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Model reduction of discrete-time systems in limited intervals

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Abstract: Model order reduction (MOR) is a process of obtaining a lower order surrogate model that accurately approximates the original high-order system. Since no actuator or plant operates over the entire time and frequency ranges, the reduced-order model should be accurate in the actual range of operation. In this paper, model reduction techniques for discrete time systems are presented that ensure less reduction error in the specified time and frequency intervals. The techniques are tested on the benchmark numerical examples and their efficacy is shown.

Key words: Balanced transformation, limited Gramians, model reduction

1. Introduction

Model order reduction (MOR) has been studied for the past few decades, especially the design and analysis of large-scale circuits and complicated systems [1–4]. MOR techniques aim to find a fairly accurate lower order approximation of the original system, so that its fundamental characteristics are retained.

Balanced truncation (BT) [5] is one of the most commonly used MOR techniques, known for characteristics such as less approximation error and preservation of stability. The discrete time counterpart of BT was presented in [6,7].

BT [5] does not follow any time or frequency domain error criteria. However, in many situations, the approximation error at a certain desired time and frequency interval is of critical importance. For instance, if a lower order approximation of a large-order filter is required, the approximation error should be much less in certain frequency regions [8]. Frequency weights in BT [5] were introduced in [9] to emphasize the desired frequency region. The use of input-, output-, or both-sided frequency weighting is possible; however, guaranteed stable reduced-order models (ROMs) are only achieved in the single-sided case [9]. Several modifications in [9] to guarantee stability have been presented in the literature [10–13].

In [9], frequency weights have to be initially defined in order to emphasize the desired frequency region, which is a problem in itself. Gawronski and Juang [14] proposed an alternative, where the desired frequency interval should be defined instead of the frequency weights. In addition, they studied MOR in the limited time interval.

Wang and Zilouchian [15] proposed the discrete time counterpart of Gawronski and Juang's technique [14], which yields less approximation error in the desired frequency interval. However, the ROMs may not be stable [16], as they are in Enns' [9] and Gawronski and Juang's techniques [14]. Ghafoor and Sreeram [16]

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proposed two algorithms to overcome this shortcoming, whereas Imran and Ghafoor [3] proposed a technique to ensure stability.

Aldhaheri defined a limited frequency interval cross Gramian for discrete time single input single output (SISO) systems in [8], and proposed a frequency-limited MOR algorithm based on this Gramian. Similarly, Jazlan et al. [17] proposed cross Gramian-based algorithms for MOR of continuous and discrete time systems in the limited time interval. These algorithms are less computational, since only one Gramian needs to be computed instead of two. To the best of our knowledge, no MOR algorithm for discrete-time systems exists in the literature that satisfies both limited time and limited frequency requirements at the same time.

This paper presents MOR algorithms for discrete time systems, which tackle the limited time and frequency scenario simultaneously, and yield ROMs such that the reduction error is small in the desired time and frequency intervals. Numerical results show that the ROMs yielded by these algorithms are accurate within the specified time and frequency intervals.

2. Preliminaries

Consider a stable and minimal n-th order discrete time system:

$$H\left(z\right) = C\left(zI - A\right)^{-1} B + D, \tag{1}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, and $D \in R^{p\times m}$.

The goal of MOR in limited intervals is to find a ROM

$$H_r(z) = C_r \left(zI - A_r\right)^{-1} B_r + D$$

of order $r \quad (r < n)$, such that the approximation error $(H(z) - H_r(z))$ is small in the desired time $T = [n_1, n_2]$ and frequency intervals $\Omega = [\omega_1, \omega_2]$ rad/s, such that $0 \leq \omega_1 \leq \omega_2 \leq \pi$.

2.1. Wang and Zilouchian's technique

The limited frequency interval controllability P_{zw} and observability Q_{zw} Gramians for the desired frequency interval Ω are defined as

$$P_{zw} = \frac{1}{2\pi}\int_{\delta\omega} \left(I - Ae^{-j\omega}\right)^{-1} BB^T \left(I - A^T e^{j\omega}\right)^{-1} d\omega \tag{2}$$

$$Q_{zw} = \frac{1}{2\pi}\int_{\delta\omega} \left(I - A^T e^{j\omega}\right)^{-1} C^T C \left(I - Ae^{-j\omega}\right)^{-1} d\omega, \tag{3}$$

which are obtained by solving the following Lyapunov equations:

$$AP_{zw}A^T - P_{zw} + X_{zw} = 0 \tag{4}$$

$$A^T Q_{zw} A - Q_{zw} + Y_{zw} = 0, \tag{5}$$

where

$$X_{zw} = SBB^T + BB^T S^H \tag{6}$$

$$Y_{zw} = S^H C^T C + C^T C S \tag{7}$$

$$S = -\frac{\omega_2 \ - \omega_1}{4\pi} I + \frac{1}{2\pi} \int_{\delta\omega} \left(I - A e^{-j\omega} \right)^{-1} d\omega \tag{8}$$

