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Outputs bounds for linear systems with repeated input signals: existence, computation and application to vehicle platooning

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Abstract: This paper investigates the effect of repeated time-limited input signals on the output excursion of stable, linear time-invariant systems. It is first shown that the maximum norm of the output signal remains bounded if the repeated input signals are separated by a nonzero dwell time. Then a novel method for computing a tight bound on the output signal norm is proposed. The setting of the paper is motivated by a vehicle platooning application, where vehicles repeatedly open/close gaps in order to perform lane changes. The developed method analyzes driving safety by computing a bound on the spacing error between vehicles when performing repeated open/close gap maneuvers.

Key words: Linear systems, time-limited inputs, vehicle following

1. Introduction

Platooning is a promising concept for improving the road capacity and traffic safety [1–4]. Platooning is based on vehicle following at a small distance, which is realized by cooperative adaptive cruise control (CACC) in the recent literature [5–7]. In addition, platoons need to be modified when performing vehicle maneuvers such as lane changes. This requires designing longitudinal maneuvers such as opening gaps for vehicles entering a platoon. Since such maneuvers are disturbances for CACC systems, their adverse effect on driving safety within a platoon needs to be analyzed.

Accordingly, the main subject of this paper is designing longitudinal maneuvers in a vehicle platoon and analyzing their effect on driving safety. The first contribution is the development of a general framework for maneuvers that are represented by a set of time-limited input signals and that are applied to linear time-invariant (LTI) systems. In this framework, a method for quantifying the effect of repeated time-limited input signals on the output signal norm of stable LTI systems is proposed. As the second contribution, it is shown that the application of an arbitrary number of such input signals leads to a bounded output signal if the input signals are separated by a nonzero dwell time. The third contribution is a novel computational method for calculating a tight bound on the output signal norm. Using this method, the effect of repeated open/close gap maneuvers on driving safety in vehicle platoons is analyzed.

The existing literature does not consider the vehicle-following application studied in this paper. Related work focuses on the suitable timing of lane changes using simplified vehicle models [8–10]. In addition, there is no existing method for quantifying the effect of repeated input signals on the output signal norm of LTI

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systems. Only the response of LTI systems to certain types of input signals is investigated in several research works. Bounds for the maximum singular value of the impulse response matrix are determined in [11,12], while [13] computationally evaluates the L_∞-induced norm of LTI systems. Different operator norms are defined in [14,15], while [16] provides explicit formulas for their evaluation. Several conditions of the L_2 and L_∞ norm of the input signal and its slope are used in [17,18]. Different from the setting in this paper, the cited approaches do not address the application of repeated input signals and do not consider time-limited input signals.

The remainder of the paper is organized as follows. Section 2 motivates the considered platooning application and formalizes the problem statement. The existence and computation of bounds on the output response for repeated time-limited input signals is studied in Section 3 and illustrated by a vehicle-platooning example. Section 4 gives conclusions.

2. Motivation

2.1. Lane change maneuver

The problem considered in this paper is motivated by the application of vehicle following in dense traffic, as illustrated in Figure 1. Here, each vehicle i must follow its predecessor vehicle $i-1$ in a platoon at a small safe distance $d_{r,i}$. In the recent literature [5–7], $d_{r,i}$ is specified by the headway time h, the desired distance at standstill r_i, the length L_i, and the velocity v_i of vehicle i as

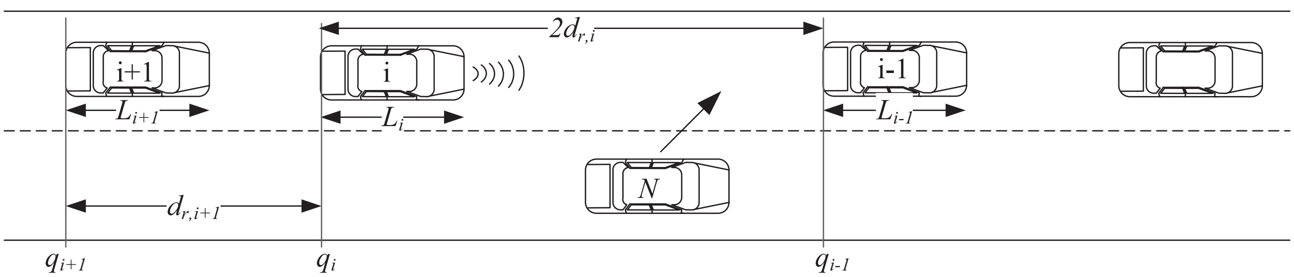

Figure 1. Vehicle platoon: vehicle following and gap opening for lane changes.

$$d_{r,i} = r_i + L_i + hv_i \tag{1}$$

In addition to vehicle following, gaps between vehicles have to be opened/closed if vehicles enter or leave an existing platoon, as illustrated in Figure 1. Here, vehicle i at position q_i opens a gap of length $2d_{r,i}$ to vehicle $i-1$, such that the new vehicle N can safely enter the platoon.

