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Robust optimal stabilization of balance systems with parametric variations

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Abstract: This paper proposes a method of robust optimal stabilization for balance systems such as rocket and missile systems, Segway human transportation systems, and inverted pendulum systems. Constant-gain controllers such as linear-quadratic regulators may not guarantee stability let alone optimality for balance systems affected by parametric variations. The robust stability and robust performance achieved through the proposed variable-gain controller are better than those of the linear-quadratic regulator. The proposed controller consists of two components, one of which is designed offline for nominal values of parametric variations and one of which is updated online for off-nominal values of parametric variations. A salient feature of the proposed method is a linear transformation that converts the vector control input of balance systems into scalar control input for application of the proposed method. A fourth-order linearized model of an inverted pendulum system is simulated to show the efficacy of the proposed method.

Key words: Robust control, optimal control, and parametric variation

1. Introduction

Balance systems appear in commercial, industrial, military, and academic applications. Segway human transportation systems, missile and rocket guidance, and inverted pendulums are examples of balance systems that have gained importance in their respective application fields. All balance systems share a generic mathematical model commonly known as the linear parameter-varying (LPV) system. A common control methodology for LPV systems may suffice for all balance systems. Control of balance systems affected by parametric variations is critical in certain applications. One such example is the control of missile systems experiencing rapid change in total mass due to fuel consumption. LPV control of quadcopters affected by mass variation and battery drainage is discussed in [1]. LPV control of wind power systems is given in [2], in which the variable speed of wind makes the rotor inside the turbine variable, too.

The control method for balance systems proposed in this paper advances our research presented on the subject in an earlier publication [3]. In [3], we proposed switchable linear-quadratic regulators (LQRs) that could provide optimal stabilization for LPV systems when parametric variations switch between a set of nominal values and a set of perturbed values. The proposed controller in this paper allows parametric variations to take on any values around nominal values because it incorporates parametric variations into its state-feedback gain. Since the control methods in [1] and [2] both involve notions of optimality in the design of their respective controllers, we discuss these now for comparison. In [1], the control law is a linear combination of gains of four LQRs designed for four sets of values of mass and battery drainage. A particular LQR becomes dominant when

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these parametric variations are inside its corresponding set. The control law in [1] has not proven to be optimal. In [2], the controller is a gain-scheduled proportional-integral controller. Since gains of proportional and integral parts are functions of rotor velocity, the controller does not lose the desired performance with varying rotor velocity.

Model predictive control (MPC) is a suboptimal control method that has been utilized in the control of LPV systems. An example is min–max MPC as a solution to the problem of controlling uncertain discrete-time linear systems [4]. MPC of LPV systems with rate constraints on parameter variations was given in [5]. Recently, an MPC approach yielding a controller that adapts to parametric variations was discussed in [6].

The theoretical novelty of the work presented in this paper is that it extends the linear-quadratic regulation theory analytically to LPV systems. This theory is well-established in cases of linear-time-invariant (LTI) and linear-time-varying (LTV) systems. Mathematically, this novelty is manifested as an extension of the algebraic Riccati equation (ARE) into a perturbed algebraic Riccati equation (PARE). The solution of this PARE forms a robust optimal controller for LPV systems just as the solution of the ARE forms a robust optimal controller, i.e. an LQR, for LTI systems.

The proposed controller consists of two components, one of which guarantees robust stability when parametric variations are at their nominal values, whereas the other ensures robust performance when parametric variations are at their off-nominal values. This two-component controller is constructed by employing the linear-quadratic regulation theory and the Lyapunov theory. A linear-invertible transformation is used to convert the vector control input of an LPV system into scalar control input for application of the proposed method. The proposed transformation is invertible to invert back the proposed control from transformed coordinates to original coordinates. Robust performance achieved with the proposed controller is found to be superior to constant-gain LQRs as shown in the simulation results.

The aforementioned treatment of the LPV system distinguishes the proposed work from modern LMI-based optimal control approaches for LPV systems on three accounts. First, the proposed work extends the historical evolution of optimal control theory to LPV systems purely analytically. Second, parametric variations and corresponding controllers are not constrained to be polytopic in nature as is usually the case with LMI-based approaches [7,8]. Third, the computational complexity of solving an LMI is greater than that of solving an ARE.

This paper is organized as follows: Section 2 discusses the problem statement; Section 3 presents the main result; Section 4 discusses the construction of a two-component controller; Section 5 discusses the stability analysis of the closed-loop system with the proposed controller; Section 6 discusses a motivating example of a fourth-order model of an inverted pendulum system on a cart; Section 7 provides simulation results for the motivating example; and Section 8 discusses the simulation results qualitatively. Finally, Section 9 concludes the paper.

2. Problem statement

A SISO linear system affected by parametric variations can be defined in the LPV framework [7] as given below:

$$\dot{x} = A(\theta)x + Bu, \tag{1}$$

where $x \in R^n$ and $u \in R^1$. System matrices are defined as $A(\theta) \in R^{n\times n}$ and $B \in R^{n\times 1}$. Note that input matrix B is such that the LPV systems are assumed to be driven by scalar control input. This means that input matrix B can be either $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for a second-order LPV system. The variable nature of system matrix $A(\theta)$ is due to stationary-exogenous parameters with fixed mean. A second-order system matrix $A(\theta)$ with maximum possible parametric variations is given as $A\left(\theta\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where parameters a, b, c, d are not known in advance but are assumed to be observable. A robust optimal controller for LPV systems must satisfy the following two requirements:

- It must achieve asymptotic stability and optimality of the equilibrium point $x = 0$ without any bounds on time.
- It must nullify the effect of parametric variations, which makes it difficult to achieve the first requirement. The desired controller is required to have a one-to-one relationship with parametric variations to guarantee optimality.