S^H is Hermitian of S and $\delta\omega = [-\omega_2, -\omega_1] \cup [\omega_1 \omega_2]$. The transformation matrix T_{zw} is calculated as $T_{zw}^T Q_{zw} T_{zw} = T_{zw}^{-1} P_{zw} T_{zw}^{-T} = diag\{\sigma_1, \sigma_2, \cdots, \sigma_n\}$, where $\sigma_k \geq \sigma_{k+1}$ and $k = 1, 2, \cdots, n-1$. σ_k are the Hankel singular values, which are the quantitative measure of the contribution of each state in the energy transfer. The states with the least contribution are then truncated. The transformed system $A_t, B_t, C_t, D_t = \{T_{zw}^{-1} A\} T_{zw}, T_{zw},^{-1} B, C T_{zw}, D\}$ is then given by

$$A_t = \begin{bmatrix} A_r & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \ B_t = \begin{bmatrix} B_r \\ B_2 \end{bmatrix}, \ C_t = \begin{bmatrix} C_r & C_2 \end{bmatrix}, \ D_t = D$$

$H_r(z) = C_r \left(zI - A_r \right)^{-1} B_r + D$ is the r^{th} order ROM.

Remark 1 *Since X_{zw} and Y_{zw} may be indefinite, the ROMs are not guaranteed to be stable [16].*

2.2. Aldhaheri's technique

Aldhaheri [8] defined frequency-limited cross Gramian R_a as:

$$R_a = \frac{1}{2\pi} \int_{\delta\omega} \left(I - A e^{-j\omega} \right)^{-1} BC \left(I - A^T e^{-j\omega} \right)^{-1} d\omega, \tag{9}$$

which is the solution to the following Sylvester equation:

$$A R_a A - R_a + SBC + BCS = 0 \tag{10}$$

The transformation matrix T_a is calculated as $T_a^{-1} R_a T_a = diag \xi_1, \xi_2, \cdots, \xi_n$, where $\xi_k \geq \xi_{k+1}$ and $k = 1, 2, \cdots, n-1$. ROM is obtained by truncating the transformed system up to the desired order.

2.3. Jazlan et al.'s technique

Jazlan et al. [17] defined time-limited cross Gramian R_j as

$$R_j = \sum_{i=0}^{n_2 - 1} A^i BC A^i - \sum_{i=0}^{n_1 - 1} A^i BC A^i, \tag{11}$$

which is the solution to the following Sylvester equation:

$$A R_j A - R_j + A^{n_1} \ BC A^{n_1} - A^{n_2} BC A^{n_2} = 0 \tag{12}$$

The transformation matrix T_j is calculated as $T_j^{-1} R_j T_j = diag \xi_1 \xi_2, \cdots, \xi_n$, where $\xi_k \geq \xi_{k+1}$ and $k = 1, 2, \cdots, n-1$. ROM is obtained by truncating the transformed system up to the desired order.

3. Main work

In this section, time- and frequency-limited Gramians are first defined. These Gramians contain the information of the input–output behavior of the system at the specified time and frequency intervals. MOR algorithms are then presented based on these definitions. ROMs thus obtained ensure accuracy in the desired time and frequency intervals.

Since A is Hurwitz, the Lyapunov Eqs. (4) and (5) can be written in summation form as

$$P_{zw} = \sum_{i=0}^{\infty} A^i X_{zw} \left(A^T\right)^i \tag{13}$$

$$Q_{zw} = \sum_{i=0}^{\infty} \left(A^T\right)^i Y_{zw} A^i \tag{14}$$

Definition 1 *The controllability and observability Gramians of the limited time and frequency intervals in the specified time, and frequency intervals $T = [n_1 n_2]$ and $\Omega = [\omega_1 \omega_2]$ rad/s, respectively, are defined as*

$$P_{T,\Omega} = \sum_{i=n_1}^{n_2-1} A^i X_{zw} \left(A^T\right)^i \tag{15}$$

$$Q_{T,\Omega} = \sum_{i=n_1}^{n_2-1} \left(A^T\right)^i Y_{zw} A^i \tag{16}$$

Theorem 1 *$P_{T,\Omega}$ and $Q_{T,\Omega}$ can be computed from P_{zw} and Q_{zw}, respectively, as*

$$P_{T,\Omega} = A^{n_1} P_{zw} \left(A^T\right)^{n_1} - A^{n_2} P_{zw} \left(A^T\right)^{n_2} \tag{17}$$

$$Q_{T,\Omega} = \left(A^T\right)^{n_1} Q_{zw} A^{n_1} - \left(A^T\right)^{n_2} Q_{zw} A^{n_2} \tag{18}$$

