Turkish Journal of Electrical Engineering & Computer Sciences

http://journals.tubitak.gov.tr/elektrik/

Research Article

Turk J Elec Eng & Comp Sci
(2018) 26: 257 – 269

doi:10.3906/elk-1612-92

Novel representations for a coherent threshold reliability system: a tale of eight signal flow graphs

Ali Muhammad RUSHDI, Alaa Mohammad ALTURKI*
Department of Electrical and Computer Engineering, Faculty of Engineering, King Abdulaziz University, Jeddah, Saudi Arabia

Received: 07.12.2016 • **Accepted/Published Online:** 20.06.2017 • **Final Version:** 26.01.2018

Abstract: A threshold system is a reliability system whose success/failure is a threshold switching function in the successes/failures of its components. A coherent system (CS) is one that is causal, monotone, and with relevant components. The coherent threshold system (CTS), typically called the weighted k-out-of-n system, is consequently described by strictly positive weights and threshold. This paper presents recursive relations as well as boundary conditions for eight entities pertaining to a CTS. These are (a) expressions of monoform literals as well as disjoint or probability-ready expressions for either system success or failure, and (b) all-additive formulas as well as inclusion-exclusion ones for either system reliability or unreliability. These entities are obtained according to the best policy of implementing the Boole–Shannon expansion with respect to a higher-weight component before it is made with respect to a lower-weight one. With this best policy, the success and failure expressions with monoform literals are both minimal and shellable. Each of the eight entities considered is represented by an acyclic (loopless) signal flow graph (SFG). The SFG for system success or failure is isomorphic to a reduced ordered binary decision diagram, which is the optimal data structure for a Boolean function. The interrelations between the SFGs demonstrate optimal procedures for implementing (a) the probability (real) transform of a Boolean function, (b) inversion or complementation of a Boolean function, and (c) disjointing or orthogonalization of a sum-of-products expression of a Boolean function. The SFGs discussed herein for a CT can be extended to a general coherent system. They reduce to elegant symmetric regular graphs for the special case of a partially redundant system (k-out-of-n system).

Key words: Reliability, coherent threshold system, weighted k-out-of-n system, recursive relations, boundary conditions, signal flow graph, reduced ordered binary decision diagram

1. Introduction

A threshold system is one composed of n statistically independent 2-state components such that the success $S(\mathbf{X}) = S(X_1, X_2, \ldots, X_n)$ of the system is a threshold switching function in the successes $\mathbf{X}$ of the system components [1–3]. By definition, a switching function is a threshold function [1–4] i.f.f. there exists a set of real numbers $W_1, W_2, \ldots, W_n$ called weights, and T, called a threshold, such that

$$S(\mathbf{X}) = 1 \; i.f.f. \; \sum_{i=1}^{n} W_i X_i \geq T. \tag{1}$$

*Correspondence: aalturki0016@stu.kau.edu.sa

Similarly to [2,3], a function S(**X**) satisfying Eq. (1) is denoted herein by H(n; **X**; **W**; T), while the corresponding reliability is called R(n; **p**; **W**; T). A threshold system can be neither symmetric nor coherent [2]. However, threshold systems of significant practical utility are typically coherent. Therefore, we will deal herein with coherent threshold systems (CTSs). A coherent system is causal, monotone, and of relevant components [5]. A monotone system is one whose reliability function is a nondecreasing function of each component reliability. The coherent threshold system is typically referred to in the literature as the weighted k-out-of-n:G system [4]. The k-out-of-n:G system can be defined as a CTS with a common positive weight for its components and a threshold equal to k multiplied by this common weight [2,3,5]. A quadratic-time iterative algorithm for computing the k-out-of-n reliability is based on binary recursive relations together with appropriate boundary conditions [5]. The algorithm achieves recursion removal by utilizing a signal flow representation of these relations and conditions. Later, this algorithm was adapted for computing the reliability of a threshold system [2] and further explained in [3], where a comparison is given for various policies of arranging components. The policy used in [2] is called the best policy in [3] and demands that a component of a higher weight be handled before one of a lower weight. This paper presents recursive relations as well as boundary conditions for eight entities pertaining to a CTS. These are (a) expressions of monoform literals as well as disjoint or probability-ready expressions for system success and failure, and (b) all-additive formulas as well as inclusion-exclusion ones for system reliability and unreliability. The system components are arranged according to the best policy of taking a higher-weight component before a lower-weight one. With this best policy, the success and failure expressions with monoform literals are both minimal and shellable. Each of the eight entities considered is represented by an acyclic (loopless) signal flow graph (SFG). The SFG for system success or failure is isomorphic to a reduced ordered binary decision diagram (ROBDD) [5]. The interrelations between the SFGs demonstrate optimal procedures implementing (a) the probability (real) transform of a Boolean function, (b) inversion or complementation of a Boolean function, and (c) disjointing or orthogonalization of a SOP expression of a Boolean function. The SFGs discussed herein for a CTS (weighted k-out-of-n system) can be extended to a general coherent system. They reduce to elegant symmetric regular graphs for the special case of a partially redundant system (k-out-of-n system). The organization of the remainder of this paper is as follows. Section 2 presents some useful nomenclature necessary for understanding the rest of the paper. Section 3 presents useful expansions for general switching (Boolean) functions and specifies these to monotonically nondecreasing and monotonically nonincreasing switching functions. Section 4 develops the aforementioned expansions to ones for the success, failure, reliability, and unreliability of a CTS and translates them (together with boundary conditions) into appropriate SFGs. This is followed by a detailed discussion in Section 5 on the merits and interrelations of the SFGs introduced in Section 4. Section 5 also points out what happens when symmetry is imposed on the TS to reduce it to an ordinary k-out-of-n system. Section 6 concludes the paper.

2. Nomenclature

Probability-Ready Expression (PRE): A Boolean expression that is directly convertible, on a one-to-one basis, to a probability expression called the probability or real transform [6]. In a PRE:

1. All ORed terms (products) are disjoint, and

2. All ANDed alterms (sums) are statistically independent.

The transition from a PRE to a probability expression is achieved by replacing Boolean variables by their expectations, AND operations by multiplications, and OR operations by additions [6–8].

Shellability: A shelling of the sum-of-products formula (DNF)

$$\bigvee_{k=1}^{m} C_k \tag{2}$$

is a permutation ($C_{\pi(1)}, C_{\pi(2)}, \ldots, C_{\pi(m)}$) of its terms such that, for each k = 2, 3, ..., m, the expression

$$\bar{C}_{\pi(1)}\bar{C}_{\pi(1)}\ldots\bar{C}_{\pi(k-1)}C_{\pi(k)} \tag{3}$$

is equal to an elementary conjunction or to 0. A DNF is called shellable if it admits a shelling [9].