All the aforementioned stipulations are to be met by having a variable Lyapunov function in the structure of LQR controller as shown below:

$$u = -R^{-1}.B^T \Psi\left(P, a\right) x \tag{2}$$

where P is the gain of a traditional LQR and a is representative of all parametric variations. This controller must simplify into an LQR when parametric variations tend to their nominal values, as given below:

$$u = \lim_{*a \to a_0} \left(-R^{-1} B^T \Psi\left(P, a\right) x\right) = -R^{-1} B^T P x, \text{ as } u = \lim_{*a \to a_0} \Psi\left(P, a\right) = P,$$

where a_0 is representative of all parametric variations at their nominal values.

3. Main results

The main result of this paper is as follows: if the parametric variations do not affect the controllability of the LPV system in Eq. (1), then with the following decomposition of system matrix $A(\theta)$ for a second-order LPV system,

$$A\left(\theta\right) = A_0 + A_\delta \Rightarrow A\left(\theta\right) = \begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix} + \begin{bmatrix} a - a_0 & b - b_0 \\ c - c_0 & d - d_0 \end{bmatrix}, \tag{3}$$

where A_0 represents nominal values of parametric variations and A_δ represents off-nominal values of parametric variations, the controller

$$u = -R^{-1}.B^T.(P + P_u).x,$$

where P and P_u are respective solutions of

$$A_0^T P + P A_0 + Q - P B R^{-1} B^T P = 0 \text{ and}$$

$$A^T\left(\theta\right) P_u + P_u A\left(\theta\right) + A_\delta^T P + P A_\delta - P B R^{-1} B^T P_u - P_u B R^{-1} B^T P - P_u B R^{-1} B^T P_u = 0,$$

guarantees the robust optimal stabilization of Eq. (1).

A salient result of this paper is as follows: a linear transformation $T : x \rightarrow x^*$ converts input matrix B from vector form to scalar form, i.e. $B_{scalar} = T^{-1} B_{vector}$, facilitating the application of the proposed method.

4. Synthesis of two-component controller

The problem of robust optimal stabilization of an LPV system can be dealt with by assuming two Lyapunov functions. One Lyapunov function, $V^* = x^T P x$ with $P > 0$, is used for optimal stability at nominal values of parametric variations, whereas the other Lyapunov function, $V_u = x^T P_u x$ with $P_u \leq \begin{bmatrix} 0 & p_{u2} \\ p_{u2} & p_{u3} \end{bmatrix}$, is used for optimal stability for off-nominal values of parametric variations. This choice is due to the requirement that the proposed controller act as an LQR at nominal values of parametric variations. Cumulative $P + P_u$ is to remain positive definite and thus $V = V^* + V_u = x^T(P + P_u)x$ qualifies as a valid Lyapunov function for $\dot{x} = A(\theta)x + Bu$. Synthesis of the proposed controller begins with equating a quadratic cost functional with the aforementioned cumulative Lyapunov function, also used in [8] and [9], as shown below:

$$\int_{t_0}^{T} \left(x(t)^T Q x(t) + u(t)^T R u(t)\right) dt \;=\; V\left(x(t)\right) = x\left(t\right)^T \left(P + P_u\right) x\left(t\right) \tag{4}$$

which, in differential form, for time-invariant infinite-horizon control problems, i.e. from $t = 0$ to $t = \infty$, is given as:

$$x(t)^T Q x(t) + u(t)^T R u(t) = d(x(t)^T (P + P_u) x(t))/dt|_0^\infty .$$

If the system $\dot{x} = A(\theta)x + Bu$ is controllable with fixed end, i.e. $x(\infty) = 0$, it simplifies to

$$\begin{aligned} &x^T(0)\,Qx(0) + u^T(0)\,Ru(0) \\ &= -x^T(0)\left\{A^T(\theta)\,(P + P_u) + (P + P_u)\,A(\theta)\right\}x(0) \; - 2B^T(P + P_u)\,x(0)\,u(0) \end{aligned} , \tag{5}$$

whose rearranged form is given as

$$\begin{aligned} &u^T(0)\,Ru(0) + 2B^T(P + P_u)\,x(0)\,u(0) + x^T(0)\left(A^T(\theta)\,(P + P_u) + (P + P_u)\,A(\theta) + Q\right)x(0) \\ &= 0. \end{aligned} \tag{6}$$

The above equation can be considered as a case of static optimization at initial conditions resulting in the following objective function:

$$F\left(u(0), x(0)\right) = \min_{u^*(0)} \left\{ \begin{array}{l} u^T(0)\,Ru(0) \; + 2B^T(P + P_u)\,x(0)\,u(0) \\ +x^T(0)\left[A(\theta)\,(P + P_u) + (P + P_u)\,A(\theta) \; + Q\right]x(0) \end{array} \right\}, \tag{7}$$

which can be made time-invariant, by knowing that this objective function is to be minimized at all times from $\mathrm{t} = 0$ to $\mathrm{t} = \infty$, by an optimal control signal u^*, thereby converting the static optimization problem into a dynamic optimization problem. Optimization theory is used to solve optimal control problems of dynamic systems, as shown in [10]. It was also shown in [11] that optimal control theory and convex optimization are dual problems. Hence, finally the objective function becomes the following.

