that arise due to modeling errors, variations in system parameters, finite resolution of the measuring equipments or some ignored factors [17]. The existence of parameter uncertainties may lead to instability in the designed system.

Delays are another source of instabilities that are frequently encountered in many physical, industrial, and engineering applications such as processing of images, cold rolling mills, decision making of manufacturing systems [17, 18] etc. Time delays are an outcome of the finite computational time or transportation delay

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among the various parts of the system [17]. Delays can be constant or time-varying in nature. In general, the stability criteria for discrete state-delayed systems can be classified into delay-independent [17, 19, 20] and delay-dependent [11, 16, 17, 21–28]. It is well known that the delay-dependent criteria generally lead to less conservative results as compared to the delay-independent criteria with a computational overhead [22, 25]. The development of the techniques for the delay-dependent stability criteria has been targeting the conservativeness and computational complexity of the stability conditions. Based on Lyapunov’s stability theory, selection of a suitable Lyapunov functional is one of the primary steps to reduce the conservativeness, and a large amount of work (see, for instance, [22, 23, 29, 30]) has been dedicated in this aspect in order to obtain improved results. The commonly used techniques to obtain delay-dependent stability criteria include the free-weighting matrices method to relax the matrix cross-products [16, 22, 27] and the bounding techniques of the cross-terms and sum terms in the forward difference of the Lyapunov functional [23, 26, 28]. Although the free-weighting matrices method [16, 22, 27] is an effective way to reduce conservatism of the stability criteria, introduction of too many free matrix variables makes the criteria mathematically complex and computationally less effective and complicates the system analysis/synthesis procedure. Therefore, it remains a challenge to derive improved stability results without increasing the computational burden.

The problem of establishing delay-dependent criteria for the global asymptotic stability of uncertain 2- D discrete systems described by the Fornasini–Marchesini second local state-space (FMSLSS) model with time-varying delay subject to various combinations of quantization and overflow nonlinearities in their physical models is a realistic and challenging task. Recently, a delay-dependent stability criterion for such systems has been reported in [16]. Due to the utilization of free-weighting matrices, the criterion in [16] generally leads to heavier computational burden. Thus, there is still some room to improve the stability criterion in [16].

In this paper, we investigate the problem of global asymptotic stability of 2- D uncertain discrete systems described by the FMSLSS model involving finite wordlength nonlinearities and interval-like time-varying delay in the state. The main contributions of the proposed method are highlighted as follows: (a) The 2- D system under consideration is comprehensive and employs various combinations of quantization/overflow nonlinearities, time-varying delays, and parameter uncertainties. (b) To derive the delay-dependent stability conditions, unlike [16], a tighter bounding technique based on the reciprocally convex method [28, 31] is used in this paper to deal with the sum terms, which reduces the computational demand and simplifies system analysis/synthesis procedure. (c) Motivated by the work on one-dimensional (1- D) systems presented in [26], a 2- D Lyapunov functional is introduced to derive computationally tractable conditions under which the global asymptotic stability of the addressed system is guaranteed.

This paper is organized as follows. Section 2 describes the system under consideration and presents the lemmas used. A new delay-dependent linear matrix inequality (LMI) based criterion for global asymptotic stability of uncertain 2-D discrete systems described by the FMSLSS model with interval-like time-varying state-delays employing various combinations of quantization and overflow nonlinearities is proposed in Section 3. In Section 4, with the help of examples, the proposed criterion is compared with a recently reported criterion [16].

2. System description

Notations: The notations used throughout this paper are standard. \mathbb{R}^p denotes the p -dimensional Euclidean space; $\mathbb{R}^{p \times q}$ is the set of $p \times q$ real matrices; $\mathbf{0}$ represents null matrix or null vector of appropriate dimension;

\mathbf{I} is the identity matrix of appropriate dimension; \mathbf{B}^T stands for the transpose of the matrix (or vector) \mathbf{B} ; $\mathbf{B} > \mathbf{0}$ ($\geq \mathbf{0}$) means that \mathbf{B} is a positive definite (semidefinite) symmetric matrix; $\mathbf{B} < \mathbf{0}$ represents that \mathbf{B} is a negative definite symmetric matrix; $diag\{g_1, g_2, \dots, g_n\}$ is a diagonal matrix with diagonal elements g_1, g_2, \dots, g_n ; \mathbb{Z}_+ is the set of nonnegative integers; $\mathbf{Q}(\cdot)$ represents quantization nonlinearities; $\mathbf{O}(\cdot)$ denotes overflow nonlinearities; $\mathbf{f}(\cdot)$ is the composite nonlinear function; $\|\cdot\|$ represents any vector or matrix norm; $sup\{\cdot\}$ denotes the supremum or least upper bound of a set; the symbol ‘ $*$ ’ represents the symmetric terms in a symmetric matrix.

