

Robust H_∞ control for chaotic supply chain networks

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Abstract: A supply chain network (SCN) is a complex nonlinear system involving multiple entities. The policy of each entity in decision-making and the uncertainties of demand and supply (or production) significantly affect the complexity of its behavior. Although several studies have presented information about the measurement of chaos in the supply chain, there has not been an appropriate way to control the chaos in it. In this paper, the chaos control problem is considered for a SCN with a time-varying delay between its entities. The innovation of this paper is the more comprehensive modeling, analysis, and control of chaotic behavior in the system. The proposed model has a control center to determine the orders of entities and control their inventories. Customer demand is modeled as an unknown exogenous disturbance. A robust H_∞ control method is designed to control its chaotic behavior in terms of a certain linear matrix inequality technique that can be readily solved using the MATLAB LMI toolbox. By using this technique and calculating the maximum Lyapunov exponent, decision parameters are determined in such a way that the behavior of the SCN is stable.

Key words: Supply chain network, chaotic behavior, unknown demand, robust H_∞ control, linear matrix inequality, Lyapunov exponent

1. Introduction

The dynamics of the supply chain have been modeled by several authors [1,2]. A number of studies have been conducted that investigated the existence and formation of chaotic behaviors in supply chains. Mosekilde and Larsen [3] and Thomsenet et al. [4] adopted a deterministic supply chain and showed the existence of chaotic behavior in it.

Marketing and competition activities are dynamic behaviors that create interaction between suppliers and customers and may generate chaotic behavior in the supply chain [5]. Wu and Zhang [6] showed the chaotic behavior of the supply chain by simulating the interaction between customers and suppliers. Hwarng and Xie [7] also showed the chaotic behavior of the supply chain by introducing five main factors that influence the supply chain, namely demand pattern, ordering policy, order information sharing, lead time, and supply chain level. They also quantified the degree of chaos using the maximum Lyapunov exponent (LE) across all levels of the supply chain.

Previous studies have generally focused on serial multilevel supply chains in which each level has only one entity; however, the fact remains that supply chains are vast networks with many entities at each level. Tarokh et al. [8] investigated the role of supply network configuration in creating chaotic behavior. Their proposed

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model consisted of more than one entity at each network level with two agents between each two successive levels for receiving and dispatching orders.

Usually there are two categories of methods for controlling chaotic systems. The first category uses their inherent characteristics. The linearization of Poincare map (OGY: Ott, Grebogi, and Yorke) method [9] and delayed feedback control [10] are the most important methods in this category. The second category looks at chaotic systems as nonlinear systems and, without considering their inherent characteristics, the controller is designed. If these methods prove stability in the general case, they will also be valid for special (chaotic) systems. Adaptive control [11], sliding mode control [12], neural network control [13], fuzzy control [14], and robust control [15] are also in this category. Fradkov and Evans [16] examined various methods of controlling chaos and studied their applications in engineering.

Inventory control is an important issue in supply chain management (SCM). The chaotic behavior of the supply chain network (SCN) can lead to inventory instability in the entities and disrupt decision-making. When customer demand increases, manufacturers are often unable to support downstream levels; when customer demand slows down, manufacturers often continue to overproduce, which results in overstock. It has been indicated that the bullwhip effect [17] (the magnification of demand variability as orders move up the supply chain) is the culprit behind chaos in inventory levels and production strategies [18]. The negative impacts of the bullwhip effect can be summed up as surplus or short capacities, inadequate customer service, excess inventories, and high inventory costs [18,19]. Therefore, controlling chaos in the SCN is crucial and requires extensive research.

Several studies have been conducted for controlling chaos in economic systems [20–22]. The development of more accurate simplified linear representations of complex nonlinear supply chain models was studied in [23]. The objective was to reduce model complexity and to assist in gaining supply chain dynamics insights. Considering the uncertainty of the macroeconomic environment, Chunxiang et al. [24] studied the robust optimization method for constructing and designing the automotive supply chain network. They built a robust optimization model for an integrated supply chain network design to effectively reduce investment risks. A robust environmental closed-loop supply chain network was investigated in [25] that includes multiple plants, collection centers, demand zones, and products and consists of both forward and reverse supply chains. The first objective of the study was to minimize economic costs and the second was to minimize the environmental influence. The synchronization of two identical chaotic SCM systems based on their mathematical model was presented in [26]. Linear feedback controllers were designed and added to the nonlinear SCM to achieve control of the system. The robustness of the proposed methods was then verified by computer simulations. However, a study aiming to control a chaotic supply chain does not yet exist in the literature. The main objective of this paper is to propose more comprehensive modeling and robust control of chaotic behavior in the SCN.

In this paper, a SCN with a centralized decision-making structure is modeled that has four successive levels. Each level has various entities and a central controller is used to control the inventory of entities. In the model, a robust control method by the linear matrix inequality technique (LMI) [27] is used to control chaotic behavior in the SCN. With this approach, decision parameters are adjusted in such a way that the behavior of the SCN is stable.