$$F(u, x) = \min_{u^*} \left\{u^T R u + 2B^T(P + P_u)\,xu + x^T\left[A(\theta)\,(P + P_u) + (P + P_u)\,A(\theta) \; + Q\right]x\right\} \tag{8}$$

This objective function can be minimized geometrically and its argument can be solved for the control signal u as a quadratic equation of one variable. A single-variable quadratic equation is given as

$$az^2 + bz + c = 0. \tag{9}$$

Quadratic matrix and quadratic vector equations also appear in the literature [12]. One such quadratic matrix equation is the famous ARE, which is quadratic in a variable matrix P. Single-variable quadratic equations can be solved by the following quadratic formula: $z = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$. Multivariable quadratic equations can be solved through this quadratic formula as shown in [12]. The quadratic formula for the proposed control of LPV system is shown below:

$$u = \frac{(-2B^T(P+P_u)x \pm \sqrt{((2B^T(P+P_u)x)^2 - 4Rx^T[A^T(\theta)(P+P_u)+(P+P_u)A(\theta)+Q]x))}}{2R}, \tag{10}$$

where the discriminant is denoted by D for further discussions. An immediate benefit by looking at Eq. (10) is that if its discriminant is set to zero, the form of optimal control appears as a variable-gain state-feedback law. This is in accordance with the desired form of the proposed controller as $u = -R^{-1}B^T(P+P_u)x$. Setting the discriminant in Eq. (10) equal to zero results in an equation that can be considered as a PARE, shown below.

$$A^T(\theta)(P+P_u) + (P+P_u)A(\theta) + Q - (P+P_u)BR^{-1}B^T(P+P_u) = 0 \tag{11}$$

The difference between the PARE and an ARE lies in the presence of modified system dynamics as $A_0 + A_\delta$ and the solution matrix as $P+P_u$. This solution matrix has a desired form that is used in the proposed controller. Before moving on to solve the PARE, more information can be extracted from the objective function as shown below.

- Existence of minimum: In order to ensure that the objective function in Eq. (10) has a minimum, the coefficient R of the nonlinear term u^2 must be positive definite.
- Asymptotic stabilization: In order to ensure asymptotic stability of the LPV system, setting $D = 0$ yields two benefits, which are the form of control law u that appears as that of an LQR, and the minimum of the objective function, which lies at the u-axis having value $u = -R^{-1}B^TPx$, as shown in Figure 1a.

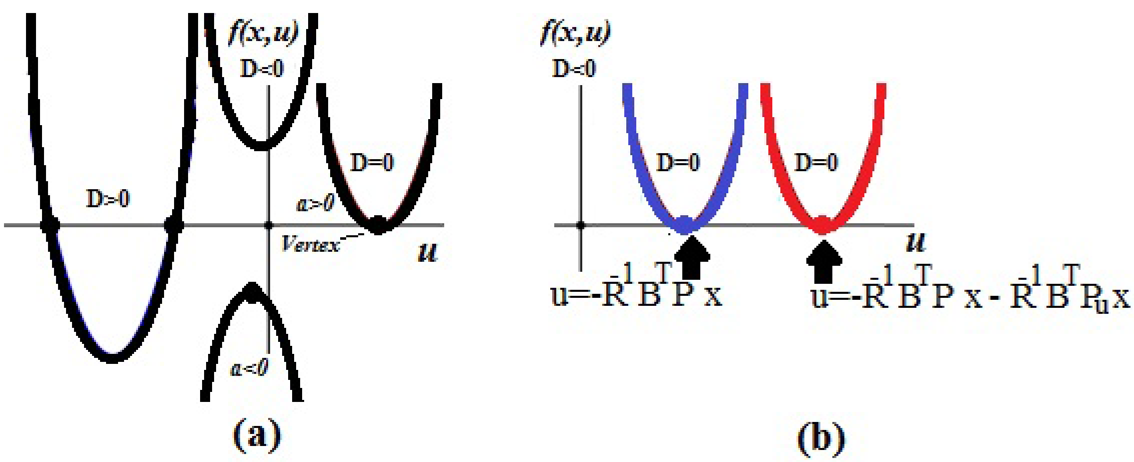

Figure 1. Minimum of the objective function lying at x = 0 when D = 0.