Consider a class of 2- D discrete uncertain systems represented by the FMSLSS model [32] with interval-like time-varying delays under various combinations of quantization and overflow nonlinearities. Specifically, the system under consideration is described by

$$\begin{aligned} \mathbf{x}(i+1, j+1) &= \mathbf{O}\{\mathbf{Q}(\mathbf{y}(i, j))\} = \mathbf{f}(\mathbf{y}(i, j)) \\ &= [f_1(y_1(i, j)) \quad f_2(y_2(i, j)) \quad \cdots \quad f_n(y_n(i, j))]^T, \end{aligned} \tag{1a}$$

$$\begin{aligned} \mathbf{y}(i, j) &= (\mathbf{A}_1 + \Delta\mathbf{A}_1)\mathbf{x}(i, j+1) + (\mathbf{A}_2 + \Delta\mathbf{A}_2)\mathbf{x}(i+1, j) \\ &\quad + (\mathbf{A}_{d_1} + \Delta\mathbf{A}_{d_1})\mathbf{x}(i-\alpha(i), j+1) + (\mathbf{A}_{d_2} + \Delta\mathbf{A}_{d_2})\mathbf{x}(i+1, j-\beta(j)) \\ &= [y_1(i, j) \quad y_2(i, j) \quad \cdots \quad y_n(i, j)]^T \end{aligned} \tag{1b}$$

where $i \in \mathbb{Z}_+$ and $j \in \mathbb{Z}_+$ are horizontal coordinate and vertical coordinate, respectively; $\mathbf{x}(i, j) \in \mathbb{R}^n$ is the local state vector; $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_{d_1}, \mathbf{A}_{d_2} \in \mathbb{R}^{n \times n}$ are the known constant matrices; $\Delta\mathbf{A}_1, \Delta\mathbf{A}_2, \Delta\mathbf{A}_{d_1}, \Delta\mathbf{A}_{d_2} \in \mathbb{R}^{n \times n}$ are the unknown matrices representing parametric uncertainties in the state matrices; $\alpha(i)$ and $\beta(j)$ are time-varying delays along horizontal direction and vertical direction, respectively. Assume that $\alpha(i)$ and $\beta(j)$ satisfy

$$\alpha_l \leq \alpha(i) \leq \alpha_h, \quad \beta_l \leq \beta(j) \leq \beta_h, \tag{1c}$$

where α_l and β_l are constant nonnegative integers representing the lower delay bounds along horizontal and vertical directions, respectively; α_h and β_h are constant nonnegative integers representing the upper delay bounds along horizontal and vertical directions, respectively.

In the event of $\mathbf{Q}(\cdot)$ being either magnitude truncation or roundoff, $\mathbf{f}(\cdot)$ is confined to the sector $[k_o, k_q]$, i.e.

$$f_k(0) = 0, \quad k_o y_k^2(i, j) \leq f_k(y_k(i, j)) y_k(i, j) \leq k_q y_k^2(i, j), \quad k = 1, 2, \dots, n, \tag{2a}$$

where

$$k_q = \begin{cases} 1, & \text{for magnitude truncation} \\ 2, & \text{for roundoff} \end{cases}, \quad k_o = \begin{cases} 0, & \text{for zeroing or saturation} \\ -\frac{1}{3}, & \text{for triangular} \\ -1, & \text{for two's complement.} \end{cases} \tag{2b}$$

The uncertainties in the state matrices are assumed to be of the form [14, 16, 19]

$$[\Delta\mathbf{A}_1 \quad \Delta\mathbf{A}_2 \quad \Delta\mathbf{A}_{d_1} \quad \Delta\mathbf{A}_{d_2}] = [\mathbf{HFE}_1 \quad \mathbf{HFE}_2 \quad \mathbf{HFE}_{d_1} \quad \mathbf{HFE}_{d_2}] \tag{3}$$

where $\mathbf{H} \in \mathbb{R}^{n \times p}$, $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_{d_1}, \mathbf{E}_{d_2} \in \mathbb{R}^{q \times n}$ are known constant matrices and $\mathbf{F} \in \mathbb{R}^{p \times q}$ is an unknown matrix that satisfies $\mathbf{F}^T \mathbf{F} \leq \mathbf{I}$.