In this setting, vehicle following is realized by an extension of the CACC architecture in Figure 2, derived from [6]. Vehicle $i+1$ follows vehicle i, assuming that both vehicles have the plant transfer function $G(s) = \frac{1}{(1+\tau s)s^2}$ with the time constant τ of the driveline dynamics. Vehicle $i+1$ receives the input signal u_i via a filter transfer function K_{ff} from vehicle i by vehicle-to-vehicle communication. In addition, vehicle $i+1$ measures the intervehicle spacing $d_{i+1} = q_i - q_{i+1} - L_{i+1}$, where d_{i+1} is used to control the distance error

$$e_{i+1} = q_i - q_{i+1} - d_{r,i+1} = d_{i+1} - r_{i+1} - hv_{i+1} \tag{2}$$

with the controller transfer function K_{fb} and the spacing policy transfer function $H(s) = 1 + hs$. Since the controller design for vehicle following in the described architecture is not the subject of this paper, the existing

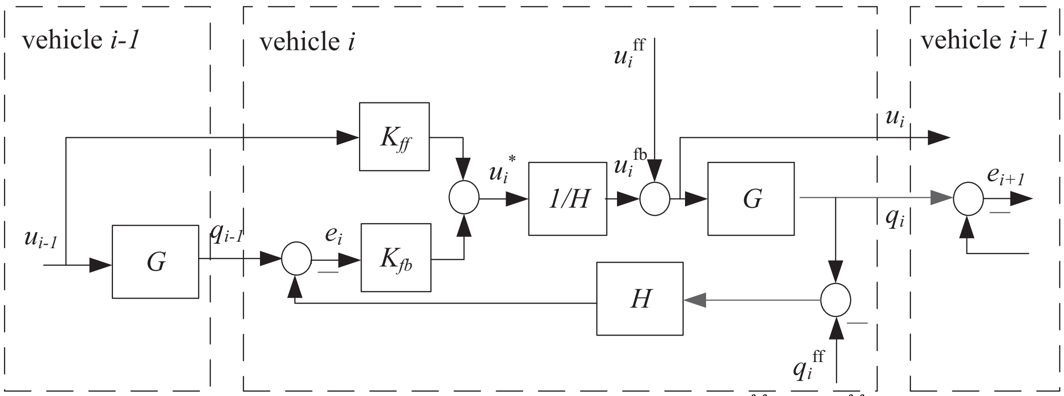

Figure 2. Feedback loop for CACC extended by u_i^{ff} and q_i^{ff}.

H_∞ controller design in [6] is used for the computation of K_{ff} and K_{fb}. The controllers used in this paper are

$$K_{ff} = \frac{1.04s^4 + 37.8s^3 + 350s^2 + 1047s + 734}{s^4 + 36.6s^3 + 336s^2 + 1036s + 734} \text{ and } K_{fb} = \frac{2.7s^4 + 93s^3 + 747s^2 + 884s + 228}{s^4 + 36.6s^3 + 336s^2 + 1036s + 734}.$$

The remaining parameters are $\tau = 0.1$, $h = 0.7$, $L_i = 5$, and $r_i = 5$. In order to perform gap opening and closing maneuvers of a vehicle i in the described architecture, a feedforward input signal u_i^{ff} and a feedforward reference signal q_i^{ff} for vehicle i are introduced. Here, q_i^{ff} and u_i^{ff} are computed such that $Q_i^{ff}(s) = G(s)U_i^{ff}(s)$. Hence, the feedback loop for vehicle following is not affected by the application of u_i^{ff}.

2.2. Input signals

If vehicle i opens/closes a gap, the vehicle distance d_i should be increased/decreased by the velocity-dependent value $d_{r,i}$ within a certain time T. This behavior can be formulated in the form of a linear optimal control problem with state constraints:

$$\min J = \int_0^T F\left(z_i, u_i^{ff}, t\right) dt \tag{3}$$

subject to the constraints

$$\dot{q}_i = v_i;\ \dot{v}_i = a_i;\ \dot{a}_i = -\frac{1}{\tau}a_i + \frac{1}{\tau}u_i^{ff} \tag{4}$$

$$q_i(0) = 0,\ v_i(0) = v,\ a_i(0) = 0,\ q_i(T) = d_{r,i},\ v_i(T) = v,\ a_i(T) = 0, \tag{5}$$

$$v_{min} \le v_i(t) \le v_{\max},\ a_{min} \le a_i(t) \le a_{\max} \tag{6}$$

J denotes the objective function with the terminal time T and Eq. (4) is a state space realization of $G(s)$ for vehicle i with the state $z_i = [\, q_i \ \ v_i \ \ a_i \,]'$. Eq. (5) states initial and terminal conditions assuming that the platoon travels at a constant velocity v. In order to maintain driving comfort, the acceleration and velocity variation during a maneuver are limited using Eq. (6). Depending on the desired maneuver, different objective functions can be used. In this paper, $F_1\left(z_i, u_i^{ff}, t\right) = 1$ minimizes the maneuver time and $F_2\left(z_i, u_i^{ff}, t\right) = \left(u_i^{ff}\right)^2$ minimizes the accumulated input signal. Example input signals for opening gaps at different velocities and with different objective functions are generated using the PROPT solver [19] according to the Table and are shown together with the created gap and acceleration in Figure 3. The same signals can be used for closing gaps when multiplying by -1.