4.1. Solution of the PARE

Solving the PARE in Eq. (11), denoted as φ, for a closed-form solution of P, P_u, or $P+P_u$ can be challenging but it can be conveniently done by a rearrangement of the PARE. Hence, the PARE is rearranged with labeling as follows:

$$\overbrace{A_0^T + PA_0 + Q - PBR^{-1}B^T P}^{ARE} + A^T(\theta)\,P_u + P_u A(\theta) + A_\delta^T P + PA_\delta = P_u BR^{-1}B^T P_u + PBR^{-1}B^T P_u + P_u BR^{-1}B^T P. \tag{12}$$

The rearranged PARE consists of an expression that is exactly the same as that of an ARE; thus, its division into two components is proposed. One component, $\varphi_{acausal}$, is concerned with nominal values of parametric variations, whereas the other component, φ_{causal}, is concerned with off-nominal values of parametric variations. These components are given below.

$$\phi_{acausal}\,(A_0, P, Q, B, R) \triangleq A_0^T P + PA_0 + Q - PBR^{-1}B^T P \tag{13}$$

$$\phi_{causal}\,(A(\theta), P, P_u) \triangleq A_u^T(\theta)P + PA(\theta) + A_\delta^T P + PA_\delta - PBR^{-1}B^T P_u - P_u BR^{-1}B^T P - P_u BR^{-1}B^T P_u \tag{14}$$

Sequentially solving $\phi_{acausal}\,(A_0, P, Q, B, R)$ and $\phi_{causal}\,(A(\theta), P, P_u)$ for a second-order system provides variable gain, $P + P_u$, in the following control law:

$$u = -R^{-1}B^T(P + P_u)x. \tag{15}$$

It is to be noted that since $u \in R$, $u = -R^{-1}B^T P_u x$ can be considered as a correction factor for $u = -R^{-1}B^T Px$,making the proposed controller in Eq. (15) optimal for LPV systems. Graphically, as shown in Figure 1b, this correction factor $u = -R^{-1}B^T P_u x$ augmented to $u = -R^{-1}B^T Px$ renders $u = -R^{-1}B^T(P + P_u)x$ optimal for LPV systems. The robust performance and robust stability achieved with the proposed controller are now explained through a second-order system.

4.2. Robust performance

The solution of $\phi_{causal}\,(A(\theta), P, P_u)$ in Eq. (14) with the following system matrices:

$$A(\theta) = \begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix} + \begin{bmatrix} a - a_0 & b - b_0 \\ c - c_0 & d - d_0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}, \quad P_u = \begin{bmatrix} 0 & p_{u2} \\ p_{u2} & p_{u3} \end{bmatrix},$$

$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $R = 1$, in terms of p_{u2} and p_{u3}, is given as:

$$p_{u2} = -p_2 + c_0 + c_\delta \pm \sqrt{p_2^2 - 2c_0p_2 + c_0^2 + 2p_1a_\delta + 2c_0c_\delta + c_\delta^2} \tag{16}$$

and

$$p_{u3} = -p_3 + d_0 + d_\delta \pm \sqrt{p_3^2 - 2d_0p_3 + d_0^2 + 2d_0d_\delta + d_\delta^2 + 2b_\delta p_2 + 2b_0p_{u2} + 2b_\delta p_{u2}}. \tag{17}$$

Expressions for p_{u2} and p_{u3} look more complicated than they actually are, as will be shown in a motivating example. These two expressions when inserted into the proposed controller result in

$$\begin{aligned} u &= -(p_2 + p_{u2})x_1 - (p_3 + p_{u3})x_2 \\ &= -(c_0 + c_\delta \pm \sqrt{p_2^2 - 2c_0p_2 + c_0^2 + 2p_1a_\delta + 2c_0c_\delta + c_\delta^2})x_1 \\ &\quad -(d_0 + d_\delta \pm \sqrt{p_3^2 - 2d_0p_3 + d_0^2 + 2d_0d_\delta + d_\delta^2 + 2b_\delta p_2 + 2b_0p_{u2} + 2b_\delta p_{u2}})x_2. \end{aligned} \tag{18}$$

When parametric variations are at their nominal values, then all the factors $a_\delta, b_\delta, c_\delta,$ and d_δ become zero, rendering the control law in Eq. (18) as

$$u = -p_2 x_1 - p_3 x_2 \tag{19}$$

and conforming with the requirement that $u = \lim_{*a \to a_0} \left(-R^{-1} B^T \Psi\left(P, a\right) x\right) = -R^{-1} B^T P x$.

4.3. Robust stability

If an off-nominal value of parametric variation tends to destabilize the closed-loop system, then the $\phi_{acausal}$ (A_0, P, Q, B, R) is proposed to be solved for heavier state penalizations, leading to stronger state-feedbacks in the form of p_2 and p_3 in $u = -p_2 x_1 - p_3 x_2$.

5. Stability analysis of the closed-loop system

There are two methods of control system design [13]. Traditionally, controllers are first designed for systems followed by stability analysis of the closed-loop system. The other approach is to set conditions of stability in the beginning, followed by controller design within constraints for stability. The proposed controller is constructed through the latter approach; however, for the sake of completeness, stability analysis is now carried out through the former approach, too.