It is assumed [11, 16, 21, 22, 25] that system (1) has a finite set of boundary conditions, i.e. there exist two positive integers K and L such that

$$\begin{aligned} \mathbf{x}(i, j) &= \mathbf{0}, \forall i \geq K, j = -\beta_h, -\beta_h + 1, \dots, 0, \\ \mathbf{x}(i, j) &= \mathbf{u}_{ij}, \forall 0 \leq i < K, j = -\beta_h, -\beta_h + 1, \dots, 0, \\ \mathbf{x}(i, j) &= \mathbf{0}, \forall j \geq L, i = -\alpha_h, -\alpha_h + 1, \dots, 0, \\ \mathbf{x}(i, j) &= \mathbf{v}_{ij}, \forall 0 \leq j < L, i = -\alpha_h, -\alpha_h + 1, \dots, 0, \\ \mathbf{u}_{00} &= \mathbf{v}_{00}. \end{aligned} \tag{4}$$

Eqs. (1)–(4) can be used to represent a large class of uncertain discrete dynamical 2- D systems operating under the influence of quantization and overflow nonlinearities and delays in the state. Such systems include finite wordlength implementation of 2- D digital control systems, dynamical processes represented by the Darboux equation [33, 34], river pollution modeling [2], networked control systems or event controlled systems via communication network [35], wireless sensor platforms employing fixed-point digital processors [36], and so on.

The following definition and lemmas are needed in the proof of our main result.

Definition 1 [1, 32] *The system described by (1) is asymptotically stable if $\lim_{l \rightarrow \infty} x_l = 0$ for all boundary conditions in (4), where $x_l = \sup \{ \|\mathbf{x}(i, j)\| : i + j = l, i, j \geq 1 \}$.*

Lemma 1 [28, 31] *For any vectors ξ_1, ξ_2 , matrices \mathbf{R}, \mathbf{S} and real numbers $\alpha_1 \geq 0, \alpha_2 \geq 0$ satisfying*

$$\begin{bmatrix} \mathbf{R} & \mathbf{S} \\ * & \mathbf{R} \end{bmatrix} \geq \mathbf{0}, \quad \alpha_1 + \alpha_2 = 1, \tag{5}$$

$$\xi_i = \mathbf{0}, \text{ if } \alpha_i = 0 \quad (i = 1, 2) \tag{6}$$

then

$$-\frac{1}{\alpha_1} \xi_1^T \mathbf{R} \xi_1 - \frac{1}{\alpha_2} \xi_2^T \mathbf{R} \xi_2 \leq - \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}^T \begin{bmatrix} \mathbf{R} & \mathbf{S} \\ * & \mathbf{R} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}. \tag{7}$$

Lemma 2 [23] *For any positive definite matrix $\mathbf{J} \in \mathbb{R}^{n \times n}$, two positive integers r and r_0 satisfying $r \geq r_0 \geq 1$, and vector function $\chi(i, j) \in \mathbb{R}^n$, one has*

$$\left(\sum_{i=r_0}^r \chi(i, j) \right)^T \mathbf{J} \left(\sum_{i=r_0}^r \chi(i, j) \right) \leq (r - r_0 + 1) \sum_{i=r_0}^r \chi^T(i, j) \mathbf{J} \chi(i, j). \tag{8}$$

3. Main result

In this section, an LMI-based criterion for the global asymptotic stability of the system (1)–(4) is established. The main result may be stated as follows.

Table. Comparison of numerical complexity.

Methods	Numerical complexity using SeDuMi solver ($M^2L^{\frac{5}{2}} + L^{\frac{7}{2}}$)		
	$n = 2$	$n = 3$	$n = 4$
Theorem 1 in [16]	1.62×10^9	1.95×10^{10}	1.17×10^{11}
Proposed method (Theorem 1)	53154295	5.59×10^8	3.14×10^9

Remark 2. The Lyapunov functional (see (A-1)–(A-4)) used in the proof of Theorem 1 may be treated as an extension of the 1- D Lyapunov functional employed in [26] for delay-dependent stability analysis of discrete-time systems with time-varying delay.

Remark 3. In our approach, the matrix variables S_1 and S_2 in (9) are the degrees of freedom and they play an important role in reducing the conservativeness of Theorem 1.

Remark 4. Condition (10) is dependent on the parameters k_o and k_q . Note that k_o and k_q are independent of wordlength used to implement the 2- D system (1). Thus, Theorem 1 can also be used as a global asymptotic stability test for 2- D systems implemented with different wordlengths (or variable wordlengths) for various signals (resulting in different quantization step sizes and/or different overflow levels). Moreover, using Theorem 1, it may be possible to determine the various combinations of quantization and overflow that would be required to ensure the absence of limit cycles in the 2- D system.

4. Comparative evaluation

To demonstrate the effectiveness of the present result and compare it with a previous result [16], we now consider the following examples.