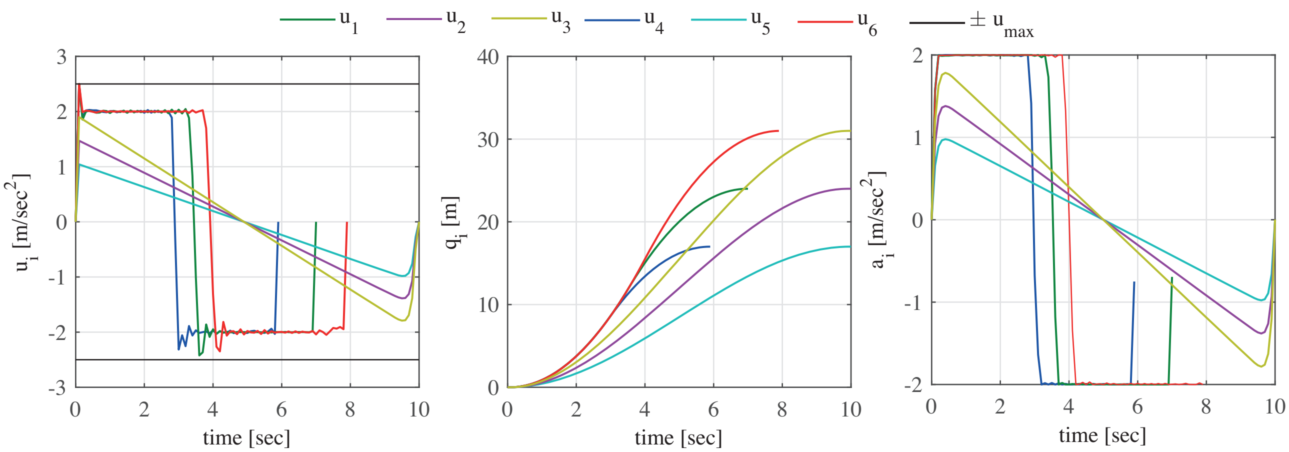

Figure 3. Different input signals for $T \leq 10$ s and related output responses.

Table. Input signals for different velocities and objective functions.

$v = 10\frac{m}{s}$, F$_1$	$v = 20\frac{m}{s}$ F$_1$	$v = 30\frac{m}{s}$, F$_1$	$v = 10\frac{m}{s}$, F$_2$	$v = 20\frac{m}{s}$, F$_2$	$v = 30\frac{m}{s}$, F$_2$
u_1	u_2	u_3	u_4	u_5	u_6

2.3. Problem statement

It has to be noted that, while u_i^{ff} is computed for the maneuver of vehicle i, there is an effect on the distance error e_{i+1} of the follower vehicle $i+1$ via the transfer function

$$\frac{E_{i+1}(s)}{U_i(s)} = \frac{G - K_{ff}G}{1 + K_{fb}G} \tag{7}$$

This effect is small (below 0.1 m) when opening a single gap, as seen in Figure 4.

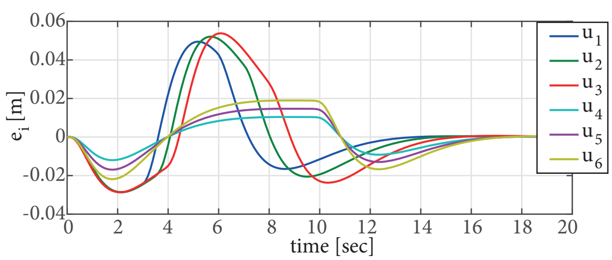

Figure 4. Error signal when opening a gap for different input signals.

However, it cannot be directly deduced how/if the distance error accumulates with a potentially negative effect on driving safety in the case of arbitrarily repeated open/close gap maneuvers of vehicle i. Accordingly, the problem addressed in this paper is to quantify the effect of repeated open/close gap maneuvers on the distance error e_{i+1}. Hereby, it has to be noted that the system model in Eq. (7) is linear and the input signals designed in Section 2.2 are time-limited in the sense that they are nonzero only for a certain time interval, as seen in Figure 3.

In order to formalize the stated problem, the paper focuses on the repeated application of time-limited input signals to LTI systems with the state space model

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{8}$$

$A \in R^{n \times n}$ is the dynamics matrix, $B \in R^{n \times p}$ is the input matrix, $C \in R^{q \times n}$ is the output matrix, $x(t) \in R^n$ is the system state, $u(t) \in R^p$ is the input signal, and $y(t) \in R^q$ is the output signal. The impulse response matrix of the system in Eq. (8) is written as γ. The input signals are time-limited with a maximum magnitude $u_{\max}$ and a time-limit $t_l < \infty$ such that the signal value is zero after t_l. Writing $||\cdot||$ for the vector 2-norm, the set of time-limited input signals is defined as

$$U_{u_{\max},t_l} = \{u : R \to R^p | \, \|u\,(t)\| \leq u_{\max} \text{ for } 0 \leq t \leq t_l, \, u\,(t) = 0 \, \text{otherwise}\} \tag{9}$$

Considering the application example, it can be seen from Figure 3 that the considered gap opening/closing scenarios require input signal levels that are bounded by ± 2.5 m/s^2 and their duration is below 10 s. That is, the set of time-limited input signals $U_{2.5,10}$ can be employed for this application example.

In order to formulate the repeated application of input signals in $U_{u_{\max},t_l}$ to an LTI system as in Eq. (8), a minimum dwell time Δ between two input signal applications is assumed. This assumption is justified by the practical fact that open gap maneuvers for different lane changes are separated in time. The time instants for input signal applications are formalized as the set Q_Δ of monotonically increasing infinite time sequences with dwell time Δ:

$$Q_\Delta = \{(t_\nu)_{\nu=0}^{\infty} \, |t_0 \geq 0, \, t_{\nu+1} - t_\nu \geq \Delta, \nu{=}0, 1, \ldots\} \tag{10}$$

Then the repeated application of input signals $u_\nu \in U_{u_{\max},t_l}$ for a given time sequence $(t_\nu)_{\nu=0}^{\infty} \in Q_\Delta$ is represented by the signal

$$u_{(t_\nu)_{\nu=0}^{\infty}}\,(t) = \sum_{\nu=0}^{\infty} u_\nu(t - t_\nu) \tag{11}$$

In this expression, the time-limited input signal $u_\nu \in U_{u_{\max},t_l}$ is applied at t_ν. Using the notions introduce above, the aim of the paper is to determine a bound on the output signal norm $\|y\,(t)\|$ over time when applying a repeated input signal $u_{(t_\nu)_{\nu=0}^{\infty}}\,(t)$ to the LTI system in Eq. (8) for arbitrary input signals $u_\nu \in U_{u_{\max},t_l}$ and sequences $(t_\nu)_{\nu=0}^{\infty} \in Q_\Delta$.