Taking the time-derivative of the Lyapunov function $V = x^T(P + P_u)x$ where $P + P_u$ is assumed to be positive definite in consistency with the Lyapunov theory yields

$$\dot{V} = \dot{x}^T(P + P_u)x + x^T(P + P_u)\dot{x} = (A(\theta)x + Bu)^T(P + P_u)x + x^T(P + P_u)(A(\theta)x + Bu)$$

Inserting $u = -R^{-1}B^T(P + P_u)x$ above gives the following.

$$\begin{aligned} \dot{V} &= \{A^T(\theta)x - B.(R^{-1}B^T(P + P_u)x)\}^T(P + P_u)x + x^T(P + P_u)\{A(\theta)x - B.(-R^{-1}B^T(P + P_u)x)\} \\ \dot{V} &= x^T\{A^T(\theta)(P + P_u) + (P + P_u)A(\theta) - (P + P_u)R^{-1}B^T(P + P_u))\}x \\ &\quad (P + P_u)A(\theta) - (P + P_u)R^{-1}B^T(P + P_u) - x^T\{(P + P_u)R^{-1}B^T(P + P_u)\}x \end{aligned}$$

Adding and subtracting x^TQx gives the following.

$$\dot{V} = x^T\{A^T(\theta)(P + P_u) + (P + P_u)A(\theta) - (P + P_u)R^{-1}B^T(P + P_u) + Q\}x - x^T\{(P + P_u)R^{-1}B^T(P + P_u) + Q\}x$$

The first two terms in the above expression are φ, which, as described in Eq. (12), has to be set to zero; therefore:

$$\dot{V} = -x^T\left\{Q + (P + P_u)BR^{-1}B^T(P + P_u)\right\}x \tag{20}$$

Without loss of generality, in the case of a second-order LPV system, the time-derivative of the Lyapunov function will read as:

$$\dot{V} = -x^T\left\{\begin{bmatrix} P_1 & 0 \\ 0 & q_2 \end{bmatrix} + \begin{bmatrix} p_1 & p_{u2} \\ p_{u2} & p_{u3} \end{bmatrix} BR^{-1}B^T \begin{bmatrix} p_1 & p_{u2} \\ p_{u2} & p_{u3} \end{bmatrix}\right\}x$$

resulting in

$$\dot{V} = -x^T\begin{bmatrix} q_1 + P_{u2}^2 & p_{u2}p_{u3} \\ p_{u2} & q_2 + p_{u3}^2 \end{bmatrix}x \tag{21}$$

Note that the since p_{u2} and p_{u3} both contain parametric variations in their expressions the time-derivative of the Lyapunov function may not remain negative definite for lower values of q_1 and q_2. Negative-definiteness of the time-derivative of the Lyapunov function can be guaranteed with large enough values of q_1 and q_2.

6. Motivating example

Let us consider a fourth-order linearized model of an inverted pendulum on a cart (IPS), as in Figure 2. The IPS is representative of more complex balance systems such as rocket and missile systems. At nominal values of parametric variations:

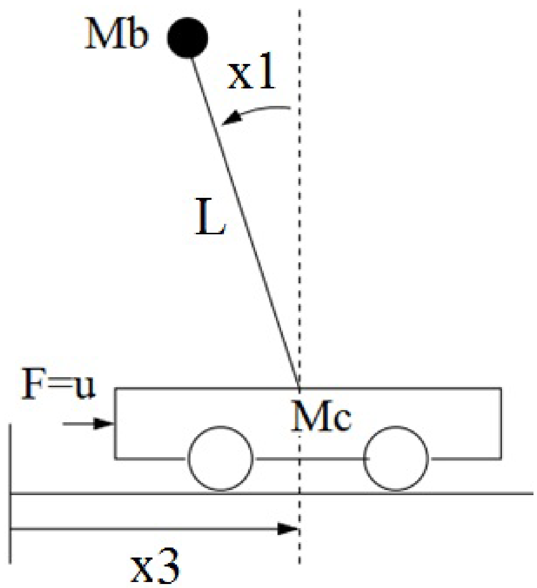

Figure 2. Model of inverted pendulum.

Nominal mass of cart = M_{c0} = 2 kg,
Nominal mass of ball = M_{b0} = 0.5 kg,
Length of rod connecting cart to ball = L = 0.5 m, and
Acceleration due to gravity = g = 9.8 m/s^2,

$$\begin{bmatrix} \dot{x}^*_{1(pendulum\ angle)} \\ \dot{x}^*_{2(pendulum\ angular\ velocity)} \\ \dot{x}^*_{3(cart\ displacement)} \\ \dot{x}^*_{4(cart\ velocity)} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \gamma & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x^*_1 \\ x^*_2 \\ x^*_3 \\ x^*_4 \end{bmatrix} + \begin{bmatrix} 0 \\ -2\beta \\ 0 \\ \beta \end{bmatrix} u \ with \quad Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, R = 1 \quad , \tag{22}$$

where $\alpha = \frac{(M_c+M_b)g}{M_c L}$, $\gamma = -\frac{M_b g}{M_c}$, and $\beta = \frac{1}{M_c}$.

The problem is to ensure robust optimal stabilization when the mass of the ball, M_{b0}, varies around the nominal value of 0.5 kg. The linearized model of the IPS has a vector control input; hence, this model is not suitable for the method proposed for synthesizing the proposed controller. A coordinate transformation is being proposed in this paper that will transform the current model to one with a scalar control input; the transformation and the transformed system are given below:

$$x^* = Tx \Rightarrow x = T^{-1}x^*, T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2\beta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \beta \end{bmatrix},$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -2\beta \\ 2g & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ \psi & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u,$$

where $\psi = \frac{\gamma}{\beta} = -gM_b$. Note that the cost functional, $\int_{t_0}^{T} (x^{*T}Qx^* + u^T Ru)dt$, needs to be transformed too into the new coordinate system, which is shown as follows: $\int_{t_0}^{T} (x^T Q_N x + u^T Ru)dt$, where $Q_N = TQT$. Now the parts of the systems matrix $A(\theta)$ for nominal and off-nominal values of parametric variations are:

$$A_0 + A_\delta = \begin{bmatrix} 0 & 1 & 0 & -2\beta_0 \\ 2g & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_0 \\ \psi_0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \psi - \psi_0 & 0 & 0 & 0 \end{bmatrix}.$$

The steps for synthesizing the proposed controller for the IPS are shown below.