Shadow: Let $A_1, A_2, \ldots, A_m$ be an ordered list of subsets of {1, 2, ..., n} . For k = 1, 2, ..., m, the shadow of A_k depends on the order in which these subsets are listed and is given by the set $S(A_1, A_2, \ldots A_k)$ = {j ∈ {1, 2, . . ., n} : there exists $\ell < k \leq m$ such that A_ℓ / A_k = {j} } [9].

3. On expansions of Boolean functions

An arbitrary Boolean function $f(\mathbf{X}) = f(X_1, X_2, \ldots, X_{i-1}, X_i, X_{i+1}, \ldots, X_n)$ can be expanded as

$$f(\mathbf{X}) = \overline{X}_i f_0 \vee X_i f_1 = \overline{X}_i f_0 \oplus X_i f_1 = f_0 \oplus X_i f_2 = \overline{X}_i f_2 \oplus f_1, \tag{4}$$

where f_0 and f_1 are restrictions, subfunctions, ratios, quotients, or cofactors of $f(\mathbf{X})$ given by

$$f_0 = f(\mathbf{X}|0_i) = f(X_1, X_2, \ldots, X_{i-1}, 0, X_{i+1}, \ldots, X_n), \tag{5}$$

$$f_1 = f(\mathbf{X}|1_i) = f(X_1, X_2, \ldots, X_{i-1}, 1, X_{i+1}, \ldots, X_n), \tag{6}$$

$$f_2 = f_0 \oplus f_1. \tag{7}$$

Eq. (4) expresses $f(\mathbf{X})$ by two versions of the Boole–Shannon expansion [10], the positive Davio expansion [11], and the negative Davio expansion [11], respectively. A coherent system is characterized by a monotonically nondecreasing success function $S(\mathbf{X})$ and a monotonically nonincreasing failure function $\overline{S}(\mathbf{X})$ [3]. The subfunctions $S_1 = S(\mathbf{X}|1_i)$ and $S_0 = S(\mathbf{X}|0_i)$ for such a system satisfy the pairs of identities shown in Table 1, which are evident from the Venn diagram in Figure 1, and hence the success of a coherent system is given via Eqs. (4) and (8e1) as

Table 1. Pairs of identities satisfied by the subfunctions of a coherent success.

$\overline{S}_1 \leq \overline{S}_0,$	(8a2)	$S_0 \leq S_1,$	(8a1)
$\overline{S}_0 = \overline{S}_0 \vee \overline{S}_1,$	(8b2)	$S_1 = S_1 \vee S_0,$	(8b1)
$S_0 = S_0 S_1,$	(8c2)	$\overline{S}_1 = \overline{S}_1 \overline{S}_0,$	(8c1)
$\overline{S}_0 \vee S_1 = 1,$	(8d2)	$S_0 \overline{S}_1 = 0,$	(8d1)
$\overline{S}_2 = \overline{S}_0.\overline{S}_1 = S_0 \vee \overline{S}_1$	(8e2)	$S_2 = S_0 \oplus S_1 = \overline{S}_0 S_1,$	(8e1)

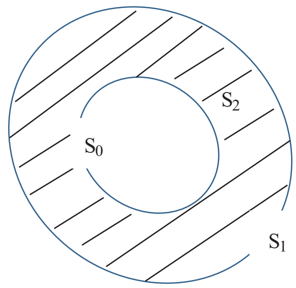

Figure 1. Venn diagram representing the subfunctions S_0 and S_1 of the success of a coherent system, together with their Boolean difference S_2.

$$S(\mathbf{X}) = S_0 \oplus X_i\overline{S}_0 S_1 = S_0 \vee X_i\overline{S}_0 S_1 = S_0 \vee X_i S_1, \tag{9}$$

where the XOR operator ($\oplus$) is replaced by the OR operator ($\vee$) since the terms S_0 and $X_i\overline{S}_0\ S_1$ are disjoint and the literal $\overline{S}_0$ is dropped by virtue of the reflection law [12]. Eq. (9) indicates that S($\mathbf{X}$) can be expressed solely in terms of uncomplemented literals X_i $\{1 \leq i \leq n\}$. Similarly, the failure $\overline{S}(\mathbf{X})$ of a coherent system is given via Eqs. (4) and (8b2) as

$$\overline{S}(X) = \overline{X}_i\overline{S}_0 \vee X_i\overline{S}_1 = \overline{X}_i(\overline{S}_0 \vee \overline{S}_1) \vee X_i\overline{S}_1 = \overline{X}_i\overline{S}_0 \vee (\overline{X}_i \vee X_i)\overline{S}_1 = \overline{X}_i\overline{S}_0 \vee \overline{S}_1. \tag{10}$$

Eq. (10) indicates that $\overline{S}(\mathbf{X})$ can be expressed solely in terms of complemented literals $\overline{X}_i$.

4. Application to a coherent threshold system

The subfunctions of threshold system success are also successes of threshold systems [2,3], namely

$$S_0 = S(\mathbf{X}|0_i) = H(n-1, \mathbf{X}/X_i,\ \mathbf{W}/W_i, T), \tag{11}$$

$$S_1 = S(\mathbf{X}|1_i) = H(n-1, \mathbf{X}/X_i,\ \mathbf{W}/W_i, T - W_i). \tag{12}$$

We now want to develop expansions for eight entities pertaining to a CTS. These eight entities are displayed in the Karnaugh-map-like structure of Figure 2 [13]. The structure is characterized by:

	Probability Domain		Boolean Domain	
Negative Expression	Inclusion-Exclusion Formula for Unreliability: U_{IE}	All-Additive Formula for Unreliability: U	PRE Failure: $\overline{S}_{PRE}$	Minimal and Shellable Failure: $\overline{S}$
Positive Expression	Inclusion-Exclusion Formula for Reliability: R_{IE}	All-Additive Formula for Reliability: R	PRE Success: S_{PRE}	Minimal and Shellable Success: S
		Orthogonal expression		Disjointness

Figure 2. A Karnaugh-map-like depiction of the eight entities discussed in this paper.

1. The set of four entities in the right half of the map is in the Boolean domain with a related set of corresponding four entities in the probability domain in the left half of the map. Transfer from each entity in the former set to the corresponding one in the latter set is obtained via the probability or real transform [6].

2. The set of four entities in the top half of the map is negative entities (failures or unreliabilities), with a related set of corresponding four positive entities (successes or reliabilities) in the bottom half of the map. Bidirectional transfer between two entities corresponding to one another is obtained via inversion or complementation [3,12,14].