This paper is organized as follows: Section 2 briefly introduces the chaotic systems and describes the LMI approach to robust control. In Section 3, the chaotic behavior of the SCN is modeled and analyzed. In Section 4, a robust H_∞ control method using the LMI technique is designed to control the chaos in the SCN.

In Section 5, a ten-entity SCN is simulated and its chaotic behavior is shown and controlled. Finally, the paper ends with a conclusion.

2. Control of chaotic systems

A chaotic system is often characterized by a number of features [16,28] such as nonrandomness and nonlinearity, sensitivity to initial conditions, strange attractors, nonperiodic or nonquasiperiodic states, impossibility of the prediction of long-term behavior of the system, and noninvertibility.

Generally, there are two ways, namely graphical methods and quantifiers [28], to identify chaos or show whether a system's behavior is stable, periodic, quasiperiodic, or chaotic. Graphical methods such as time series and phase plots are visible but less accurate. The LE is a standard quantifier that measures the sensitivity of initial conditions and classifies the behavior of nonlinear systems. In practice, a wide range of LEs can be obtained, but what matters is the largest LE, calculated as follows:

$$\lambda_{\max} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \ln |\Delta R_{n+1} / \Delta R_n|, \quad (1)$$

where ΔR_n and ΔR_{n+1} are the distances between two nearby trajectories at times n and $n + 1$, respectively. A sufficient condition for chaotic systems is that at least one LE or the largest LE is positive.

The specific control objective for chaotic systems is stabilizing the unstable equilibrium or at least creating a stable limit cycle [16]. The robust H_∞ control is one of the most important methods of controlling chaos. LMI is a powerful technique in the design of robust controllers. One of the major applications of the LMI technique is to find the stability conditions of a nonlinear system (continuous-time or discrete-time). First, a Lyapunov functional candidate is constructed. For system stability, the derivative of the Lyapunov function should be less than zero, which leads to matrix inequality. The general form of the LMI technique is as follows [27]:

$$N^T L(x) N \leq M^T R(x) M, \quad (2)$$

where x is a decision variable to establish matrix inequality. In the MATLAB toolbox, the following problem is solved as follows:

$$\begin{aligned} & \text{Minimize } t \\ & \text{s.t. } N^T L(x) N - M^T R(x) M \leq tI \end{aligned} \quad (3)$$

LMI constraints are feasible if $t_{\min} \leq 0$ and strictly feasible if $t_{\min} < 0$.

3. Model

In this paper, the SCN is characterized by the beer distribution model. It is made up of four levels: factories, distributors, wholesalers, and retailers (Figure 1). In the SCN, products flow from factories to customers and orders propagate from customers to factories. Each level has several entities. Decisions are made globally by a control center based on information about entities such as actual and desired inventory levels, backlog, expected orders, and incoming products. There is a time-varying delay between order placement and delivery. The basic hypotheses of the model are:

- (1) An entity with sufficient inventory must fulfill orders.
- (2) Unfilled orders are kept in a backlog and will be filled when the inventory is sufficient.

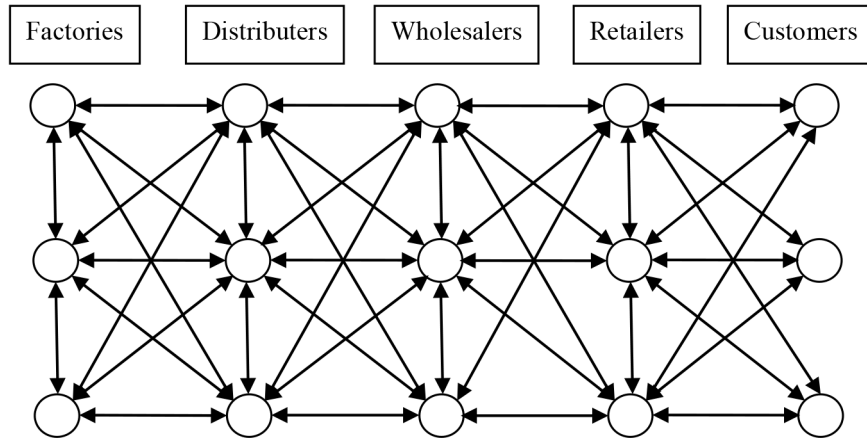


Figure 1. Configuration of the supply chain network.

- (3) Placed orders cannot be cancelled and incoming products cannot be returned.
- (4) Each retailer independently receives customer demands.

Based on the decisions of the control center, each entity receives its products after a time-varying delay from the time of placing an order. At the same time, a new demand is received and all or part of the backlog and current demand is fulfilled. The operations of entities at each level are as follows:

Retailer level:

$$x_i(t + 1) = a_i x_i(t) + c_i f_i(x_i(t - \tau(t))) - d_i(t), \quad i = 1, 2, \dots, N_R. \tag{4}$$

Other levels:

$$x_i(t + 1) = a_i x_i(t) + c_i f_i(x_i(t - \tau(t))) - \sum_{j=1}^{N^d} b_{ij} f_j^d(x_j^d(t)), \quad i = 1, 2, \dots, N_W \text{ or } N_D \text{ or } N_F, \tag{5}$$

where R , W , D , and F respectively represent the retailer, wholesaler, distributor, and factory and d indicates downstream. $x_i(t)$ is the inventory of the i th entity at time t , a_i is the attenuating factor ($0 \leq a_i \leq 1$), c_i is the inventory adjustment parameter of the i th entity ($0 \leq c_i \leq 1$), and b_{ij} is as follows:

$$b_{ij} = w_i c_j, \sum_{i=1}^{N^u} w_i = 1, \tag{6}$$

where N^u is the number of entities in an upstream level.