Example 1. Consider the system (1)–(4) with

$$\begin{aligned}
 \mathbf{A}_1 &= \begin{bmatrix} 0.02 & 1 \\ -0.09 & 0.15 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} 0.2 & 0 \\ 0.26 & 0.12 \end{bmatrix}, \mathbf{A}_{d_1} = \begin{bmatrix} 0 & 0.1 \\ 0 & 0 \end{bmatrix}, \mathbf{A}_{d_2} = \begin{bmatrix} 0.02 & 0 \\ 0 & -0.12 \end{bmatrix}, \\
 \mathbf{H} &= \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \mathbf{E}_1 = \mathbf{E}_2 = [0.01 \ 0], \mathbf{E}_{d_1} = \mathbf{E}_{d_2} = [0 \ 0.01], \alpha_l = 3, \alpha_h = 7, \beta_l = 2
 \end{aligned} \tag{16}$$

and the composite nonlinearities belong to the sector $[k_o, k_q] = [-1, 1]$ which includes saturation, zeroing, triangular, two’s complement overflow, magnitude truncation, combinations of saturation and magnitude truncation, zeroing and magnitude truncation, triangular and magnitude truncation, two’s complement overflow and magnitude truncation, etc. By using YALMIP 3.0 parser [37] and SeDuMi 1.21 solver [38], it is found from Theorem 1 that the present system is globally asymptotically stable for the delay range $3 \leq \alpha(i) \leq 7$ and $2 \leq \beta(j) \leq 10$. For the present example, Theorem 1 in [16] also succeeds in establishing the global asymptotic stability for $\beta_h = 10$, which is identical to that arrived at via Theorem 1. However, as shown in the Table, Theorem 1 has much smaller numerical complexity than Theorem 1 in [16].

Example 2. Consider a system represented by the following Darboux equation [33, 34]:

$$\frac{\partial \theta(x, t)}{\partial x} = -\frac{\partial \theta(x, t)}{\partial t} - a_0 \theta(x, t) - a_1 \theta(x, t - \tau), \quad (17)$$

where $\theta(x, t)$ is the temperature at space $x \in [0, x_f]$ and time $t \in [0, \infty)$, τ is the time delay, and a_0 and a_1 are real coefficients. Eq. (17) may be used to describe thermal processes in chemical reactors, heat exchangers, pipe furnaces [16, 21, 25], etc. As shown in [16], (17) can be transformed into the following FMSLSS model:

$$\begin{aligned} \mathbf{x}(i+1, j+1) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}(i, j+1) + \begin{bmatrix} 0 & 0 \\ \frac{\Delta t}{\Delta x} & 1 - \frac{\Delta t}{\Delta x} - a_0 \Delta t \end{bmatrix} \mathbf{x}(i+1, j) \\ &+ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{x}(i - \alpha(i), j+1) + \begin{bmatrix} 0 & 0 \\ 0 & -a_1 \Delta t \end{bmatrix} \mathbf{x}(i+1, j - \beta(j)). \end{aligned} \quad (18)$$

In the presence of finite wordlength nonlinearities and parameter uncertainties, the system (18) converts itself into the format of (1)–(4). Now choose $a_0 = 5$, $a_1 = 1.2$, $\Delta x = 0.4$, $\Delta t = 0.1$, $[k_o, k_q] = [-1, 1]$, $\mathbf{H} = [0 \ 0.1]^T$, $\mathbf{E}_1 = \mathbf{E}_2 = [0.01 \ 0]$, $\mathbf{E}_{d_1} = \mathbf{E}_{d_2} = [0 \ 0.01]$. This example was also considered in [16]. Using Theorem 1, it turns out that the present system is globally asymptotically stable over the delay range $3 \leq \alpha(i) \leq 7$ and $2 \leq \beta(j) \leq 17$. It is verified that Theorem 1 in [16] assures the global asymptotic stability for the same delay range but with a heavier computational burden.

From the above examples, it is clear that Theorem 1 may provide the same level of conservativeness with significantly reduced computational burden in comparison with [16].

5. Conclusion

By using the reciprocally convex approach, a delay-dependent LMI-based stability criterion (Theorem 1) for a class of 2- D uncertain discrete systems with interval-like time-varying delays subject to various combinations of quantization and overflow nonlinearities has been proposed. As compared to [16], the proposed criterion leads to the same level of conservativeness with significantly reduced computational burden. The 2- D stability results presented in this paper can be easily extended to m - D ($m > 2$) systems. Further work is required to reduce the conservativeness of Theorem 1 by possibly making use of frequency-dependent Lyapunov functions [30] along with more precise characterization of uncertainties, nonlinearities, and delays.