3. The set of four entities in the middle half of the map is orthogonal expressions while the ones in the two extreme columns are not. Transfer from the fourth to the third column is achieved via the disjointness operation [15].

In summary, the eight entities include expressions of monoform literals as well as disjoint or probability-ready expressions for either system success or system failure. They also include all-additive formulas as well as single-polarity ones for either system reliability or system unreliability. The eight entities are handled via recursions indicated in Table 2. The system success and failure are governed by recursions that have already appeared in Eqs. (9) and (10), respectively, while the PRE success and failure are governed by Eqs. (13) and (14), respectively, which are versions of Eq. (4). Eqs. (15) and (16) for the system reliability and unreliability are obtained by taking the expectation of both sides of each of Eqs. (13) and (14), respectively. Finally, the single-polarity expressions of Eqs. (17) and (18), for reliability and unreliability expressions, are obtained by rewriting ($q_i = 1 - p_i$) in Eqs. (15) and (16), respectively. These are essentially a rephrasing of recursive inclusion-exclusion [15], namely

Table 2. The recursions used to obtain the eight entities in Figure 2. Each of these recursions must be supplemented by appropriate limiting cases or boundary conditions.

Probability domain	Boolean domain
$U_{IE}(\mathbf{p}) = U(\mathbf{p}\|0_i) - p_i U(\mathbf{p}\|0_i) + p_i U(\mathbf{p}\|1_i), \quad 1 \leq i \leq n,$ (18)	$\overline{S}(X) = \overline{X}_i \overline{S}_0 \vee \overline{S}_1, \quad 1 \leq i \leq n,$ (10)
$R_{IE}(\mathbf{p}) = R(\mathbf{p}\|0_i) - p_i R(\mathbf{p}\|0_i) + p_i R(\mathbf{p}\|1_i), \quad 1 \leq i \leq n,$ (17)	$S(\mathbf{X}) = S_0 \vee X_i S_1, \quad 1 \leq i \leq n,$ (9)
$U(\mathbf{p}) = q_i U(\mathbf{p}\|0_i) + p_i U(\mathbf{p}\|1_i), \quad 1 \leq i \leq n,$ (16)	$\overline{S}_{PRE}(\mathbf{X}) = \overline{X}_i \overline{S}_0 \vee X_i \overline{S}_1, \quad 1 \leq i \leq n,$ (14)
$R(\mathbf{p}) = q_i R(\mathbf{p}\|0_i) + p_i R(\mathbf{p}\|1_i), \quad 1 \leq i \leq n,$ (15)	$S_{PRE}(\mathbf{X}) = \overline{X}_i S_0 \vee X_i S_1, \quad 1 \leq i \leq n,$ (13)

$$R_1 = E\{T_1\}, \tag{19a}$$

$$R_j = E\{\bigvee_{i=1}^{j} T_i\} = R_{j-1} + E\{T_j\} - E\{T_j \wedge \bigvee_{i=1}^{j-1} T_i\} \tag{19b}$$

$$= R_{j-1} + E\{T_j \wedge \bigwedge_{i=1}^{j-1} \overline{T}_i\}, 2 \leq j \leq j_T. \tag{19c}$$

$$U_1 = E\{C_1\}, \tag{20a}$$

$$U_j = E\{\bigvee_{i=1}^{j} C_j\} = U_{j-1} + E\{C_j\} - E\{C_j \wedge \bigvee_{i=1}^{j-1} C_i\} \tag{20b}$$

$$= U_{j-1} + E\{C_j \wedge \bigwedge_{i=1}^{j-1} \overline{C}_i\}, 2 \le j \le j_C. \tag{20c}$$

Note that the subfunctions used in Table 2 are obtained from Eqs. (11) and (12), or their expectations

$$R(\mathbf{p}|0_i) = E\{S(\mathbf{X}|0_i)\} = R(n-1, \mathbf{p}/p_i,\ \mathbf{W}/W_i, T), \tag{21}$$

$$R(\mathbf{p}|1_i) = E\{S(\mathbf{X}|1_i)\} = R(n-1, \mathbf{p}/p_i,\ \mathbf{W}/W_i, T - W_i), \tag{22}$$

with obviously similar results for the complementary quantities $\overline{S}$ and U. All the expansions in Table 2 are implemented for 1 ≤ i ≤ n, employing the best policy of Rushdi [2] and Rushdi and Alturki [3]. This policy requires that an expansion or recursion be made w.r.t. a higher-weight component before it is made w.r.t. a lower-weight one. The recursive relations in Table 2 are not sufficient to decide the pertinent quantities. They must be supplemented with limiting cases or boundary conditions [2,3]. A typical set of such conditions for a CTS is shown in Table 3.

Table 3. Typical sets of boundary conditions for a CTS.

For T ≤ 0	S(n; **X**; **W**; T) = 1, (23a)	$\overline{S}$(n; X; W; T) = 0, (23b)
	R(n; **p**; **W**; T) = 1.0, (23c)	U(n; **p**; **W**; T) = 0.0, (23d)
For $\sum_{i=1}^{n} W_i <$ T	S(n; **X**; **W**; T) = 0, (24a)	$\overline{S}$(n; X; W; T) = 1, (24b)
	R(n; **p**; **W**; T) = 0.0, (24c)	U(n; **p**; **W**; T) = 1.0, (24d)

The information of the recursive relations in Table 2 and the conditions of Eqs. (23), (24), (19a), and (19b) now allow us to draw a SFG for computing each of the eight entities in Table 2, for a system of n = 6, **W** = [1 2 3 4 5 5], and T = 7. Each SFG is drawn on an appropriate grid of weights and threshold. The source nodes in any of the SFGs are of two types: black nodes of value 1 or white nodes of value 0. Except for Figure 4 for computing R(6, **p**, **W**, T), which is similar to the figures of Rushdi [2] and Rushdi and Alturki [3], all other figures are novel ones that have not appeared earlier in the open literature. Though the SFGs are strikingly similar, they have subtle differences in their transmittances as well as nature of source nodes. These differences are shown in Table 4. The SFGs are acyclic (loopless), and a simple application of Mason's gain formula [16] produces the following expressions for the pertinent eight quantities:

$$S = X_5X_6 \vee X_4X_6 \vee X_4X_5 \vee X_3X_6 \vee X_3X_5 \vee X_3X_4 \vee X_2X_6 \vee X_2X_5 \vee X_1X_2X_4, \tag{25}$$