$\tau(t)$ represents the maximum transmission delay between the entities of two adjacent levels (lead time) that satisfied:

$$\tau_1 < \tau(t) < \tau_2, t \in \mathbb{N} = \{1, 2, 3, \dots\}. \tag{7}$$

$f_i(x_i(t))$ denotes the maximum order of the i th entity that is obtained as follows:

$$f_i(x_i(t)) = |\min(0, x_i(t) - \bar{x}_i)|, \tag{8}$$

where $x_i(t)$ is the actual inventory and \bar{x}_i is the desired inventory.

$d_i(t)$ indicates customer demand of the i th entity at the retailer level.

In total, the dynamic equation of the SCN is as follows:

$$\begin{aligned} x(t+1) &= Ax(t) + Bf(x(t)) + Cf(x(t-\tau(t))) + Dv(t), \\ y(t) &= Hx(t), \end{aligned} \tag{9}$$

where

$$\begin{aligned} n &= N_R + N_W + N_D + N_F, \\ x(t) &= [x_1(t) \ x_2(t) \ \dots \ x_n(t)]^T, \\ f(x(t)) &= [f_1(x_1(t)) \ f_2(x_2(t)) \ \dots \ f_n(x_n(t))]^T, \\ v(t) &= [d_1(t) \ d_2(t) \ \dots \ d_{N_R}(t)]^T, \end{aligned}$$

and $A = \text{diag}\{a_1, a_2, \dots, a_n\}$ is the attenuating matrix, $B = [b_{ij}]_{n \times n}$ is the connection weight matrix, $C = \text{diag}\{c_1, c_2, \dots, c_n\}$ is the time-delay connection weight matrix, $D = [d_{ij}]_{n \times N_R}$ is the input matrix, and $H = [h_{ij}]_{N_F \times n}$ is the controlled output matrix.

4. Stability analysis

Assume 0 is the desired inventory of the i th entity and consider the following range for the function $f_i(x_i(t))$:

$$l_i^- \leq \frac{f_i(x_i(t))}{x_i(t)} \leq l_i^+, \quad i = 1, 2, \dots, n, \tag{10}$$

where l_i^- and l_i^+ are known constant scalars and allowed to be positive, negative, or zero. These numbers are obtained due to the limitations of demand and order (or production). Before proving the robust exponential stability, we recall the following definitions:

Definition 1 *The discrete time-delay SCN (Eq. (9)) with $v(t) = 0$ is said to be robustly exponentially stable if there are constants $\alpha > 0$ and $0 < \beta < 1$ such that*

$$\|x(t)\| \leq \alpha \beta^t \sup_{-\tau_2 \leq i \leq 0} \|x(i)\|. \tag{11}$$

Definition 2 *The discrete time-delay SCN (Eq. (9)) is said to be robustly exponentially stable about its equilibrium point with disturbance attenuation level γ if it is robustly exponentially stable for $v(t) = 0$ and under zero initial conditions:*

$$\|x(t)\|_2 \leq \gamma \|v(t)\|_2. \tag{12}$$

4.1. Main results

In this subsection, the robust stability problem based on the LMI approach is analyzed and sufficient conditions are derived under which the SCN (Eq. (9)) is robustly exponentially stable.

Theorem 1 *Under the condition of Eq. (10), the SCN (Eq. (9)) with $v(t) = 0$ is robustly exponentially stable if there are symmetric positive-definite matrices P , Q_1 , Q_2 , and Z and two diagonal matrices $R > 0$ and*

$S \succ 0$ such that the following LMI holds:

$$\Phi = \begin{bmatrix} E_1 & 0 & Z & L_2 R & 0 & A^T P & \tau_2(A - I)Z \\ * & E_2 & 0 & 0 & L_2 S & 0 & 0 \\ * & * & E_3 & 0 & 0 & 0 & 0 \\ * & * & * & -R & 0 & B^T P & \tau_2 B^T Z \\ * & * & * & * & -S & C^T P & \tau_2 C^T Z \\ * & * & * & * & * & -P & 0 \\ * & * & * & * & * & * & -Z \end{bmatrix} \prec 0, \tag{13}$$

where

$$L_1 = \text{diag}\{l_1^- l_1^+, l_2^- l_2^+, \dots, l_n^- l_n^+\}, L_2 = \text{diag}\left\{\frac{l_1^- + l_1^+}{2}, \frac{l_2^- + l_2^+}{2}, \dots, \frac{l_n^- + l_n^+}{2}\right\},$$

$$E_1 = -P + (\tau_2 - \tau_1 + 1)Q_1 + Q_2 - Z - L_1 R, E_2 = -Q_1 - L_1 S, E_3 = -Q_2 - Z.$$