Problem 1. For a stable LTI system with impulse response matrix γ, find K_y such that

$$\sup_{(t_v)_{\nu=0}^{\infty} \in Q_\Delta, t \geq 0} \|y\,(t)\| = \sup_{(t_v)_{\nu=0}^{\infty} \in Q_\Delta, t \geq 0} \left\|\gamma(t) \star u_{(t_\nu)_{\nu=0}^{\infty}}\,(t)\right\| \leq K_y < \infty \tag{12}$$

Solving Problem 1 for the vehicle following example with input signals in $U_{2.5,10}$ and the output signal $y = e_{i+1}$ quantifies the effect of repeated open/close gap maneuvers on the distance error in order to evaluate driving safety.

3. Bound existence and computation

3.1. Bound existence

The first important question regarding Problem 1 is if a finite bound K_y in Eq. (12) exists. Theorem 1 shows that, indeed, $K_y < \infty$ for any $U_{u_{\max},t_l}$ and stable LTI system.

Theorem 1 *Consider a stable LTI system with the impulse response matrix γ. Let $\Delta > 0$ and $U_{u_{\max},t_l}$ be given for $t_l u_{\max} > 0$. Then there exists a $K_y < \infty$ such that Eq. (12) holds.*

Note that proofs of all formal results are given in the Appendix. Theorem 1 implies that the output signal is bounded whenever applying an arbitrary number of bounded input signals with dwell time Δ. Suitable bounds K_y are computed in the next section. Regarding the vehicle-following example, Theorem 1 ensures that the distance error is bounded when performing an arbitrary number of open/close gap maneuvers that are separated in time by at least Δ.

3.2. Prerequisite for the bound computation

This paper develops a method for computing a tight bound on the output signal norm $\|y\,(t)\|$ when applying repeated input signals in $U_{u_{\max},t_l}$ according to Eq. (12) in Problem 1. As a prerequisite for this computation, Lemma 1 assumes that the output response after applying a single input signal in $U_{u_{\max},t_l}$ is bounded by a nonnegative monotonically decreasing function $f\,(t)$. Then Lemma 1 determines a bound on the output signal norm $\|y\,(t)\|$ when applying repeated input signals in $U_{u_{\max},t_l}$.

Lemma 1 *Let $f : R \to R$ be a function with $f\,(t) = 0$ for $t < 0, f\,(t) \geq 0$ for $t \geq 0$ and $f\,(t) \geq f(t')$ for allt, t' with $t \leq t'$. Assume that $\Delta > 0$ and $\|y\,(t)\| = \|\gamma(t) \star u(t)\| \leq f(t)$ for any input signal $u \in U_{u_{\max},t_l}$. Then it holds that*

$$\sup_{(t_v)_{\nu=0}^{\infty} \in Q_\Delta, t\geq 0} \|y\,(t)\| \leq \sup_{(t_v)_{\nu=0}^{\infty} \in Q_\Delta, t\geq 0} \sum_{\nu=0}^{\infty} f\,(t - t_\nu) = \sum_{\nu=0}^{\infty} f\,(\nu\Delta) \tag{13}$$

That is, the bound in Eq. (12) can be evaluated by the sum in Eq. (13) if it is possible to find a monotonic bound $f\,(t)$ on the output signal when applying any input signal in $U_{u_{\max},t_l}$.

3.3. Monotonic bound for a single input signal

This section develops a method for determining a monotonic bound for the output response after a single input signal application (such as a single open/close gap maneuver) that is required for the computation of the output response bound according to Eq. (13). First, a bound on the output response for any input signal in $U_{u_{\max},t_l}$ is determined using a monotonic bound for the impulse response of the LTI system.

Lemma 2 *Consider a stable LTI system with the impulse response matrix γ. Let $c(t)$ be a function that is zero for $t < 0$ and nonnegative monotonically decreasing for $t \geq 0$ such that $\|\gamma(t)\| \leq c(t)$ for all $t \in R$. Then it holds for any u$\in U_{u_{\max},t_l}$ that*

$$\|y(t)\| \leq u_{\max} \int_0^{t_l} c(t-\tau) d\tau. \tag{14}$$

The bound in Eq. (14) is zero for $t<0$, has a maximum at $t=t_l$ and is nonnegative monotonically decreasing for $t \geq t_l$.

Respecting Lemma 2, a nonnegative monotonically decreasing bound for $\|y(t)\|$ is

$$\|y(t)\| \leq f(t) := u_{\max} \begin{cases} \int_0^{t_l} c(t_l-\tau)d\tau & for\, t \leq t_l \\ \int_0^{t_l} c(t-\tau)d\tau & \text{otherwise.} \end{cases} \tag{15}$$

That is, the output signal norm $\|y(t)\|$ is bounded by $f(t)$ in Eq. (15) when applying an arbitrary input signal $u \in U_{u_{\max},t_l}$.

3.4. Tight bound computation

It is now possible to evaluate the effect of repeated input signals in $U_{u_{\max},t_l}$ on the output signal norm $\|y(t)\|$ according to Eq. (12) in Problem 1. Using Lemma 1 and Eq. (15) and writing $N_0 = \lceil \frac{t_l}{\Delta} \rceil$, it holds that

$$\sup_{(t_v)_{\nu=0}^{\infty} \in Q_\Delta, t \geq 0} \|y(t)\| \leq \sum_{\nu=0}^{\infty} f(\nu\Delta) \tag{16}$$

$$= u_{\max} \left(N_0 \times \int_0^{t_l} c(t_l-\tau)d\tau + \sum_{\nu=N_0}^{\infty} \int_0^{t_l} c(\nu\Delta-\tau)d\tau \right) \tag{17}$$

Here, Eq. (16) directly follows from Eq. (13) in Lemma 1 and Eq. (17) follows from Eq. (15).