$$S_{PRE} = X_5X_6 \vee X_4\overline{\mathbf{X}}_\mathbf{5}X_6 \vee X_4X_5\overline{\mathbf{X}}_\mathbf{6} \vee X_3\overline{\mathbf{X}}_\mathbf{4}\overline{\mathbf{X}}_\mathbf{5}X_6 \vee X_3\overline{\mathbf{X}}_\mathbf{4}X_5\overline{\mathbf{X}}_\mathbf{6} \vee X_3X_4\overline{\mathbf{X}}_\mathbf{5}\overline{\mathbf{X}}_\mathbf{6} \vee X_2\overline{\mathbf{X}}_\mathbf{3}\overline{\mathbf{X}}_\mathbf{4}\overline{\mathbf{X}}_\mathbf{5}X_6$$

$$\vee X_2\overline{\mathbf{X}}_\mathbf{3}\overline{\mathbf{X}}_\mathbf{4}X_5\overline{\mathbf{X}}_\mathbf{6} \vee X_1X_2\overline{\mathbf{X}}_\mathbf{3}X_4\overline{\mathbf{X}}_\mathbf{5}\overline{\mathbf{X}}_\mathbf{6}, \tag{26}$$

$$R = p_5p_6 + p_4q_5p_6 + p_4p_5q_6 + p_3q_4q_5p_6 + p_3q_4p_5q_6 + p_3p_4q_5q_6 + p_2q_3q_4q_5p_6 +$$

$$p_2q_3q_4p_5q_6 + p_1p_2q_3p_4q_5q_6, \tag{27}$$

$$R_{IE} = p_5p_6 + p_4p_6 + p_4p_5 + p_3p_6 + p_3p_5 + p_3p_4 + p_2p_6 + p_2p_5 + p_1p_2p_4 - 2p_4p_5p_6 - 2p_3p_4p_6$$

$$-2p_3p_5p_6 + 3p_3p_4p_5p_6 - 2p_3p_4p_5 - p_2p_3p_6 - p_2p_4p_6 - 2p_2p_5p_6 + p_2p_3p_4p_6 + 2p_2p_3p_5p_6$$

Table 4. Comparison of the eight signal flow graphs.

Fig.	Represents	Horizontal weight	Inclined weight	Upper leaves	Lower Leaves	Sink node	Sink node value
3	Shellable and minimal success	1	X_i	1	0	S	Eq. (25)
4	PRE success	$\overline{X}_i$	X_i	1	0	S_{PRE}	Eq. (26)
5	Reliability	q_i	p_i	1	0	R	Eq. (27)
6	Inclusion-exclusion reliability	1, – p_i	p_i	1	0	R_{IE}	Eq. (28)
7	Shellable and minimal failure	$\overline{X}_i$	1	0	1	$\overline{S}$	Eq. (29)
8	PRE failure	$\overline{X}_i$	X_i	0	1	$\overline{S}_{PRE}$	Eq. (30)
9	Unreliability	q_i	p_i	0	1	U	Eq. (31)
10	Iunnclusi-exclusion unreliability	q_i	1, – q_i	0	1	U_{IE}	Eq. (32)

$$+2p_2p_4p_5p_6 - 2p_2p_3p_4p_5p_6 - p_2p_3p_5 - p_2p_4p_5 + p_2p_3p_4p_5 - p_1p_2p_3p_4 - p_1p_2p_4p_5 - p_1p_2p_4p_6$$

$$+p_1p_2p_3p_4p_5 + p_1p_2p_3p_4p_6 + p_1p_2p_4p_5p_6 - p_1p_2p_3p_4p_5p \tag{28}$$

$$\overline{S} = \overline{X}_4\overline{X}_5\overline{X}_6 \vee \overline{X}_2\overline{X}_3\overline{X}_5\overline{X}_6 \vee \overline{X}_2\overline{X}_3\overline{X}_4\overline{X}_5 \vee \overline{X}_2\overline{X}_3\overline{X}_4\overline{X}_6 \vee \overline{X}_1\overline{X}_3\overline{X}_5\overline{X}_6, \tag{29}$$

$$\overline{S}_{PRE} = \overline{X}_4\overline{X}_5\overline{X}_6 \vee \overline{X}_2\overline{X}_3\mathbf{X_4}\overline{X}_5\overline{X}_6 \vee \overline{X}_2\overline{X}_3\overline{X}_4\overline{X}_5\mathbf{X_6} \vee \overline{X}_2\overline{X}_3\overline{X}_4\mathbf{X_5}\overline{X}_6 \vee \overline{X}_1\mathbf{X_2}\overline{X}_3\mathbf{X_4}\overline{X}_5\overline{X}_6, \tag{30}$$

$$U = q_4q_5q_6 + q_2q_3p_4q_5q_6 + q_2q_3q_4q_5p_6 + q_2q_3q_4p_5q_6 + q_1p_2q_3p_4q_5q_6, \tag{31}$$

$$U_{IE} = q_4q_5q_6 + q_2q_3q_5q_6 - 3q_2q_3q_4q_5q_6 + q_2q_3q_4q_5 + q_2q_3q_4q_6 + q_1q_3q_5q_6$$

$$-q_1q_2q_3q_5q_6 - q_1q_3q_4q_5q_6 + q_1q_2q_3q_4q_5q_6. \tag{32}$$

In passing, we demonstrate via Table 5 that $\overline{S}$ in Eq. (29) is not only minimal, but it is also shellable. Shellability of the formula $\overline{S}$ in Eq. (29) is evident since its disjointed version $\overline{S}_{PRE}$ in Eq. (30) has exactly the original terms of Eq. (29) with each term (beyond the first) replaced by a subsuming one that has additional noncomplemented literals ANDed with it.