Proof Consider the following Lyapunov–Krasovskii functional candidate for the system of Eq. (9) with $v(t) = 0$:

$$V(t) = \sum_{i=1}^5 V_i(t), \tag{14}$$

where

$$V_1(t) = x^T(t) P x(t), V_2(t) = \sum_{i=t-\tau(t)}^{t-1} x^T(i) Q_1 x(i), V_3(t) = \sum_{j=t-\tau_2+1}^{t-\tau_1} \sum_{i=j}^{t-1} x^T(i) Q_1 x(i),$$

$$V_4(t) = \sum_{i=t-t_2}^{t-1} x^T(i) Q_2 x(i), V_5(t) = \tau_2 \sum_{j=-\tau_2}^{-1} \sum_{i=t+j}^{t-1} \eta^T(i) Z \eta(i), \quad \eta(t) = x(t+1) - x(t).$$

Define $\Delta V(t) = V(t+1) - V(t)$. Along the solution of the system of Eq. (9), the following is obtained:

$$\Delta V(t) = \sum_{i=1}^5 \Delta V_i(t), \tag{15}$$

where

$$\begin{aligned} \Delta V_1(t) &= x^T(t+1) P x(t+1) - x^T(t) P x(t) \\ &= x^T(t) (A^T P A - P) x(t) + 2x^T(t) A^T P B f(x(t)) + 2x^T(t) A^T P C f(x(t - \tau(t))) \\ &\quad + f^T(x(t)) B^T P B f(x(t)) + 2f^T(x(t)) B^T P C f(x(t - \tau(t))) \\ &\quad + f^T(x(t - \tau(t))) C^T P C f(x(t - \tau(t))), \end{aligned} \tag{16}$$

$$\begin{aligned}
 \Delta V_2(t) &= \sum_{i=t+1-\tau(t+1)}^t x^T(i)Q_1x(i) - \sum_{i=t-\tau(t)}^{t-1} x^T(i)Q_1x(i) \\
 &= \sum_{i=t+1-\tau(t+1)}^{t-\tau_1} x^T(i)Q_1x(i) + \sum_{i=t-\tau_1+1}^{t-1} x^T(i)Q_1x(i) + x^T(t)Q_1x(t) \\
 &\quad - \sum_{i=t-\tau(t)+1}^{t-1} x^T(i)Q_1x(i) - x^T(t-\tau(t))Q_1x(t-\tau(t)) \\
 &\leq \sum_{i=t+1-\tau(t+1)}^{t-\tau_1} x^T(i)Q_1x(i) + x^T(t)Q_1x(t) - x^T(t-\tau(t))Q_1x(t-\tau(t)) \\
 &\leq \sum_{i=t+1-\tau_2}^{t-\tau_1} x^T(i)Q_1x(i) + x^T(t)Q_1x(t) - x^T(t-\tau(t))Q_1x(t-\tau(t)),
 \end{aligned} \tag{17}$$

$$\begin{aligned}
 \Delta V_3(t) &= \sum_{i=t-\tau_2+2}^{t+1-\tau_1} \sum_{i=j}^t x^T(i)Q_1x(i) - \sum_{j=t-\tau_2+1}^{t-\tau_1} \sum_{i=j}^{t-1} x^T(i)Q_1x(i) \\
 &= (\tau_2 - \tau_1)x^T(t)Q_1x(t) - \sum_{i=t-\tau_2+1}^{t-\tau_1} x^T(i)Q_1x(i),
 \end{aligned} \tag{18}$$

$$\Delta V_4(t) = x^T(t)Q_2x(t) - x^T(t-\tau_2)Q_2x(t-\tau_2), \tag{19}$$

$$\begin{aligned}
 \Delta V_5(t) &= \tau_2 \sum_{j=-\tau_2}^{-1} [\eta^T(t)Z\mu(t) - \eta^T(t+j)Z\eta(t+j)] \\
 &= \tau_2^2 \mu^T(t)Z\eta(t) - \tau_2 \sum_{j=t-\tau_2}^{t-1} \eta^T(j)Z\eta(j) \\
 &\leq \tau_2^2 \eta^T(t)Z\eta(t) - \sum_{j=t-\tau_2}^{t-1} \eta^T(j)Z \sum_{j=t-\tau_2}^{t-1} \eta(j) \\
 &= \tau_2^2 \eta^T(t)Z\eta(t) + \begin{bmatrix} x(t) \\ x(t-\tau_2) \end{bmatrix}^T \begin{bmatrix} -Z & Z \\ * & -Z \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau_2) \end{bmatrix}.
 \end{aligned} \tag{20}$$