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Appendix. Proof of Theorem 1

Proof: Consider a 2- D Lyapunov functional

$$V(\mathbf{x}(i, j)) = \bar{V}(\mathbf{x}(i, j)) + \hat{V}(\mathbf{x}(i, j)), \quad (\text{A-1})$$

where

$$\begin{aligned} \bar{V}(\mathbf{x}(i, j)) = & \mathbf{x}^T(i, j) \mathbf{P}_1 \mathbf{x}(i, j) + \sum_{r=-\alpha_l}^{-1} \mathbf{x}^T(i+r, j) \mathbf{Q}_1 \mathbf{x}(i+r, j) \\ & + \sum_{r=-\alpha_h}^{-1} \mathbf{x}^T(i+r, j) \mathbf{Q}_2 \mathbf{x}(i+r, j) + \sum_{\theta=-\alpha_h}^{-\alpha_l} \sum_{r=\theta}^{-1} \mathbf{x}^T(i+r, j) \mathbf{Q}_3 \mathbf{x}(i+r, j) \\ & + \alpha_l \sum_{\theta=-\alpha_l+1}^0 \sum_{r=-1+\theta}^{-1} \boldsymbol{\eta}_1^T(i+r, j) \mathbf{Z}_1 \boldsymbol{\eta}_1(i+r, j) \\ & + \alpha_{hl} \sum_{\theta=-\alpha_h+1}^{-\alpha_l} \sum_{r=-1+\theta}^{-1} \boldsymbol{\eta}_1^T(i+r, j) \mathbf{Z}_2 \boldsymbol{\eta}_1(i+r, j), \end{aligned} \quad (\text{A-2})$$

$$\begin{aligned} \hat{V}(\mathbf{x}(i, j)) = & \mathbf{x}^T(i, j) \mathbf{P}_2 \mathbf{x}(i, j) + \sum_{r=-\beta_l}^{-1} \mathbf{x}^T(i, j+r) \mathbf{Q}_4 \mathbf{x}(i, j+r) \\ & + \sum_{r=-\beta_h}^{-1} \mathbf{x}^T(i, j+r) \mathbf{Q}_5 \mathbf{x}(i, j+r) + \sum_{\theta=-\beta_h}^{-\beta_l} \sum_{r=\theta}^{-1} \mathbf{x}^T(i, j+r) \mathbf{Q}_6 \mathbf{x}(i, j+r) \\ & + \beta_l \sum_{\theta=-\beta_l+1}^0 \sum_{r=-1+\theta}^{-1} \boldsymbol{\eta}_2^T(i, j+r) \mathbf{Z}_3 \boldsymbol{\eta}_2(i, j+r) \\ & + \beta_{hl} \sum_{\theta=-\beta_h+1}^{-\beta_l} \sum_{r=-1+\theta}^{-1} \boldsymbol{\eta}_2^T(i, j+r) \mathbf{Z}_4 \boldsymbol{\eta}_2(i, j+r) \end{aligned} \quad (\text{A-3})$$

and

$$\boldsymbol{\eta}_1(i, j+1) = \mathbf{x}(i+1, j+1) - \mathbf{x}(i, j+1) = \mathbf{f}(\mathbf{y}(i, j)) - \mathbf{x}(i, j+1), \quad (\text{A-4a})$$

$$\boldsymbol{\eta}_2(i+1, j) = \mathbf{x}(i+1, j+1) - \mathbf{x}(i+1, j) = \mathbf{f}(\mathbf{y}(i, j)) - \mathbf{x}(i+1, j). \quad (\text{A-4b})$$

Now, following [7, 14, 16], we define $\Delta V(\mathbf{x}(i, j))$ as

$$\Delta V(\mathbf{x}(i, j)) = \Delta_1 \bar{V}(\mathbf{x}(i, j)) + \Delta_2 \hat{V}(\mathbf{x}(i, j)), \quad (\text{A-5})$$

where

$$\begin{aligned}
 \Delta_1 \bar{V}(\mathbf{x}(i, j)) &= \bar{V}(\mathbf{x}(i+1, j+1)) - \bar{V}(\mathbf{x}(i, j+1)) \\
 &= \mathbf{f}^T(\mathbf{y}(i, j)) \mathbf{P}_1 \mathbf{f}(\mathbf{y}(i, j)) - \mathbf{x}^T(i, j+1) \mathbf{P}_1 \mathbf{x}(i, j+1) \\
 &\quad + \mathbf{x}^T(i, j+1) \mathbf{Q}_1 \mathbf{x}(i, j+1) - \mathbf{x}^T(i - \alpha_l, j+1) \mathbf{Q}_1 \mathbf{x}(i - \alpha_l, j+1) \\
 &\quad + \mathbf{x}^T(i, j+1) \mathbf{Q}_2 \mathbf{x}(i, j+1) - \mathbf{x}^T(i - \alpha_h, j+1) \mathbf{Q}_2 \mathbf{x}(i - \alpha_h, j+1) \\
 &\quad + (\alpha_{hl} + 1) \mathbf{x}^T(i, j+1) \mathbf{Q}_3 \mathbf{x}(i, j+1) - \sum_{r=-\alpha_h}^{-\alpha_l} \mathbf{x}^T(i+r, j+1) \mathbf{Q}_3 \mathbf{x}(i+r, j+1) \\
 &\quad + \alpha_l^2 \boldsymbol{\eta}_1^T(i, j+1) \mathbf{Z}_1 \boldsymbol{\eta}_1(i, j+1) + \alpha_{hl}^2 \boldsymbol{\eta}_1^T(i, j+1) \mathbf{Z}_2 \boldsymbol{\eta}_1(i, j+1) \\
 &\quad - \alpha_l \sum_{r=-\alpha_l}^{-1} \boldsymbol{\eta}_1^T(i+r, j+1) \mathbf{Z}_1 \boldsymbol{\eta}_1(i+r, j+1) \\
 &\quad - \alpha_{hl} \sum_{r=-\alpha_h}^{-\alpha_l-1} \boldsymbol{\eta}_1^T(i+r, j+1) \mathbf{Z}_2 \boldsymbol{\eta}_1(i+r, j+1), \tag{A-6}
 \end{aligned}$$