5. Features and interrelations of the eight SFGs

We have already pointed out certain interrelations among the eight SFGs of Figures 3–10, as they are evenly divided among the three independent dimensions of Boolean versus probability, monoform versus biform, and positive versus negative. Figures 3–6 represent the positive expressions, while Figures 7–10 represent the negative expressions. As stated earlier, the eight SFGs in Figures 3–10 are very similar with possible differences in their transmittances and nature of source nodes, as shown in Table 5. These SFGs constitute pictorial proofs for many important results:

Table 5. Demonstration that the threshold system achieves shellability for the minimal failure.

i	C_i	Set A_i	Shadow $S(A_i)$	$\bar{C}_1\bar{C}_2\ldots\bar{C}_{i-1}$	$\bar{C}_1\bar{C}_2\ldots\bar{C}_{i-1}C_i$
1	$\bar{X}_4\bar{X}_5\bar{X}_6$	{4,5,6}	ϕ	1	$\bar{X}_4\bar{X}_5\bar{X}_6$
2	$\bar{X}_2\bar{X}_3\bar{X}_5\bar{X}_6$	{2,3,5,6}	{4}	$X_4 \vee X_5 \vee X_6$	$\bar{X}_2\bar{X}_3X_4\bar{X}_5\bar{X}_6$
3	$\bar{X}_2\bar{X}_3\bar{X}_4\bar{X}_5$	{2,3,4,5}	{6}	$X_2X_4 \vee X_3X_4 \vee X_5 \vee X_6$	$\bar{X}_2\bar{X}_3\bar{X}_4\bar{X}_5 X_6$
4	$\bar{X}_2\bar{X}_3\bar{X}_4\bar{X}_6$	{2,3,4,6}	{5}	$X_2X_4 \vee X_3X_4 \vee X_5 \vee X_2X_6 \vee X_3X_6 \vee X_4X_6$	$\bar{X}_2\bar{X}_3\bar{X}_4X_5\bar{X}_6$
5	$\bar{X}_1\bar{X}_3\bar{X}_5\bar{X}_6$	{1,3,5,6}	{2, 4}	$X_2X_4 \vee X_3X_4 \vee X_2X_5 \vee X_3X_5 \vee X_4X_5 \vee X_5X_6 \vee X_2X_6 \vee X_3X_6 \vee X_4X_6$	$\bar{X}_1X_2\bar{X}_3X_4\bar{X}_5\bar{X}_6$
				$X_1X_2X_4 \vee X_3X_4 \vee X_2X_5 \vee X_3X_5 \vee X_4X_5 \vee X_5X_6 \vee X_2X_6 \vee X_3X_6 \vee X_4X_6$	

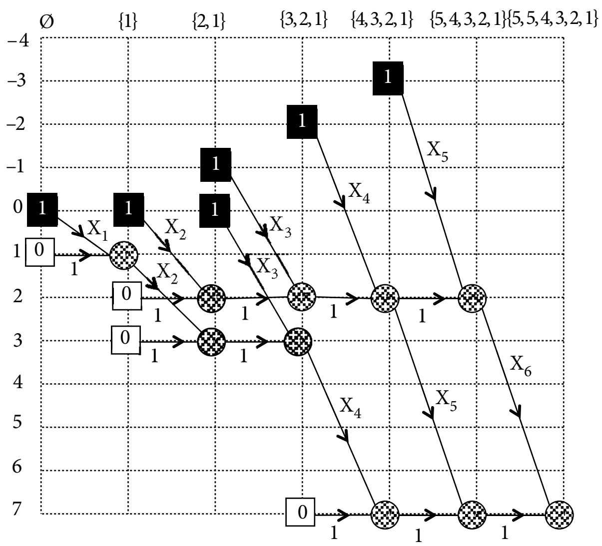

Figure 3. A shellable and minimal representation for the threshold-system success S = H(6; **X**; {1, 2, 3, 4 , 5, 5}; 7).

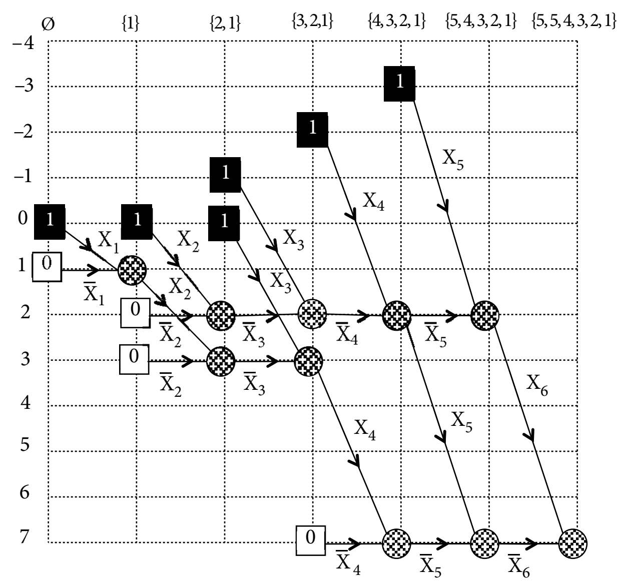

Figure 4. A PRE representation for the threshold-system success S_{PRE} = H(6; **X**; {1, 2, 3, 4 , 5, 5}; 7).

5.1. The functions S and $\overline{S}$ for a CTS system are unate

A function is called unate if it is possible to express it using only noncomplemented literals or complemented ones. In the former case, the function is said to be a monotonically nondecreasing function in its variables, while in the latter case it is said to be a monotonically nonincreasing function in its variables. The SFG in Figure 3 is pictorial proof that the success S of the CTS considered (as expressed in Eq. (25)) is a monotonically nondecreasing function in its arguments. Note that no edge transmittance in Figure 3 is a complemented variable (each edge transmittance is either 1 or X_i). Similarly, Figure 7 is pictorial proof that the failure $\overline{S}$ of the CTS considered (as expressed in Eq. (29)) is a monotonically nonincreasing function in its arguments. This is due to the fact that no edge transmittance in Figure 7 is an uncomplemented literal (each edge transmittance is either 1 or $\overline{X}_i$). Note that similar pictorial proofs for the unateness of the success and failure of a general coherent system (not necessarily threshold) are also possible.

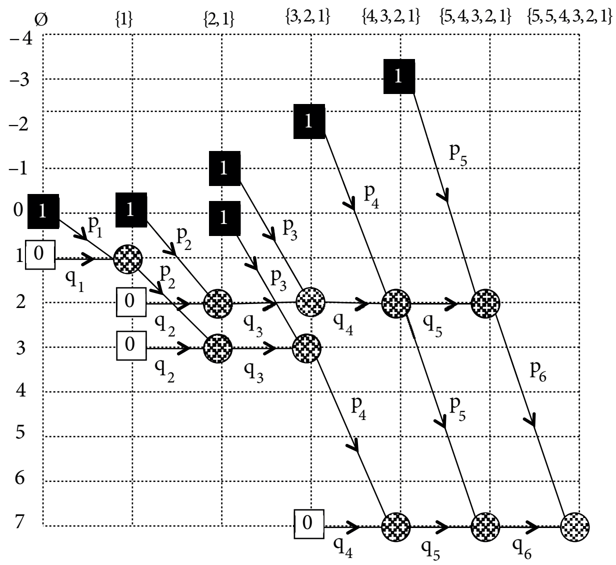

Figure 5. A representation for the threshold-system reliability R(6; **p**; {1, 2, 3, 4 , 5, 5}; 7).