Eq. (17) can be used as a bound for $\|y(t)\|$ if $c(t) \geq \|\gamma(t)\|$ can be chosen to fulfill the assumptions in Lemma 2 such that the infinite sum $\sum\limits_{\nu=N_0}^{\infty} \int_0^{t_l} c(\nu\Delta-\tau)d\tau$ converges. In addition, $c(t)$ should constitute a tight bound for $\|\gamma(t)\|$.

In [11,12], analytical bounds for $\|\gamma(t)\|$ exist in the form

$$\|\gamma(t)\| \leq b(t) := \|C\| \|B\| e^{-\mu t} \left(\sum_{k=0}^{n-1} a_k t^k \right) \tag{18}$$

whereby a_k depends on the system matrices A, B, C in Eq. (8) and n depends on the bounding method. Such a bound is nonnegative, monotonically decreasing, and tight for large enough values of t. Accordingly, a threshold value θ is selected and the bound $b(t)$ is employed only for large enough times $t \geq t_f$, such that $b(t) \leq \theta$ for $t \geq t_f$. In the remaining interval $[0, t_f]$, a monotonic bound $a(t) \geq \|\gamma(t)\|$ can be found as follows. Using a simulation run of $\|\gamma(t)\|$ for $t \in [0, t_f]$ a bounding function $a(t) \geq \|\gamma(t)\|$ for $t \in [0, t_f]$ with $a(t_f) = b(t_f)$ is determined. In this work, a suitable bounding function is

$$a(t) = me^{-\eta t} \tag{19}$$

with appropriate values of m and η; $a(t)$ is nonnegative and monotonically decreasing and $a(t) \geq \|\gamma(t)\|$ for all $t \in [0, t_f]$. The overall bound $c(t)$ according to Lemma 2 is

$$c(t) := \begin{cases} 0\, for\, t<0 \\ a(t)\;\, for\, 0 \leq t \leq t_f \\ b(t)\;\, for\, t > t_f. \end{cases} \tag{20}$$

For illustration, the bound in Eq. (20) is computed for the vehicle-following example based on a minimal realization that is determined from Figure 2. The input signal is u_i^{ff} and the output signal is $y = e_{i+1}$. Figure 5a shows the impulse response norm $\|\gamma(t)\|$ and the corresponding bound $c(t)$ in Eq. (20). The bound $b(t) = 10.1e^{-0.55t}\sum_{k=0}^{8}\frac{11.4^k t^k}{k!}$ in Eq. (18) is found using $\theta = 10^{-5}$ $(t_f = 135)$ and $a(t) = 0.016e^{-0.33t}$ is determined by simulation. In addition, Figure 5b shows the bound in Eq. (14) and the corresponding function $f(t)$ in Eq. (15) for $t_l = 10$. For comparison, these figures also show example input responses for the following time-limited input signals in the Table. Some conservatism is introduced when comparing the computed bounds and the actual simulations. This is expected, since the bound is valid for all possible input signals in $U_{2.5,10}$, whereas the specific input signals u_1 to u_6 are used for the simulations.

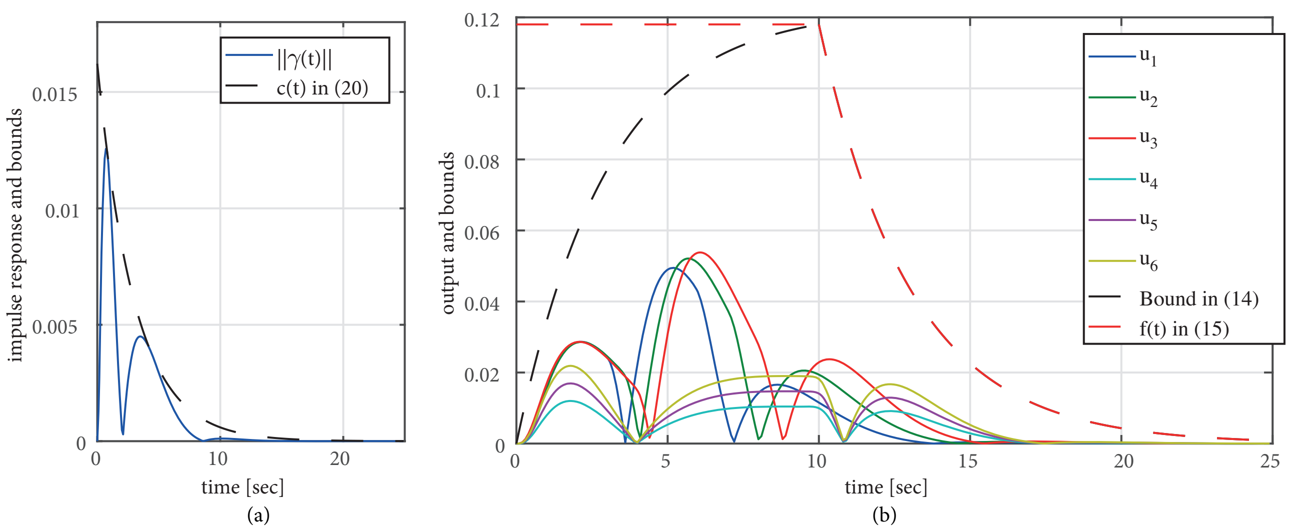

Figure 5. Bounds: a) Impulse response; b) Corresponding outputs and $f(t)$ for $t_l = 10$.

Using the monotonic bound $f(t)$ in Eq. (15) for the output signal $y(t)$ with $c(t)$ in Eq. (20), Lemma 1 can be directly applied to evaluate the bound on the output response in Eq. (17) in the case of repeated input signals. In particular, Theorem 2 shows that the infinite sum in Eq. (17) converges and can be evaluated using $c(t)$ in Eq. (20).