- Solving $\varphi_{acausal} = 0$: The ARE in Eq. (13) is solved for a 4 $\times$ 4 matrix P as shown below:

$$P = \begin{bmatrix} p_1 & p_2 & p_4 & p_7 \\ p_2 & p_3 & p_5 & p_8 \\ p_4 & p_5 & p_6 & p_9 \\ p_7 & p_8 & p_9 & p_{10} \end{bmatrix}.$$

- Solving $\varphi_{causal} = 0$: Solving Eq. (14) using Eq. (13) yields

$$\begin{aligned} &p_{u7} = -p_7 + \psi - \sqrt{p_7^2 + 2p_7\psi - 4p_7\psi_0 + \psi_0^2}, p_{u8} = 0, \quad p_{u9} = 0, \\ &p_{u10} = -p_{10} + \sqrt{p_{10}^2 - 4p_{u7}\beta_0 + 2p_{u9}\beta_0}. \end{aligned}$$

Thus, the proposed controller in transformed coordinates becomes

$$\begin{aligned} u &= -(p_7 + p_{u7})x_1 - (p_8 + p_{u8})x_2 - (p_9 + p_{u9})x_3 - (p_{10} + p_{u10})x_4 \\ u &= -(\psi - \sqrt{p_7^2 + 2p_7\psi - 4p_7\psi_0 + \psi_0^2})x_1 - p_8x_2 - p_9x_3 - (\sqrt{p_{10}^2 - 4p_{u7}\beta_0})x_4 \end{aligned}. \tag{23}$$

Note that the proposed controller in Eq. (23) remains robust and optimal only in transformed coordinates. This whole expression needs to be transformed back into the original coordinate system using the same transformation $x = T^{-1}x^*$ yielding the following:

$$u = -(\psi - \sqrt{p_7^2 + 2p_7\psi - 4p_7\psi_0 + \psi_0^2})x_1^* - p_8x_2^* - p_9x_3^* - (2p_8 + \frac{\sqrt{p_{10}^2 - 4p_{u7}\beta_0}}{\beta_0})x_4^*. \tag{24}$$

7. Simulation results

Finally, the simulation results are presented to show the efficacy of the proposed method. Two different cases of variation in the mass of the ball, M_{b0}, are considered. These are described below.

Case I: Mass of ball at nominal value

When the mass of the ball, M_{b0}, is at its nominal value, i.e. 0.5 kg, $\psi = \psi_0$ in Eq. (24) simplifies it as

$$u = -p_7 x_1^* - p_8 x_2^* - p_9 x_3^* - (2p_8 + \frac{p_{10}}{\beta_0}) x_4^* . \tag{25}$$

With the following system matrices, solving $\varphi_{acausal} = 0$ results in matrix P as

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 19.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 \\ -4.9 & 0 & 0 & 0 \end{bmatrix}, \quad Q_N = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1.25 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$R = 1, P = \begin{bmatrix} 570.1 & 118.2 & 49.6 & -73.2 \\ 118.2 & 25.6 & 11.0 & -15.7 \\ 49.6 & 11.0 & 20.3 & -3.1 \\ -73.2 & -15.7 & -3.1 & 12.4 \end{bmatrix}.$$

Figure 3 shows the convergence of states of the proposed controller and an LQR determined through Eq. (22) to the origin at the nominal value of the mass of ball, i.e. $M_{b0} = 0.5$ kg. Figure 4 shows the costs incurred through the proposed controller and an LQR at the nominal value of mass of ball, i.e. $M_{b0} = 0.5$ kg.

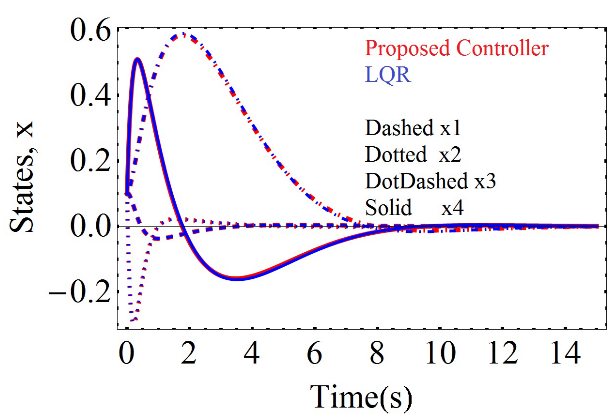

Figure 3. Convergence of states of the proposed controller and an LQR at nominal value of mass of ball, i.e. $M_{b0} = 0.5$ kg.