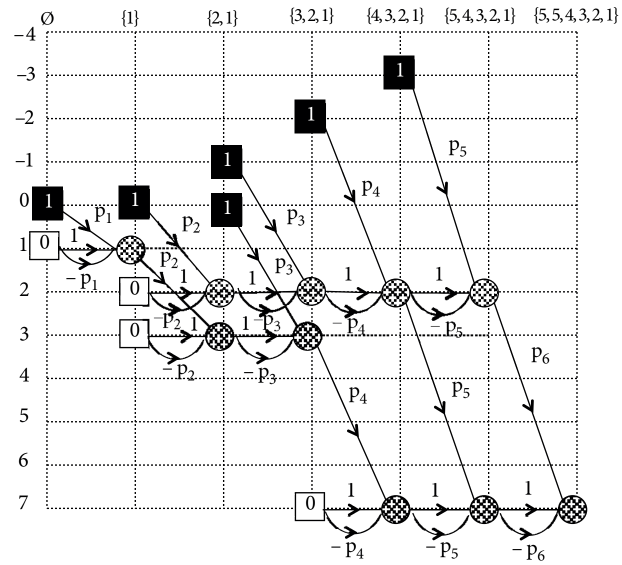

Figure 6. Inclusion-exclusion representation for the threshold-system reliability R(6; **p**; {1, 2, 3, 4, 5, 5}; 7).

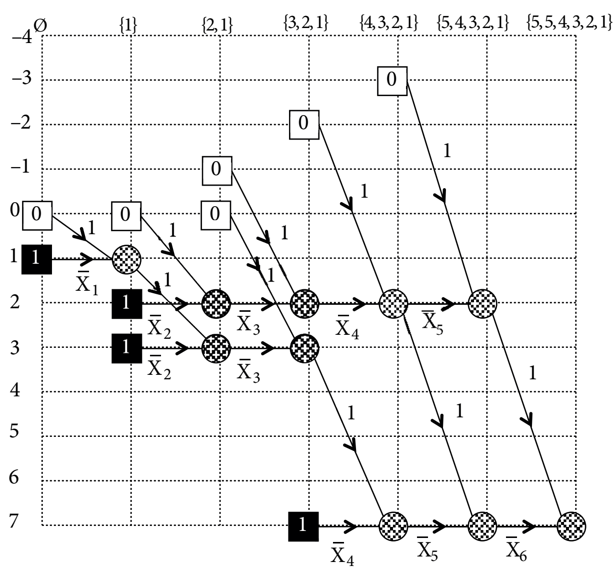

Figure 7. A shellable and minimal representation for the threshold-system failure $\bar{S} = \overline{H}$ (6; X; {1, 2, 3, 4, 5, 5}; 7).

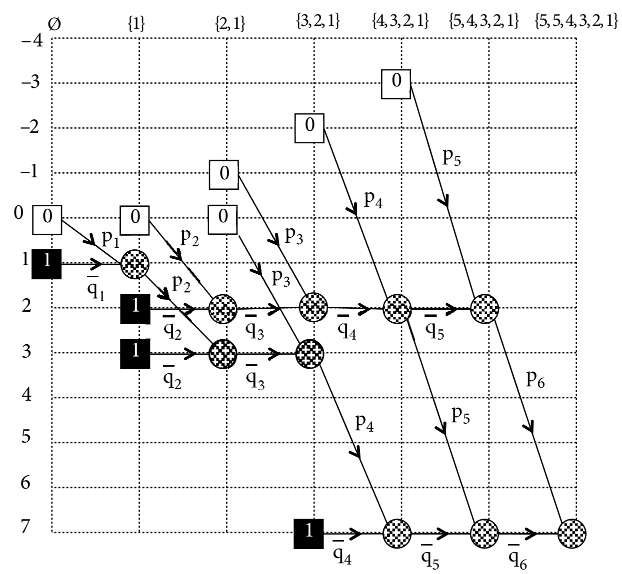

Figure 8. PRE representation for the threshold-system failure $\bar{S}_{PRE} = \overline{H}$ (6; **X**; {1, 2, 3, 4 , 5, 5}; 7).

5.2. The expressions for S and $\bar{S}$ are complete and minimal sums

Each of the expressions of Eq. (25) and Eq. (29) for S and $\bar{S}$ is a syllogistic formula (a sum-of-products formula that contains all the prime implicants of the pertinent function [17]), and also an absorptive formula (one that has no term that can be absorbed by another [17]). Hence, each of these expressions is a complete sum or a Blake canonical form (a disjunction of all prime implicants, and nothing else). Since the complete sum and minimal sum are identical for a unate function [9,12], the expressions of Eqs. (25) and (29) are also the minimal expressions for the unate functions S and $\bar{S}$.

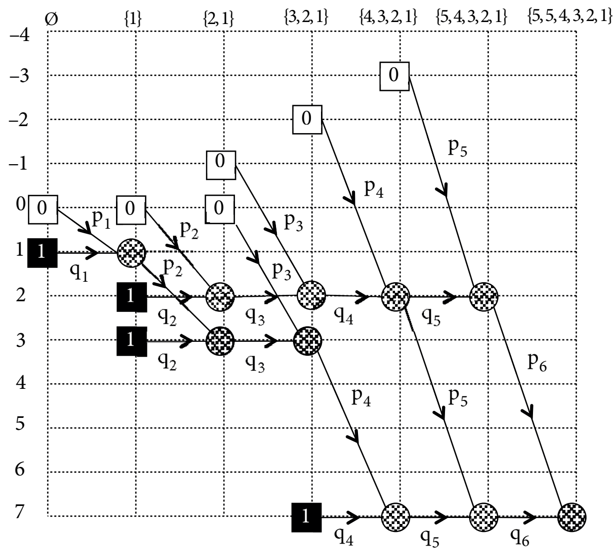

Figure 9. A representation for the threshold-system unreliability U(6; **p**; {1, 2, 3, 4 , 5, 5}; 7).