Theorem 2. Consider a stable LTI system with the set of input signals $U_{u_{\max},t_l}$ and the impulse response bound $c(t)$ in Eq. (20). Let $\Delta > 0$ and $t_f \geq 0$. Write $N_0 = \left\lceil \frac{t_l}{\Delta} \right\rceil, N_1 = \left\lceil \frac{t_f}{\Delta} \right\rceil, \mathit{and}\, N_2 = \left\lceil \frac{t_f + t_l}{\Delta} \right\rceil$. Define $c_l = \sum_{j=0}^{n-1-l} a_{l+j}\binom{l+j}{j}\int_0^{t_l}\tau^j e^{\mu t}d\tau$ for $l = 0, ..., n-1$. Then a suitable bound in Eq. (12) is given by

$$K_y = u_{\max}\frac{m}{\eta}\left(N_0\left(1 - e^{-\eta t_l}\right) + \left(e^{-\eta t_l} - 1\right)\sum_{\nu=N_0+1}^{N_2} e^{-\eta\nu\Delta}\right)$$

$$+ u_{\max}e^{-\mu N_1\Delta}\sum_{l=0}^{n-1} c_l \sum_{i=0}^{l}\binom{l}{i}\left(N_1\Delta\right)^{l-i}\left(-\Delta\right)^i \frac{d^i}{d\left(\mu\Delta\right)^i}\frac{1}{1 - e^{-\mu\Delta}} \tag{21}$$

In words, Theorem 2 computes the bound K_y based on the parameters of the impulse response bound $c(t)$ in Eq. (20), the set of time-limited input signals $U_{u_{\max},t_l}$, and the dwell time Δ. In summary, the following procedure is suitable to determine a bound for the output signal norm of the stable LTI system in Eq. (8) when repeatedly applying input signals from $U_{u_{\max},t_l}$.

P1 Determine the analytical bound $b(t)$ in Eq. (18) and t_f for a given threshold θ.

P2 Determine the bounding function $a(t)$ in Eq. (19) by simulation.

P3 Evaluate the bound K_y on $\|y\,(t)\|$ in Eq. (21).

The results in Theorem 2 and the related steps P1 to P3 are next used to compute the bound in Eq. (21) for the vehicle-following example with input signals in $U_{2.5,10}$ and the output signal e_{i+1} (distance error). Hereby, it has to be noted that steps P1 and P2 were already performed when illustrating Eq. (20). Regarding step P3, scenarios where vehicle i potentially opens a gap every $\Delta = 10$ s and $\Delta = 20$ s are chosen and the bounds $K_y = 0.25$ m and $K_y = 0.13$ m are obtained, respectively. Figure 6 shows a comparison of the computed bounds with simulations using the different repeated input signals from $U_{2.5,10}$ in the Table. The computed bounds are valid for the repeated input signals. For example, it is confirmed that the error signal of vehicle $i+1$ remains below 0.25 m even if the predecessor vehicle i performs gap opening maneuvers every 10 s. Considering that the desired distance at a speed of $v = 10$ m/s is $d_{r,i} = 17$ m, this does not cause a violation of driving safety.

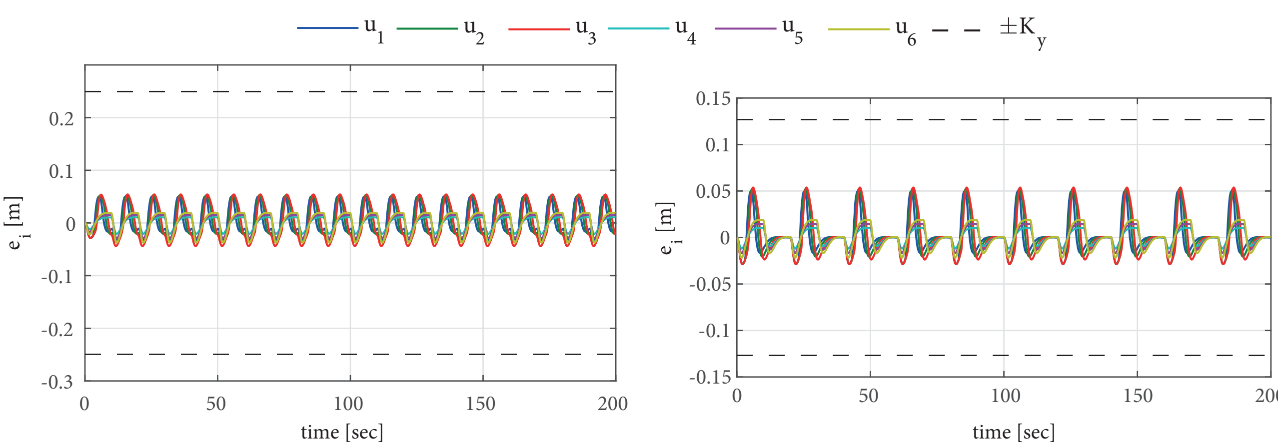

Figure 6. Comparison: bound and simulation for repeated inputs: a) $\Delta = 10$ s; b) $\Delta = 20$ s.

3.5. Discussion

In this section, the evaluation of the bound in Eq. (21) is discussed. It is first noted that the bound in Eq. (21) has two addends. The first addend is computed based on $a(t)$ in Eq. (19) that is obtained by simulation. It determines a bound for up to N_2 repetitions of input signals in $U_{u_{\max},t_l}$. The second addend depends on $b(t)$ in Eq. (18) and captures the effect of applying an arbitrary number of input signals.