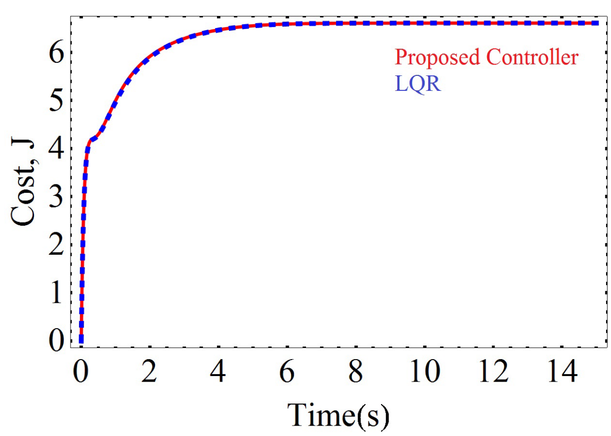

Figure 4. Costs incurred through the proposed controller and an LQR at nominal value of mass of ball, i.e. $M_{b0} = 0.5$ kg.

Case II: Mass of ball at off-nominal values

When the mass of ball, M_{b0}, is at off-nominal values then the proposed control law may not be able to stabilize the system properly. Figure 5 shows the state trajectories controlled through the proposed controller experiencing oscillatory behavior with the mass of the ball fixed at $M_b = 1.5$ kg and state penalizations of Case I.

In order to check the optimality of the proposed controller in real time, Figure 6 shows a more severe variation in the mass of ball M_b tending to its nominal value of $M_b = 0.5$ kg from an initial value of $M_b =$

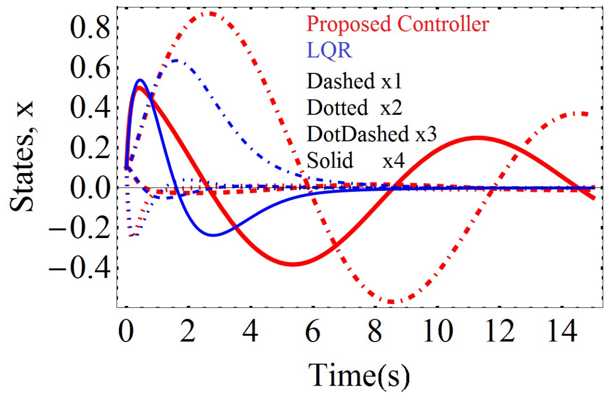

Figure 5. State trajectories controlled through the proposed controller experiencing oscillatory behavior with the mass of the ball fixed at $M_b = 1.5$ kg and state penalization of Case I.

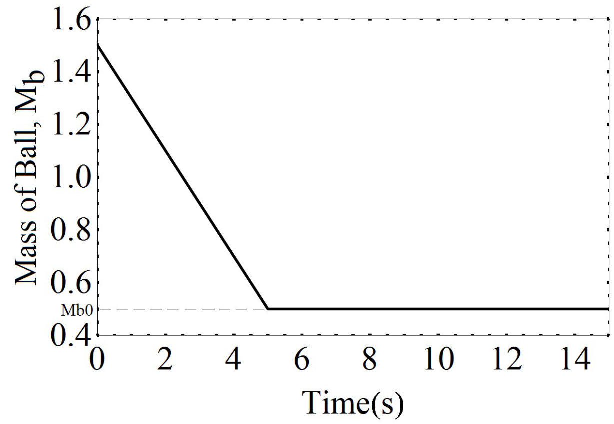

Figure 6. Mass of ball M_b tending to its nominal value of $M_b = 0.5$ kg from an initial value of $M_b = 1.5$ kg.

1.5 kg. Robust stability and robust performance can be recovered by an increase in state penalization through matrix Q in the cost functional. With $Q = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix} \Rightarrow Q_N = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & -10 \\ 0 & 0 & 10 & 0 \\ 0 & -10 & 0 & 12.5 \end{bmatrix}$, Figure 7 shows the convergence of states of the proposed controller and an LQR for M_{b0} changing from 1.5 kg to 0.5 kg. Figure 8 shows the comparison of costs incurred through the proposed controller and an LQR for M_{b0} changing from 1.5 kg to 0.5 kg. Figure 9 shows the variables gains p_{u7} and p_{u10} tending to 0 for M_{b0} changing from 1.5 kg to 0.5 kg. Finally, Figure 10 shows the proposed control signal to be continuous during stabilization for M_{b0} changing from 1.5 kg to 0.5 kg.

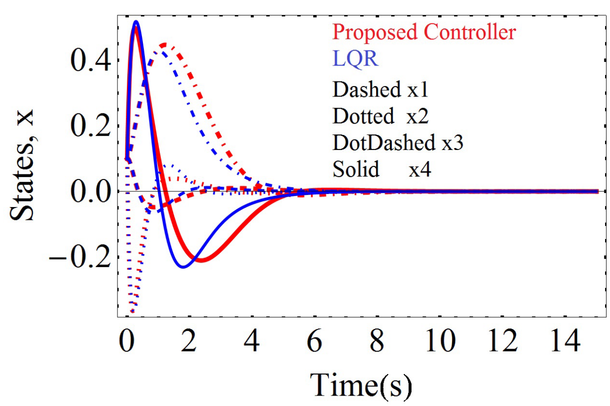

Figure 7. Convergence of states of the proposed controller and an LQR for M_{b0} changing from 1.5 kg to 0.5 kg.