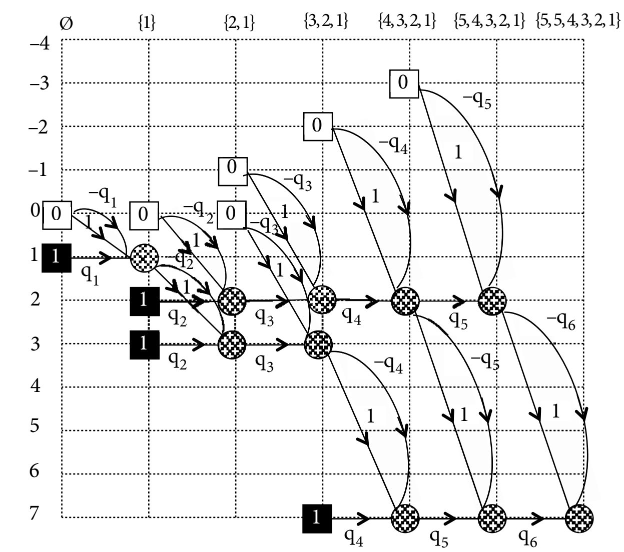

Figure 10. Inclusion-exclusion representation for the threshold-system unreliability U(6; **p**; $\{1,2,3,4,5,5\}$; 7).

5.3. The expressions for S and $\bar{S}$ are shellable

The expression S in Eq. (25) is obtained from Figure 3, while the expression S_{PRE} in Eq. (26) is obtained from Figure 4. The only difference between Figure 3 and Figure 4 is that each horizontal transmittance of 1 in Figure 3 is replaced by a complemented literal transmittance $\overline{X}_i$ in Figure 4. The paths contributing to S_{PRE} in Figure 4 are exactly those contributing to S in Figure 3 with the sole exception that the horizontal edge transmittance (for paths beyond the first) are changed from 1 to complemented literals. This means that the number of terms of the formula S in Eq. (25) does not change while going to S_{PRE} in Eq. (26), i.e. S is shellable, and the algorithm for constructing S_{PRE} via Figure 4 is a shelling for it. Likewise, the expression $\bar{s}$ obtained via Figure 7 in Eq. (29) is shellable, and the algorithm for constructing $\bar{S}_{PRE}$ in Eq. (30) via Figure 8 is a shelling for it. This is an immediate result of the fact that each oblique transmittance of 1 in Figure 7 is replaced by a noncomplemented transmittance X_i in Figure 8, and the fact that the paths contributing to $\bar{S}_{PRE}$ are identical (apart from differences in oblique transmittances) to those contributing to $\bar{S}$ in Figure 7.

5.4. The SFGs of S and $\bar{S}$ are essentially ROBDDs

The ROBDD was proposed by Bryant [18] as an extension of the BDD methodology of Akers [19]. The ROBDD deals with general switching (two-valued Boolean) functions and is now considered the state-of-the-art data structure for handling such functions, with extensive applications in reliability [20,21]. The SFGs for S in Figure 3 and $\bar{S}$ in Figure 7 are, in fact, implementations of the ROBDD for the class of unate switching functions. Apart from the unateness restriction, these SFGs have the same features as the ROBDD algorithm, namely both these SFGs and the ROBDD algorithms are based on the Boole–Shannon expansion in the Boolean domain.

1. Both entities visit the variables in a certain order, typically monotonically ascending or monotonically descending.
2. Both entities reduce the resulting expansion tree (which is exponential in size) to a rooted acyclic graph that is both canonical and hopefully compact or subexponential. The reduction rules [21] require (a)

merging isomorphic subtrees, and (b) deletion of useless nodes whose outgoing edges point to the same child node.

5.5. Efficient inverse algorithms

Availability of complementary SFGs allows for a pedagogical understanding of many existing complementation or inversion algorithms [14] both for the Boolean and probability domains. For example, one can start from the sink node of S_{PRE} in Figure 4, use expansion until the leaf nodes are reached, complement the value of each leaf node (replace 0 by 1 and 1 by 0), thereby effectively transferring to Figure 8, and finally go back to the sink of Figure 8, which is $\bar{S}_{PRE}$. Other ways for complementation are possible. One can perform the converse operation of going from $\bar{S}$ to S (expand in Figure 8, complement leaves, transfer to Figure 4, and go to its sink). One can also achieve disjoint complementation by going from S (Figure 3) to $\bar{S}_{PRE}$ (Figure 8). Complementation is also possible in the probability domain by going from R (Figure 5) to U (Figure 9) or vice versa.

5.6. Symmetric coherent threshold systems

The SFGs discussed herein for a CT (weighted k-out-of-n) system reduce to elegant symmetric regular graphs for the special case of a partially redundant system (k-out-of-n system). A sample of these graphs is given in Figures 11 and 12 for the success and failure of the symmetric coherent system $H(4; \mathbf{X}; \{1, 1, 1, 1\}; 3)$, which can be identified as a 3-out-of-4:G or a 2-out-of-4:F system. To stress symmetry, these graphs are drawn on a rectangular grid of axes $k_1 = k$ and $k_2 = n - k$.

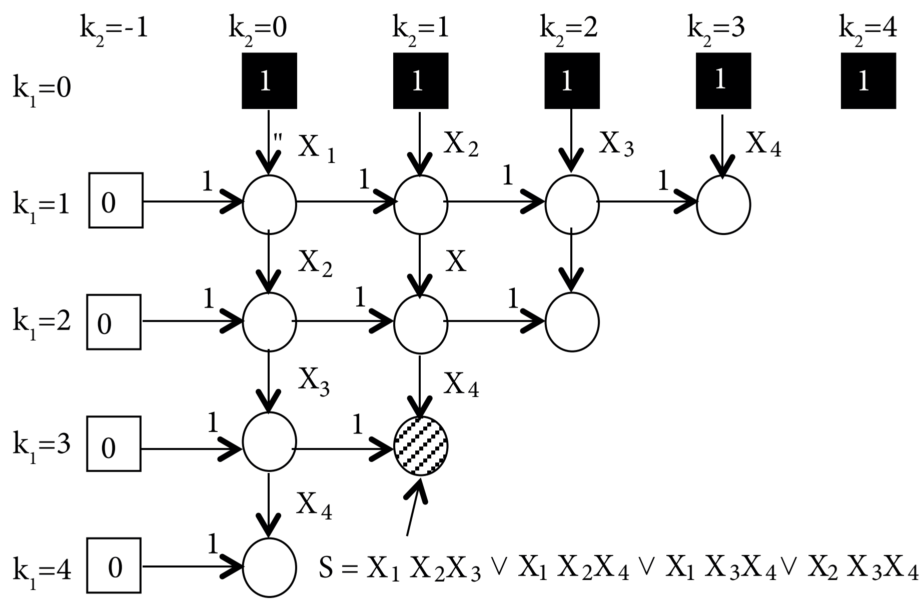

Figure 11. A shellable and minimal representation for the success of a 3-out-of-4:G system.