In principle, it could be argued that the rather intricate second addend can be avoided if it is ensured that the input signal is repeated no more than N_2 times. Nevertheless, such an assumption places a restriction on the possible system behavior. In the application example, this would mean that only a limited number of

N_2 opening/closing gap maneuvers are permitted while guaranteeing the bound on the error signal. Precisely, the advantage of the bound in Eq. (21) including the second addend is that the bound is valid for any number of input signal repetitions. In addition, the evaluation of Eq. (21) is an offline computation that only depends on the range of the possible input signals in $U_{u_{\max},t_l}$ and the impulse response bound of the LTI system in Eq. (20). Choosing θ small enough (and hence t_f large enough) ensures that the contribution of the second addend in Eq. (21) is small. For example, when computing $K_y = 0.25$ m for the input signal u_3 and $\Delta = 10$ s in Section 3.3., the first addend is 0.249 m and the second addend is 0.001 m.

Finally, the set $U_{u_{\max},t_l}$ is obtained by inspecting the expected input signals to be applied to the LTI system as illustrated in Section 2.2. A benefit of the proposed method is that any new input signal can be applied without violating the computed bound as long as it belongs to $U_{u_{\max},t_l}$.

4. Conclusions

This paper considers the repeated application of time-limited input signals to stable LTI systems. Such input signals are encountered, for example, when performing longitudinal maneuvers such as opening/closing gaps in vehicle platoons. In this context, output signals such as the distance error between vehicles should remain bounded in order to ensure driving safety even if maneuvers are repeatedly executed.

Accordingly, the paper first shows that a bound on the output signal norm exists if the repeated input signals are separated by a nonzero dwell time. Moreover, an original computational procedure for finding a tight bound on the output signal norm is developed. Using this method, a suitable bound for the distance error of vehicles in a vehicle platoon is determined. A safe driving distance is guaranteed even if an arbitrary number of longitudinal maneuvers is performed.

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Appendix

Proof of Theorem 1. Consider time instant t and $N_\Delta = \lceil \frac{t_l}{\Delta} \rceil$. If $t > t_l$, $\left\| u_{(t_\nu)_{\nu=0}^{\infty}}(t) \right\| \leq N_\Delta u_{\max}$, since at most N_Δ successive input signals can be nonzero and the norm of any input signal is bounded by $u_{\max}$. If $t < t_l$, less-than N_Δ input signals are nonzero such that $\left\| u_{(t_\nu)_{\nu=0}^{\infty}}(t) \right\| \leq N_\Delta u_{\max}$. Together, $\left\| u_{(t_\nu)_{\nu=0}^{\infty}}(t) \right\| \leq N_\Delta u_{\max}$ is bounded for all t. Since the LTI system is stable, $\|y(t)\|$ is bounded and Eq. (12) holds.

Proof of Lemma 1. Eqs. (11) – (13) and the assumption in Lemma 1 imply

$$
\begin{aligned}
\sup_{(t_v)_{\nu=0}^{\infty} \in Q_\Delta, t \geq 0} \|y(t)\| &= \sup_{(t_v)_{\nu=0}^{\infty} \in Q_\Delta, t \geq 0} \left\| \gamma(t) \star u_{(t_\nu)_{\nu=0}^{\infty}}(t) \right\| \\
&= \sup_{(t_v)_{\nu=0}^{\infty} \in Q_\Delta, t \geq 0} \left\| \sum_{\nu=0}^{\infty} \gamma(t) \star u_\nu(t-t_\nu) \right\| \leq \sup_{(t_v)_{\nu=0}^{\infty} \in Q_\Delta, t \geq 0} \sum_{\nu=0}^{\infty} \|\gamma(t) \star u_\nu(t-t_\nu)\| \\
&\leq \sup_{(t_v)_{\nu=0}^{\infty} \in Q_\Delta, t \geq 0} \sum_{\nu=0}^{\infty} f(t-t_\nu)
\end{aligned} \tag{A1}
$$

Now consider the finite sum $\sum_{\nu=0}^{k} f(t-t_\nu)$ with k addends. Then $f(t-t_\nu)$ assumes its supremum for $t_k = t$, since $f(t)$ monotonically decreases. Second, since the t_ν are separated by the dwell time Δ, the maximum value of $f(t-t_\nu)$ is obtained for $t_\nu = t_k - (k-\nu)\Delta = t - (k-\nu)\Delta$. That is, $\sum_{\nu=0}^{k} f(t-t_\nu) = \sum_{\nu=0}^{k} f(t-t+(k-\nu)\Delta) = \sum_{\nu=0}^{k} f(\nu\Delta)$. Taking the limit for $k \to \infty$, Eq. (13) in Lemma 1 directly follows.

Proof of Lemma 2. $\|y(t)\| = \|\gamma(t) \star u(t)\| = \left\| \int_0^t \gamma(t-\tau) u(\tau) d\tau \right\| \leq \int_0^t \|\gamma(t-\tau)\| \|u(\tau)\| d\tau \leq \int_0^{t_l} c(t-\tau) d\tau$. Moreover, $\int_0^{t_l} c(t-\tau) d\tau = 0$ for $t < 0$, since $c(t) = 0$ for $t < 0$. For $t \leq t_l$, it holds that $\int_0^{t_l} c(t-\tau) d\tau = \int_0^{t} c(t-\tau) d\tau$. Since $c(t)$ is nonnegative, $\int_0^{t_l} c(t-\tau) d\tau \leq \int_0^{t} c(t_l-\tau) d\tau$ for any $t \leq t_l$. Since $c(t)$ monotonically decreases, $\int_0^{t_l} c(t'-\tau) d\tau \leq \int_0^{t_l} c(t-\tau) d\tau$ for $t' \geq t \geq t_l$. Hence, $\int_0^{t_l} c(t-\tau) d\tau$ has a maximum at $t = t_l$ and monotonically decreases for $\geq t_l$.