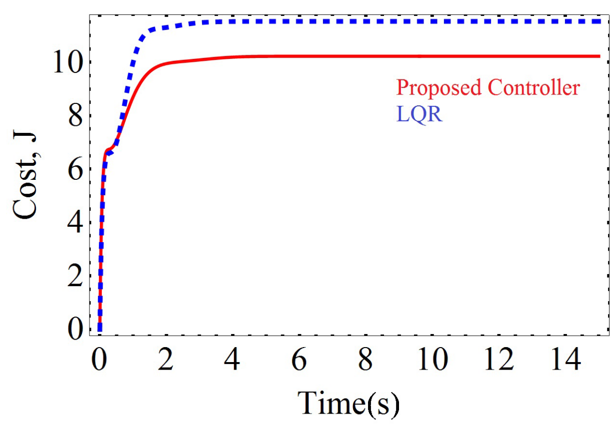

Figure 8. Costs incurred through the proposed controller and an LQR for M_{b0} changing from 1.5 kg to 0.5 kg.

8. Discussion

The quantitative aspect of the work in this paper has been highlighted in Sections 5, 6, and 7, followed by a discussion on the qualitative aspect in this section. Parametric variations have the potential to render a

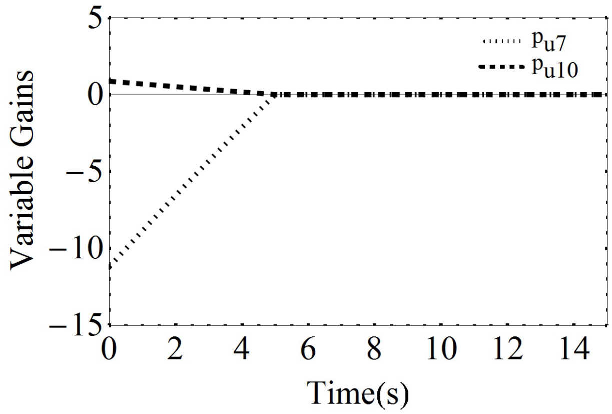

Figure 9. Variables gains p_{u7} and p_{u10} tending to 0 for M_{b0} changing from 1.5 kg to 0.5 kg.

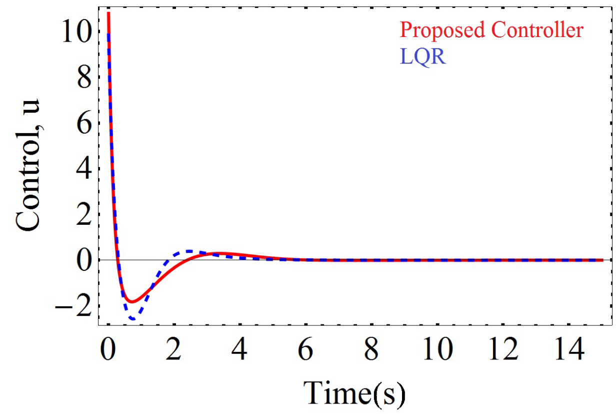

Figure 10. Proposed control signal being continuous during stabilization for M_{b0} changing from 1.5 kg to 0.5 kg.

control system suboptimal, unstable, and uncontrollable. Linear-quadratic regulation theory provides optimal stabilization of LTI and LTV systems because the system dynamics, $\dot{x} = Ax + Bu$ in the case of LTI systems and $\dot{x} = A(t)x + B(t)u$ in the case of LTV systems, are known a priori. In the case of LPV systems, where the exogenous parametric variations are unknown a priori, linear-quadratic regulation theory cannot provide optimal stabilization. Thus, this work reports that if parametric variations are assumed available through any observation method then linear-quadratic regulation can be extended for optimal stabilization of LPV systems.

Figures 3 and 4 show that the proposed controller acts just like a traditional LQR at nominal values of parametric variations; thus, the purpose of including Figures 3 and 4 is to show that the proposed controller is indeed an extension of LQR. Next, Figure 5 is used to show the destabilizing effect of fixing the mass of ball at an off-nominal value. In Figure 6, parametric variation is made even more severe by allowing the mass of the ball to vary from the fixed value towards the nominal value. Figures 7 and 8 show that the proposed controller is indeed optimally stabilizing as the cost incurred by it is less than that of LQR while the mass of ball tends to its nominal value as shown in Figure 6. Figure 9 is important as it shows the satisfaction of the following requirement:

$$u = \lim_{*a \to a_0} \left(-R^{-1}B^T\Psi\left(P,a\right)x\right) = -R^{-1}B^TPx, \text{ as } \lim_{*a \to a_0} \Psi\left(P,a\right) = P.$$

Comparing Figures 6 and 9 shows that the values of p_{u7} and p_{u10} in variable gains of the proposed controller tend to zero as the mass of ball tends to its nominal value from an off-nominal value. Thus, the proposed controller becomes a traditional LQR at nominal values.

9. Conclusion

A new method has been proposed in this paper for robust optimal stabilization of linear parameter-varying systems. The proposed controller is shown to have variable gains, dependent upon parametric variations, which simplify into a traditional LQR at nominal values of parametric variations. A coordinate transformation is utilized to cover those linear systems that have vector control inputs. The proposed controller has its controller gains influenced in real time by constant gains of LQR and instantaneous values of parametric variations. Incorporation of a variation estimation mechanism in the form of an observer is proposed as future work. Simulation results for the example of a fourth-order inverted pendulum system are also given for the efficacy of the method.

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