$$\begin{aligned}
 \Delta_2 \hat{V}(\mathbf{x}(i, j)) &= \hat{V}(\mathbf{x}(i+1, j+1)) - \hat{V}(\mathbf{x}(i+1, j)) \\
 &= \mathbf{f}^T(\mathbf{y}(i, j)) \mathbf{P}_2 \mathbf{f}(\mathbf{y}(i, j)) - \mathbf{x}^T(i+1, j) \mathbf{P}_2 \mathbf{x}(i+1, j) \\
 &\quad + \mathbf{x}^T(i+1, j) \mathbf{Q}_4 \mathbf{x}(i+1, j) - \mathbf{x}^T(i+1, j - \beta_l) \mathbf{Q}_4 \mathbf{x}(i+1, j - \beta_l) \\
 &\quad + \mathbf{x}^T(i+1, j) \mathbf{Q}_5 \mathbf{x}(i+1, j) - \mathbf{x}^T(i+1, j - \beta_h) \mathbf{Q}_5 \mathbf{x}(i+1, j - \beta_h) \\
 &\quad + (\beta_{hl} + 1) \mathbf{x}^T(i+1, j) \mathbf{Q}_6 \mathbf{x}(i+1, j) - \sum_{r=-\beta_h}^{-\beta_l} \mathbf{x}^T(i+1, j+r) \mathbf{Q}_6 \mathbf{x}(i+1, j+r) \\
 &\quad + \beta_l^2 \boldsymbol{\eta}_2^T(i+1, j) \mathbf{Z}_3 \boldsymbol{\eta}_2(i+1, j) + \beta_{hl}^2 \boldsymbol{\eta}_2^T(i+1, j) \mathbf{Z}_4 \boldsymbol{\eta}_2(i+1, j) \\
 &\quad - \beta_l \sum_{r=-\beta_l}^{-1} \boldsymbol{\eta}_2^T(i+1, j+r) \mathbf{Z}_3 \boldsymbol{\eta}_2(i+1, j+r) \\
 &\quad - \beta_{hl} \sum_{r=-\beta_h}^{-\beta_l-1} \boldsymbol{\eta}_2^T(i+1, j+r) \mathbf{Z}_4 \boldsymbol{\eta}_2(i+1, j+r). \tag{A-7}
 \end{aligned}$$

In view of Lemma 2, we have the following relations:

$$\begin{aligned}
 -\alpha_l \sum_{r=-\alpha_l}^{-1} \boldsymbol{\eta}_1^T(i+r, j+1) \mathbf{Z}_1 \boldsymbol{\eta}_1(i+r, j+1) \\
 \leq -[\mathbf{x}^T(i, j+1) - \mathbf{x}^T(i - \alpha_l, j+1)] \mathbf{Z}_1 [\mathbf{x}(i, j+1) - \mathbf{x}(i - \alpha_l, j+1)], \tag{A-8}
 \end{aligned}$$

$$\begin{aligned}
 -\beta_l \sum_{r=-\beta_l}^{-1} \boldsymbol{\eta}_2^T(i+1, j+r) \mathbf{Z}_3 \boldsymbol{\eta}_2(i+1, j+r) \\
 \leq -[\mathbf{x}^T(i+1, j) - \mathbf{x}^T(i+1, j - \beta_l)] \mathbf{Z}_3 [\mathbf{x}(i+1, j) - \mathbf{x}(i+1, j - \beta_l)]. \tag{A-9}
 \end{aligned}$$