6. Conclusions

A CTS is a very useful reliability model. Many of its features, probabilities, and algorithms are studied herein in terms of various recursive relations and boundary conditions, which are pictorially displayed in terms of various loopless SFGs. The success and failure of a CTS are shown to be unate Boolean functions whose minimal and

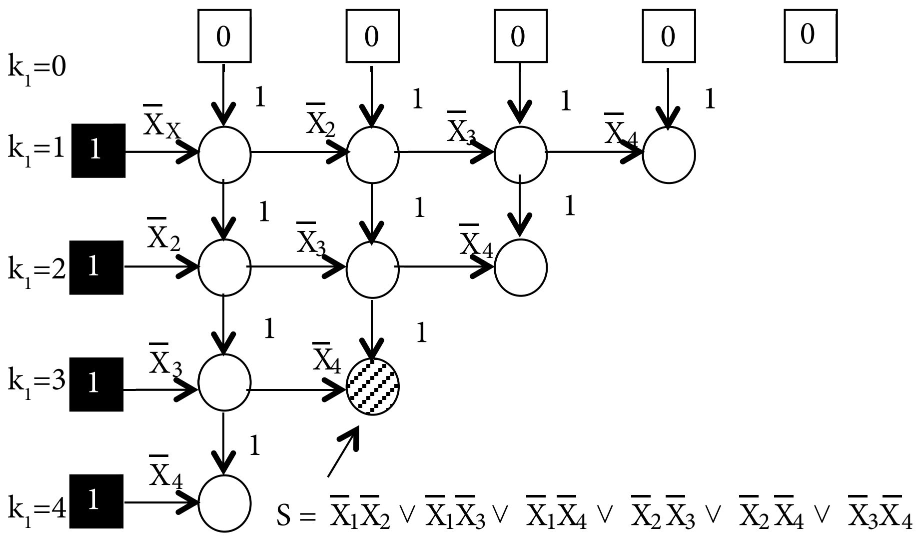

Figure 12. A shellable and minimal representation for the failure of a 2-out-of-4:F system.

complete sum expressions are identical. These expressions are shellable and allow the introduction of PRE expressions of the same number of terms. Interrelations among the SFGs demonstrate optimal procedures for mutual complementation among S and $\bar{S}$, for disjointing S and $\bar{S}$ to obtain the PRE expressions S_{PRE} and $\bar{S}_{PRE}$. The probability or real transforms of S_{PRE} and $\bar{S}_{PRE}$ (namely, the reliability R and unreliability U) are obtained by replacing logical variables by their expectations and replacing ANDing and ORing operations by arithmetic multiplication and addition. The probability transforms of S and $\bar{S}$ are exactly the same as those of S_{PRE} and $\bar{S}_{PRE}$ and can be obtained in a two-step fashion by first converting S and $\bar{S}$ to S_{PRE} and $\bar{S}_{PRE}$ and then transforming them. These transforms are also obtained directly via the conventional inclusion-exclusion principle or via a recursive version of it. The graph complexity of each of the SFGs encountered herein is exponential in the worst case (even with the best policy). However, it becomes subexponential in many prominent cases. For example, it becomes quadratic for the symmetric CTS, i.e. the k-out-of-n system.

References

[1] Ball M, Provan J. Disjoint products and efficient computation of reliability. Oper Res 1988; 36: 703-715.

[2] Rushdi A. Threshold systems and their reliability. Microelectron Reliab 1990; 30: 299-312.

[3] Rushdi A, Alturki A. Reliability of coherent threshold systems. J Appl Sci 2015; 15: 431-443.

[4] Eryilmaz S. Capacity loss and residual capacity in weighted k-out-of-n: G systems. Reliab Eng Syst Safe 2015; 136: 140-144.

[5] Rushdi A. Partially-redundant systems: examples, reliability, and life expectancy. International Magazine on Advances in Computer Science and Telecommunications 2010; 1: 1-13.

[6] Rushdi A, Ghaleb F. The Walsh spectrum and the real transform of a switching function: a review with a Karnaugh-map perspective. Journal of Engineering and Computer Sciences, Qassim University 2014; 7: 73-112.

[7] Rushdi A, Goda A. Symbolic reliability analysis via Shannon's expansion and statistical independence. Microelectron Reliab 1985; 25: 1041-1053.

[8] Rushdi A, Rushdi MA. Switching-algebraic analysis of system reliability. In: Ram M, Davim P, editors. Advances in Reliability and System Engineering. Management and Industrial Engineering Series. Cham, Switzerland: Springer International Publishing, 2016. pp. 139-161.

[9] Crama Y, Hammer P. Boolean Functions: Theory, Algorithms, and Applications. Cambridge, UK: Cambridge University Press, 2011.

[10] Rushdi A, Ghaleb F. A tutorial exposition of semi-tensor products of matrices with a stress on their representation of Boolean functions Journal of King Saud University - Computer and Information Sciences 2016; 5: 3-30.

[11] Sasao T, Butler J. The eigenfunction of the Reed-Müller transformation. In: Proceedings of the Workshop on Applications of the Reed Muller Expansion in Circuit Design, and Representation and Methodology of Future Computing Technology; 16 May 2007; Oslo, Norway.

[12] Muroga S. Logic Design and Switching Theory. New York, NY, USA: Wiley, 1979.

[13] Rushdi A. Karnaugh map. In: Hazewinkel M, editors. Encyclopaedia of Mathematics, Supplement Volume I. Boston, MA, USA: Kluwer Academic Publishers, 1997. pp. 327-328.

[14] Rushdi A. Map derivation of the minimal sum of a switching function from that of its complement. Microelectron Reliab 1985; 25: 1055-1065.

[15] Rushdi A, Al-Khateeb D. A review of methods for system reliability analysis: a Karnaugh-map perspective. In: Proceedings of the First Saudi Engineering Conference; 1983; Jeddah, Saudi Arabia. pp. 57-95.

[16] Golnaraghi F, Kuo B. Automatic Control Systems. 9th ed. New York, NY, USA: Wiley, 2009.

[17] Brown F. Boolean Reasoning: The Logic of Boolean Equations. Boston, MA, USA: Kluwer Academic Publishers, 1990.

[18] Bryant R. Graph-based algorithms for Boolean function manipulation. IEEE T Comput 1986; 35: 677-691.

[19] Akers S. Binary decision diagrams. IEEE T Comput 1960; C-27: 509-516.

[20] Rauzy A. Binary decision diagrams for reliability studies. In: Misra KB, editor. Handbook of Performability Engineering. London, UK: Springer, 2008.

[21] Mo Y, Xing L, Amari S, Dugan J. Efficient analysis of multi-state k-out-of-n system. Reliab Eng Syst Safe 2015; 133: 95-105.