Substituting Eqs. (16)–(10) into Eq. (15) leads to:

$$\Delta V(t) \leq \zeta^T(t)\Phi_1\zeta(t), \tag{21}$$

where

$$\zeta(t) = [x^T(t) \quad x^T(t-\tau(t)) \quad x^T(t-\tau_2) \quad f^T(x(t)) \quad f^T(x(t-\tau(t)))],$$

$$\Phi_1 = \begin{bmatrix} E_0 & 0 & Z & 0 & 0 \\ * & -Q_1 & 0 & 0 & 0 \\ * & * & -Q_2 - Z & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix} + \begin{bmatrix} A \\ 0 \\ 0 \\ B^T \\ C^T \end{bmatrix} P \begin{bmatrix} A \\ 0 \\ 0 \\ B^T \\ C^T \end{bmatrix}^T + \begin{bmatrix} A - I \\ 0 \\ 0 \\ B^T \\ C^T \end{bmatrix} (\tau_2^2 Z) \begin{bmatrix} A - I \\ 0 \\ 0 \\ B^T \\ C^T \end{bmatrix}^T,$$

$$E_0 = -P + (\tau_2 - \tau_1 + 1)Q_1 + Q_2 - Z.$$

From Eq. (10), it follows that [29]:

$$\begin{bmatrix} x(t) \\ f(x(t)) \end{bmatrix} \begin{bmatrix} L_1R & -L_2R \\ * & R \end{bmatrix} \begin{bmatrix} x(t) \\ f(x(t)) \end{bmatrix} \leq 0, \tag{22}$$

$$\begin{bmatrix} x(t - \tau(t)) \\ f(x(t - \tau(t))) \end{bmatrix} \begin{bmatrix} L_1S & -L_2S \\ * & S \end{bmatrix} \begin{bmatrix} x(t - \tau(t)) \\ f(x(t - \tau(t))) \end{bmatrix} \leq 0. \tag{23}$$

Thus,

$$\begin{aligned} \Delta V(t) &\leq \zeta^T(t)\Phi_1\zeta(t) \\ &\quad - \begin{bmatrix} x(t) \\ f(x(t)) \end{bmatrix}^T \begin{bmatrix} L_1R & -LR \\ * & R \end{bmatrix} \begin{bmatrix} x(t) \\ f(x(t)) \end{bmatrix} \\ &\quad - \begin{bmatrix} x(t - \tau(t)) \\ f(x(t - \tau(t))) \end{bmatrix}^T \begin{bmatrix} L_1S & -L_2S \\ * & S \end{bmatrix} \begin{bmatrix} x(t - \tau(t)) \\ f(x(t - \tau(t))) \end{bmatrix} \\ &= \zeta^T(t)\Phi_2\zeta(t), \end{aligned} \tag{24}$$

where

$$\Phi_2 = \begin{bmatrix} E_1 & 0 & Z & L_2R & 0 \\ * & E_2 & 0 & 0 & L_2S \\ * & * & E_3 & 0 & 0 \\ * & * & * & -R & 0 \\ * & * & * & * & -S \end{bmatrix} + \begin{bmatrix} A^T \\ 0 \\ 0 \\ B^T \\ C^T \end{bmatrix} P \begin{bmatrix} A^T \\ 0 \\ 0 \\ B^T \\ C^T \end{bmatrix}^T + \begin{bmatrix} A - I \\ 0 \\ 0 \\ B^T \\ C^T \end{bmatrix} (\tau_2^2 Z) \begin{bmatrix} A - I \\ 0 \\ 0 \\ B^T \\ C^T \end{bmatrix}^T.$$

$E_1, E_2,$ and E_3 are defined in Eq. (13). According to the research conducted in [30], Φ is equivalent to Φ_2 and we have:

$$\Delta V(t) \leq \zeta^T(t)\Phi\zeta(t). \tag{25}$$

It is clear from $\Phi < 0$ that there exists a small scalar $\varepsilon_0 > 0$ such that:

$$\Phi + \varepsilon_0 \text{diag}\{I_{n \times n}, 0\} < 0. \tag{26}$$

From Eqs. (25) and (26), it follows that:

$$\Delta V(t) \leq -\varepsilon_0 \|x(t)\|^2. \tag{27}$$

Therefore, Lyapunov stability for the SCN (Eq. (9)) is achieved. The exponential stability analysis of the network of Eq. (9) is similar to the method in [31]. This completes the proof. \square

Next, the H_∞ performance of the SCN (Eq. (9)) will be analyzed.