Define $\bar{\gamma}_1(i, j) = \mathbf{x}(i - \alpha_l, j + 1) - \mathbf{x}(i - \alpha(i), j + 1)$ and $\bar{\gamma}_2(i, j) = \mathbf{x}(i - \alpha(i), j + 1) - \mathbf{x}(i - \alpha_h, j + 1)$. Using Lemma 2, we obtain

$$\begin{aligned} & -\alpha_{hl} \sum_{r=-\alpha_h}^{-\alpha_l-1} \boldsymbol{\eta}_1^T(i+r, j+1) \mathbf{Z}_2 \boldsymbol{\eta}_1(i+r, j+1) \\ &= -\alpha_{hl} \sum_{r=-\alpha(i)}^{-\alpha_l-1} \boldsymbol{\eta}_1^T(i+r, j+1) \mathbf{Z}_2 \boldsymbol{\eta}_1(i+r, j+1) - \alpha_{hl} \sum_{r=-\alpha_h}^{-\alpha(i)-1} \boldsymbol{\eta}_1^T(i+r, j+1) \mathbf{Z}_2 \boldsymbol{\eta}_1(i+r, j+1) \\ &\leq -\frac{1}{\frac{\alpha(i) - \alpha_l}{\alpha_{hl}}} \bar{\gamma}_1^T(i, j) \mathbf{Z}_2 \bar{\gamma}_1(i, j) - \frac{1}{\frac{\alpha_h - \alpha(i)}{\alpha_{hl}}} \bar{\gamma}_2^T(i, j) \mathbf{Z}_2 \bar{\gamma}_2(i, j). \end{aligned} \quad (\text{A-10})$$

Note that $\bar{\gamma}_1(i, j) = \mathbf{0}$, if $\{\alpha(i) - \alpha_l\} / \alpha_{hl} = 0$ and $\bar{\gamma}_2(i, j) = \mathbf{0}$, if $\{\alpha_h - \alpha(i)\} / \alpha_{hl} = 0$. In view of Lemma 1 and (A-10), one can obtain that if there exists a matrix \mathbf{S}_1 satisfying $\bar{\Theta} \geq \mathbf{0}$ then

$$-\alpha_{hl} \sum_{r=-\alpha_h}^{-\alpha_l-1} \boldsymbol{\eta}_1^T(i+r, j+1) \mathbf{Z}_2 \boldsymbol{\eta}_1(i+r, j+1) \leq -\begin{bmatrix} \bar{\gamma}_1(i, j) \\ \bar{\gamma}_2(i, j) \end{bmatrix}^T \bar{\Theta} \begin{bmatrix} \bar{\gamma}_1(i, j) \\ \bar{\gamma}_2(i, j) \end{bmatrix}. \quad (\text{A-11})$$

Similarly, one can show that if there exists a matrix \mathbf{S}_2 satisfying $\hat{\Theta} \geq \mathbf{0}$ then

$$-\beta_{hl} \sum_{r=-\beta_h}^{-\beta_l-1} \boldsymbol{\eta}_2^T(i+1, j+r) \mathbf{Z}_4 \boldsymbol{\eta}_2(i+1, j+r) \leq -\begin{bmatrix} \hat{\gamma}_1(i, j) \\ \hat{\gamma}_2(i, j) \end{bmatrix}^T \hat{\Theta} \begin{bmatrix} \hat{\gamma}_1(i, j) \\ \hat{\gamma}_2(i, j) \end{bmatrix} \quad (\text{A-12})$$

where $\hat{\gamma}_1(i, j) = \mathbf{x}(i+1, j - \beta_l) - \mathbf{x}(i+1, j - \beta(j))$ and $\hat{\gamma}_2(i, j) = \mathbf{x}(i+1, j - \beta(j)) - \mathbf{x}(i+1, j - \beta_h)$. It is easy to verify that

$$-\sum_{r=-\alpha_h}^{-\alpha_l} \mathbf{x}^T(i+r, j+1) \mathbf{Q}_3 \mathbf{x}(i+r, j+1) \leq -\mathbf{x}^T(i - \alpha(i), j+1) \mathbf{Q}_3 \mathbf{x}(i - \alpha(i), j+1), \quad (\text{A-13})$$

$$-\sum_{r=-\beta_h}^{-\beta_l} \mathbf{x}^T(i+1, j+r) \mathbf{Q}_6 \mathbf{x}(i+1, j+r) \leq -\mathbf{x}^T(i+1, j - \beta(j)) \mathbf{Q}_6 \mathbf{x}(i+1, j - \beta(j)). \quad (\text{A-14})$$

Let

$$\bar{\mathbf{A}}_1 = \mathbf{A}_1 + \Delta \mathbf{A}_1, \quad \bar{\mathbf{A}}_2 = \mathbf{A}_2 + \Delta \mathbf{A}_2, \quad \bar{\mathbf{A}}_{d_1} = \mathbf{A}_{d_1} + \Delta \mathbf{A}_{d_1}, \quad \bar{\mathbf{A}}_{d_2} = \mathbf{A}_{d_2} + \Delta \mathbf{A}_{d_2}. \quad (\text{A-15})$$