Proof of Theorem 2. Using Eq. (17), it is computed as

$$
\begin{aligned}
\sup_{(t_v)_{\nu=0}^{\infty} \in Q_\Delta, t \geq 0} \|y(t)\| &\leq u_{\max} \left(N_0 \times \int_0^{t_l} c(t_l-\tau) d\tau + \sum_{\nu=N_0}^{\infty} \int_0^{t_l} c(\nu\Delta-\tau) d\tau \right) \\
&= u_{\max} \left(N_0 \times \int_0^{t_l} c(t_l-\tau) d\tau + \sum_{\nu=N_0}^{N_1-1} \int_0^{t_l} a(\nu\Delta-\tau) d\tau \right. \\
&\quad + \sum_{\nu=N_1}^{N_2} \left(\int_0^{\nu\Delta t_f} b(\nu\Delta-\tau) d\tau + \int_{\nu\Delta-t_f}^{t_l} a(\nu\Delta-\tau) d\tau \right) \\
&\quad \left. + \sum_{\nu=N_2+1}^{\infty} \int_0^{t_l} b(\nu\Delta-\tau) d\tau \right)
\end{aligned} \tag{A2}
$$

This computation considers that the convolution integral is applied to $a(t)$ before $t = t_f$ (until $\nu = N_1 - 1$), to $a(t)$ and $b(t)$ for $t_f \le t \le t_f + t_l (N_1 \le \nu \le N_2)$, and to $b(t)$ for $t \ge t_f + t_l (\nu > N_2)$. Further noting that $a(\mathrm{t})$ and $b(\mathrm{t})$ are nonnegative, it also holds that

$$\sup_{(t_v)_{\nu=0}^{\infty} \in Q_\Delta, t \ge 0} \|y(t)\| \le u_{\max} \left(N_0 \times \int_0^{t_l} a(t_l - \tau) d\tau + \sum_{\nu = N_0}^{N_2} \int_0^{t_l} a(\nu\Delta - \tau) d\tau + \sum_{\nu = N_1}^{\infty} \int_0^{t_l} b(\nu\Delta - \tau) d\tau \right) \tag{A3}$$

It can be directly computed for $t \ge t_l$ that

$$\int_0^{t_l} a(t_l - \tau) d\tau = \frac{m}{\eta} \left(1 - e^{-\eta t_l}\right) \; and \; \int_0^{t_l} a(t - \tau) d\tau = \frac{m}{\eta} \left(e^{\eta t_l} - 1\right) e^{-\eta t} \tag{A4}$$

In order to evaluate $\int_0^{t_l} b(\nu\Delta - \tau) d\tau$, Eq. (12) and the binomial theorem are used and written:

$$\int_0^{t_l} b(\nu\Delta - \tau) d\tau = \int_0^{t_l} \sum_{k=0}^{n-1} a_k (t-\tau)^k e^{-\mu(t-\tau)} d\tau = e^{-\mu t} \int_0^{t_l} \sum_{k=0}^{n-1} a_k \sum_{i=0}^{k} \binom{k}{i} t^{k-i} (-\tau)^i e^{\mu t} d\tau$$

Reorganizing the summations and the integral according to powers of t leads to

$$\begin{aligned} \int_0^{t_l} b(t - \tau) d\tau &= e^{-\mu t} \sum_{l=0}^{n-1} t^l \sum_{j=0}^{n-1-l} a_{l+j} \binom{l+j}{j} \int_0^{t} (-\tau)^j e^{\mu\tau} d\tau \\ & \quad e^{-\mu t} \sum_{l=0}^{n-1} t^l \sum_{j=0}^{n-1-l} a_{l+j} \binom{l+j}{j} \int_0^{t_l} (-\tau)^j e^{\mu\tau} d\tau \\ &\le e^{-\mu t} \sum_{l=0}^{n-1} t^l = e^{-\mu t} \sum_{l=0}^{n-1} c_l t^l \end{aligned}$$

Then the infinite sum in Eq. (A2) results in

$$\begin{aligned} \sum_{\nu=N_1}^{\infty} \int_0^{t_l} b(\nu\Delta - \tau) d\tau &= \sum_{\nu=N_1}^{\infty} e^{-\mu\nu\Delta} \sum_{l=0}^{n-1} c_l (\nu\Delta)^l \\ &= e^{-\mu N_1 \Delta} \sum_{\nu=0}^{\infty} e^{-\mu\nu\Delta} \sum_{l=0}^{n-1} c_l (N_1\Delta + \nu\Delta)^l \\ &= e^{-\mu N_1 \Delta} \sum_{l=0}^{n-1} c_l \Delta^l \sum_{\nu=0}^{\infty} e^{-\mu\nu\Delta} (N_1 + \nu)^l \\ &= e^{-\mu N_1 \Delta} \sum_{l=0}^{n-1} c_l \Delta^l \sum_{i=0}^{l} \binom{l}{i} N_1^{l-i} \sum_{\nu=0}^{\infty} \nu^i e^{-\mu\nu\Delta} \\ &= e^{-\mu N_1 \Delta} \sum_{l=0}^{n-1} c_l \Delta^l \sum_{i=0}^{l} \binom{l}{i} N_1^{l-i} (-1)^i \frac{d^i}{d(\mu\Delta)^i} \frac{1}{1 - e^{-\mu\Delta}} \end{aligned} \tag{A5}$$

Here, the last two identities are derived based on the binomial theorem and the geometric series. Using Eqs. (A3) and (A4), the result in Eq. (21) directly follows. Since all the summations in Eq. (21) are finite, $K_y < \infty$.