Theorem 2 Under the condition of Eq. (10), the SCN (Eq. (9)) is robustly exponentially stable for $v(t) = 0$ and satisfies $\|y(t)\|_2 \leq \gamma \|v(t)\|_2$ under the zero initial condition for any nonzero $v \in l_2[0, \infty)$ if there exist

symmetric positive-definite matrices P , Q_1 , Q_2 , and Z and two diagonal matrices $R \succ 0$ and $S \succ 0$ such that the following LMI holds:

$$\Psi = \begin{bmatrix} \Omega & 0 & Z & L_2 R & 0 & 0 & A^T P & \tau_2(A - I)Z \\ * & E_2 & 0 & 0 & L_2 S & 0 & 0 & 0 \\ * & * & E_3 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -R & 0 & 0 & B^T P & \tau_2 B^T Z \\ * & * & * & * & -S & 0 & C^T P & \tau_2 C^T Z \\ * & * & * & * & * & -\gamma^2 I & D^T P & \tau_2 D^T Z \\ * & * & * & * & * & * & -P & 0 \\ * & * & * & * & * & * & * & -Z \end{bmatrix} \prec 0, \tag{28}$$

where E_2, E_3, L_1 , and L_2 are defined as in Theorem 1 and

$$\Omega = -P + (\tau_2 - \tau_1 + 1)Q_1 + Q_2 - Z - L_1 R + H^T H. \tag{29}$$

Proof Construct the same Lyapunov–Krasovskii functional candidate $V(t)$ as in Theorem 1 for the system of Eq. (9). A similar calculation as in Theorem 1 leads to:

$$\Delta V(t) \leq \xi^T(t) \Psi_1 \xi(t), \tag{30}$$

where

$$\xi(t) = [x^T(t) \quad x^T(t - \tau(t)) \quad x^T(t - \tau_2) \quad f^T(x(t)) \quad f^T(x(t - \tau(t))) \quad v(t)],$$

$$\Psi_1 = \begin{bmatrix} -P + (\tau_2 - \tau_1 + 1)Q_1 + Q_2 - Z & 0 & Z & 0 & 0 & 0 \\ * & -Q_1 & 0 & 0 & 0 & 0 \\ * & * & -Q_2 - Z & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix} + \begin{bmatrix} A \\ 0 \\ 0 \\ B^T \\ C^T \\ D^T \end{bmatrix} P \begin{bmatrix} A \\ 0 \\ 0 \\ B^T \\ C^T \\ D^T \end{bmatrix}^T$$

$$+ \begin{bmatrix} A - I \\ 0 \\ 0 \\ B^T \\ C^T \\ D^T \end{bmatrix} (\tau_2^2 Z) \begin{bmatrix} A - I \\ 0 \\ 0 \\ B^T \\ C^T \\ D^T \end{bmatrix}^T.$$

In order to deal with the H_∞ performance of the system of Eq. (9), J_N is defined:

$$J_N = \sum_{t=0}^N [y^T(t)y(t) - \gamma^2 v^T(t)v(t)], \tag{31}$$

where $N > 0$ is an integer. Noting the zero initial condition, from Eqs. (22), (23), (30), and (31), we get:

$$\begin{aligned}
 J_N &= \sum_{t=0}^N [y^T(t)y(t) - \gamma^2 v^T(t)v(t) + \Delta V(t)] - V(N+1) \\
 &\leq \sum_{t=0}^N [x^T(t)H^T Hx(t) - \gamma^2 v^T(t)v(t) + \Delta V(t)] \\
 &\leq \sum_{t=0}^N \xi^T(t)\Psi_2\xi(t) ,
 \end{aligned} \tag{32}$$

where

$$\begin{aligned}
 \Psi_2 &= \begin{bmatrix} -P + (\tau_2 - \tau_1 + 1)Q_1 + Q_2 - Z - L_1R + H^T H & 0 & Z & L_2R & 0 & 0 \\ * & -Q_1 & 0 & 0 & L_2S & 0 \\ * & * & -Q_2 - Z & 0 & 0 & 0 \\ * & * & * & -R & 0 & 0 \\ * & * & * & * & -S & 0 \\ * & * & * & * & * & -\gamma^2 I \end{bmatrix} \\
 &+ \begin{bmatrix} A \\ 0 \\ 0 \\ B^T \\ C^T \\ D^T \end{bmatrix} P \begin{bmatrix} A \\ 0 \\ 0 \\ B^T \\ C^T \\ D^T \end{bmatrix}^T + \begin{bmatrix} A - I \\ 0 \\ 0 \\ B^T \\ C^T \\ D^T \end{bmatrix} (\tau_2^2 Z) \begin{bmatrix} A - I \\ 0 \\ 0 \\ B^T \\ C^T \\ D^T \end{bmatrix}^T .
 \end{aligned}$$

Now, by the same line as in the proof of Theorem 1, it follows from Eq. (28) that $\Psi_2 < 0$, which together with Eq. (32) ensures that $\|y(t)\|_2 \leq \gamma \|v(t)\|_2$ holds under the zero initial condition. This completes the proof of the theorem. □

4.2. Managerial perspective

Decision-making in the SCN can be centralized or decentralized. In the decentralized approach, each entity makes a decision locally. However, the reality is that different levels in the SCN are related, and the decision made by each entity affects the behavior of the entire system. It is impossible to control SCN behavior, especially the chaotic behavior, with this approach. Therefore, effective management of the SCN requires coordination between all the entities in adjusting decision parameters. This can be done through a centralized approach with a central controller. It determines the range of decision parameters in order to stabilize the SCN behavior and facilitates optimal decision-making.