Employing (A-5) – (A-15), we have the following inequality

$$\Delta V(\mathbf{x}(i, j)) \leq \boldsymbol{\xi}^T(i, j) \boldsymbol{\Psi}_1 \boldsymbol{\xi}(i, j) - 2\delta \quad (\text{A-16})$$

where $\boldsymbol{\xi}(i, j) = [\mathbf{x}^T(i, j+1) \quad \mathbf{x}^T(i+1, j) \quad \mathbf{x}^T(i - \alpha(i), j+1) \quad \mathbf{x}^T(i+1, j - \beta(j)) \quad \mathbf{x}^T(i - \alpha_l, j+1) \quad \mathbf{x}^T(i - \alpha_h, j+1) \quad \mathbf{x}^T(i+1, j - \beta_l) \quad \mathbf{x}^T(i+1, j - \beta_h) \quad \mathbf{f}^T(\mathbf{y}(i, j))]^T$ and

$$\begin{aligned} \delta &= \sum_{l=1}^n g_l [k_q y_l(i, j) - f_l(y_l(i, j))] [f_l(y_l(i, j)) - k_o y_l(i, j)] \\ &= [k_q \mathbf{y}(i, j) - \mathbf{f}(\mathbf{y}(i, j))]^T \mathbf{G} [\mathbf{f}(\mathbf{y}(i, j)) - k_o \mathbf{y}(i, j)], \end{aligned} \quad (\text{A-17})$$

$$\bar{\mathbf{E}} = [\mathbf{E}_1 \quad \mathbf{E}_2 \quad \mathbf{E}_{d_1} \quad \mathbf{E}_{d_2} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0}] \text{ and}$$

$$\mathbf{M} = \begin{bmatrix}
 \Xi_{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Z}_1 & \mathbf{0} & \mathbf{0} \\
 * & \Xi_{22} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Z}_3 \\
 * & * & \Xi_{33} & \mathbf{0} & \mathbf{Z}_2 - \mathbf{S}_1^T & \mathbf{Z}_2 - \mathbf{S}_1 & \mathbf{0} \\
 * & * & * & \Xi_{44} & \mathbf{0} & \mathbf{0} & \mathbf{Z}_4 - \mathbf{S}_2^T \\
 * & * & * & * & -\mathbf{Q}_1 - \mathbf{Z}_1 - \mathbf{Z}_2 & \mathbf{S}_1 & \mathbf{0} \\
 * & * & * & * & * & -\mathbf{Q}_2 - \mathbf{Z}_2 & \mathbf{0} \\
 * & * & * & * & * & * & -\mathbf{Q}_4 - \mathbf{Z}_3 - \mathbf{Z}_4 \\
 * & * & * & * & * & * & * \\
 * & * & * & * & * & * & * \\
 * & * & * & * & * & * & * \\
 \mathbf{0} & \Xi_{19} + k_q \mathbf{A}_1^T \mathbf{G} & -k_q \sqrt{-2k_o} \mathbf{A}_1^T \mathbf{G} \\
 \mathbf{0} & \Xi_{29} + k_q \mathbf{A}_2^T \mathbf{G} & -k_q \sqrt{-2k_o} \mathbf{A}_2^T \mathbf{G} \\
 \mathbf{0} & k_q \mathbf{A}_{d_1}^T \mathbf{G} & -k_q \sqrt{-2k_o} \mathbf{A}_{d_1}^T \mathbf{G} \\
 \mathbf{Z}_4 - \mathbf{S}_2 & k_q \mathbf{A}_{d_2}^T \mathbf{G} & -k_q \sqrt{-2k_o} \mathbf{A}_{d_2}^T \mathbf{G} \\
 \mathbf{0} & \mathbf{0} & \mathbf{0} \\
 \mathbf{0} & \mathbf{0} & \mathbf{0} \\
 \mathbf{S}_2 & \mathbf{0} & \mathbf{0} \\
 -\mathbf{Q}_5 - \mathbf{Z}_4 & \mathbf{0} & \mathbf{0} \\
 * & \Xi_{99} + \left(\frac{k_o}{2k_q} \right) \mathbf{G} & \sqrt{\frac{-k_o}{2}} \mathbf{G} \\
 * & * & -k_q \mathbf{G}
 \end{bmatrix}. \tag{A-22}$$

By using Lemma 1 of [14], (A-21) is equivalent to

$$\mathbf{M} + \epsilon^{-1} \bar{\mathbf{H}} \bar{\mathbf{H}}^T + \epsilon \bar{\mathbf{E}}^T \bar{\mathbf{E}} < \mathbf{0}. \tag{A-23}$$

Using Schur's complement, (A-23) leads to (10). This completes the proof of Theorem 1.