In the centralized approach, all entities report their inventory to the control center. The elements of matrix A , i.e. attenuating factors, are determined by the entities of the SCN and coordinated by the controller. The elements of matrix B represent product distribution from higher-level to lower-level entities and depend on their demand. Product distribution must be fair in order to avoid instability in the inventory of entities. The elements of matrix C are the parameters used to adjust the inventory of entities. Setting these parameters

is crucial to the stability of supply chain behavior. The matrix D expresses the relationship between customers and the SCN at the retail level. Customer demand for the SCN is unknown. The controlled output matrix H selects the inventory of factories as an output used in H_∞ performance. The main responsibility of the control center is to adjust the elements of matrix C , the inventory adjustment parameters. Using the LMI technique, an H_∞ robust controller determines the range of these parameters for each entity. Each entity then independently selects its inventory adjustment parameter within the range and thus controls its inventory and indirectly contributes to the stabilization of supply chain behavior. It should be noted that there is a time-varying delay for product transfer between entities. The designer of the control center sets a suitable range for it after studying the SCN and uses it in implementing a H_∞ robust controller.

5. Simulation

Consider a SCN with ten entities: one factory, two distributors, three wholesalers, and four retailers. It is assumed that all entities use the same inventory adjustment parameters ($c_i = \alpha, i = 1, \dots, n$). The total initial inventory and the demand of each level are equally distributed among its entities. Initial values and parameters are set according to Table 1. The model is simulated with MATLAB software, and 2000 data points are used to calculate the maximum LE.

Table 1. Initial data and parameters.

Item	Value
Total initial inventory (in each level)	24
Customer demand (for each retailer)	4
Lead time	3
Inventory adjustment parameter	$0 \leq \alpha \leq 1$
Attenuating factor	1

First, the presence of chaos in the SCN is investigated. With $\alpha = 0.26$, the maximum LE is negative (-0.0054) and the behavior of the SCN is stable. The time series plot of the distributors' total inventory (DTI) and the factories' total inventory (FTI) is shown in Figure 2. Figure 3 shows the phase plot of DTI-FTI. If α is slightly increased, i.e. by 0.02, the maximum LE becomes positive ($\lambda_{\max} = 0.0066$) and the behavior of the network becomes chaotic (Figure 4). Figure 5 shows the phase plot of DTI-FTI in this case. These figures indicate that the behavior of the SCN is very sensitive to small changes of α (the inventory adjustment parameter) and it may exhibit chaotic behavior. It must be noted that different levels in the SCN are related, and poor decisions made in one entity can lead to chaotic behavior and disrupt inventory control in other entities. As a result, a control center is needed to coordinate between all entities in adjusting decision parameters.

The effect of the inventory adjustment parameter on the behavior of the SCN is now investigated. Assume that the parameter α is changed from 0 to 1. By calculating the maximum LE, the chaotic behavior of the SCN is studied. Figure 6 shows that the maximum LE is positive in a large range. This figure demonstrates that α , as an endogenous parameter, plays an important role in the creation of chaotic behavior in the SCN. Therefore, parameter α should be set in such a way as to avoid chaotic behavior in all of the entities through a centralized approach with a central controller.

Customer demand is an unknown exogenous disturbance. Under certain conditions, the network parameters should be adjusted in such a way that the behavior of the SCN is stable. Using the MATLAB LMI toolbox, a robust H_∞ control method is designed to control the chaotic behavior of the SCN. The control parameters are set according to Table 2. Figure 7 shows that by changing α from 0 to 0.17, the behavior of the SCN is stable.

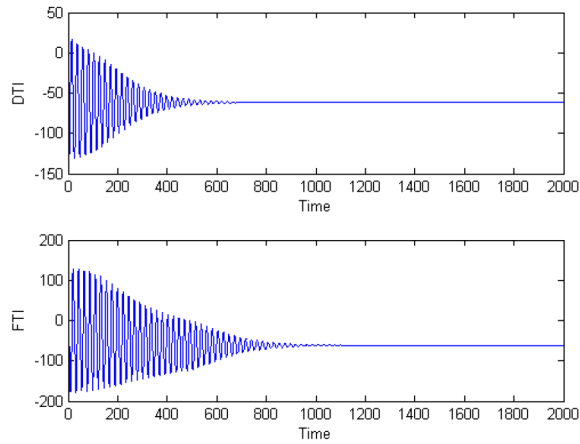


Figure 2. Time series plot of DTI and FTI in the stable state.

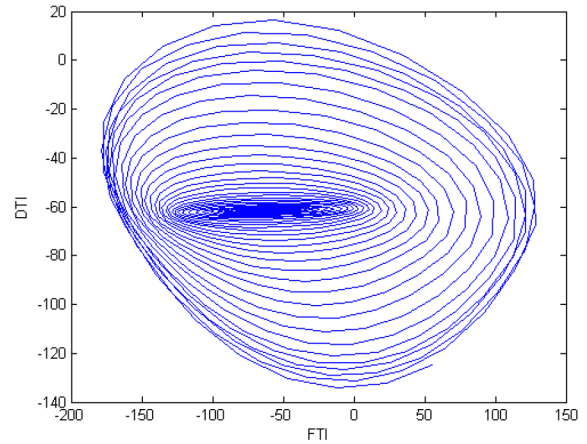


Figure 3. Phase plot of DTI-FTI in the stable state.

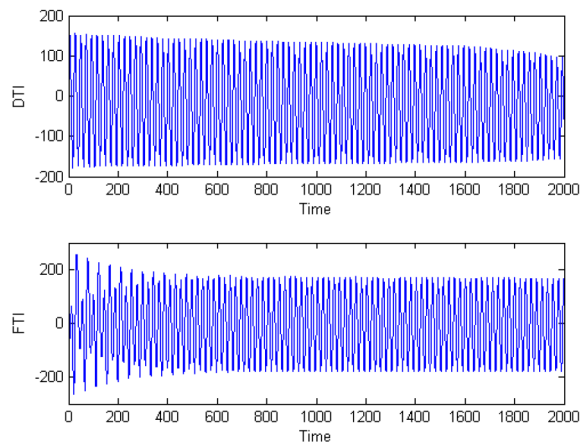


Figure 4. Time series plot of DTI and FTI in the chaotic state.

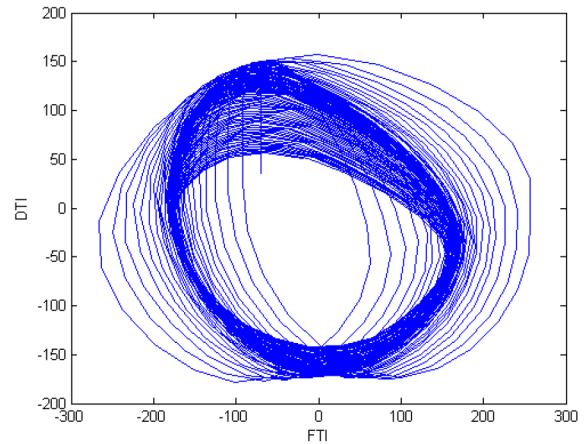


Figure 5. Phase plot of DTI-FTI in the chaotic state.

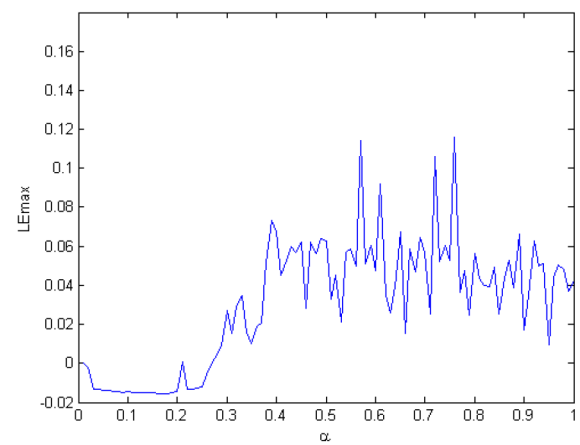


Figure 6. Effect of the inventory adjustment parameter.

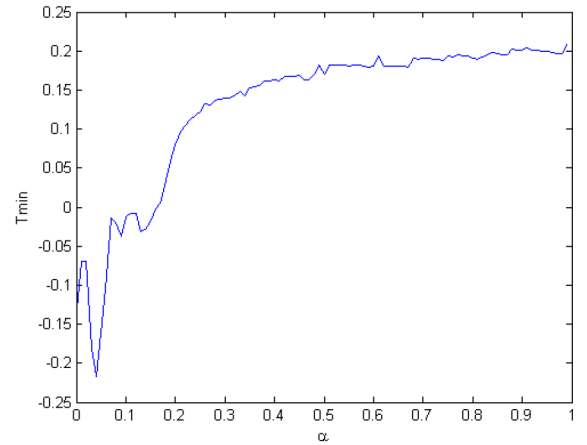


Figure 7. Stabilizing range of the inventory adjustment parameter.

Table 2. Parameters of the central controller.

Item	Value
a_i	0.5 for $i = 1, 2, \dots, 10$.
c_i	$0 \leq \alpha \leq 1$ for $i = 1, 2, \dots, 10$.
γ	5 for $i = 1, 2, \dots, 10$.
l_i^-	0 for $i = 1, 2, \dots, 10$.
l_i^+	1 for $i = 1, 2, \dots, 10$.

6. Conclusion

A SCN is a complex nonlinear system that may exhibit chaotic behavior. The inventory adjustment parameter is the main parameter for decision-making in each entity. It is an internal decision variable and has a major role in controlling the inventory. A central controller is able to find the range of the inventory adjustment parameters so that the behavior of the network is stable.

Customer demand is an external input in each entity of the retailer level. It is influenced by market activities and unnatural events. The SCN should therefore be robust against these kinds of changes.

Robust H_∞ control with the LMI technique is an appropriate method to design robust controllers and find stability conditions for chaotic networks. It can be used when there are uncertainties and unknown factors in the network. By using this method in the control center, the behavior of the SCN is stable in the specific range of inventory adjustment parameters.

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