Proof Consider the term $A^n P_{zw} \left(A^T\right)^n$ first.

$$\begin{aligned}
P_{zw} - A^n P_{zw} \left(A^T\right)^n &= \sum_{i=0}^{\infty} A^i X_{zw} \left(A^T\right)^i - A^n \sum_{i=0}^{\infty} A^i X_{zw} \left(A^T\right)^i \left(A^T\right)^n \\
&= \sum_{i=0}^{\infty} A^i X_{zw} \left(A^T\right)^i - \sum_{i=0}^{\infty} A^{n+i} X_{zw} \left(A^T\right)^{n+i} \\
&= \sum_{i=0}^{\infty} A^i X_{zw} \left(A^T\right)^i - \sum_{i=n}^{\infty} A^n X_{zw} \left(A^T\right)^n \\
&= \sum_{i=0}^{n-1} A^i X_{zw} \left(A^T\right)^i
\end{aligned}$$

Therefore, $A^n P_{zw} \left(A^T\right)^n = P_{wz} - \sum_{i=0}^{n-1} A^i X_{zw} \left(A^T\right)^i$. □

The right side of Eq. (17) can then be written as

$$\begin{aligned} A^{n_1} P_{zw} \left(A^T\right)^{n_1} - A^{n_2} P_{zw} \left(A^T\right)^{n_2} &= \sum_{i=0}^{n_2-1} A^i X_{zw} \left(A^T\right)^i - \sum_{i=0}^{n_1-1} A^i X_{zw} \left(A^T\right)^i \\ &= \sum_{i=n_1}^{n_2-1} A^i X_{zw} \left(A^T\right)^i \\ &= P_{T,\Omega} \end{aligned}$$

Similarly, it can be proven that $Q_{T,\Omega} = \left(A^T\right)^{n_1} Q_{zw} A^{n_1} - \left(A^T\right)^{n_2} Q_{zw} A^{n_2}$.

Theorem 2 *$P_{T,\Omega}$ and $Q_{T,\Omega}$ satisfy the following Lyapunov equations:*

$$A P_{T,\Omega} A^T - P_{T,\Omega} + X_{T,\Omega} = 0 \tag{19}$$

$$A^T Q_{T,\Omega} A - Q_{T,\Omega} + Y_{T,\Omega} = 0, \tag{20}$$

where

$$X_{T,\Omega} = A^{n_1} SBB^T \left(A^T\right)^{n_1} + A^{n_1} BB^T S^H \left(A^T\right)^{n_1} - A^{n_2} SBB^T \left(A^T\right)^{n_2} - A^{n_2} BB^T S^H \left(A^T\right)^{n_2} \tag{21}$$

$$Y_{T,\Omega} = \left(A^T\right)^{n_1} S^H C^T C A^{n_1} + \left(A^T\right)^{n_1} C^T C S A^{n_1} - \left(A^T\right)^{n_2} S^H C^T C A^{n_2} - \left(A^T\right)^{n_2} C^T C S A^{n_2} \tag{22}$$

Proof By putting Eq. (13) in Eq. (17), we get

$$\begin{aligned} P_{T,\Omega} &= A^{n_1} \left(\sum_{i=0}^{\infty} A^i X_{zw} \left(A^T\right)^i \right) \left(A^T\right)^{n_1} - A^{n_2} \left(\sum_{i=0}^{\infty} A^i X_{zw} \left(A^T\right)^i \right) \left(A^T\right)^{n_2} \\ &= \sum_{i=0}^{\infty} A^i A^{n_1} \left(SBB^T + BB^T S^H\right) \left(A^T\right)^{n_1} \left(A^T\right)^i - \sum_{i=0}^{\infty} A^i A^{n_2} \left(SBB^T + BB^T S^H\right) \left(A^T\right)^{n_2} \left(A^T\right)^i \\ &= \sum_{i=0}^{\infty} A^i \left(A^{n_1} SBB^T \left(A^T\right)^{n_1} + A^{n_1} BB^T S^H \left(A^T\right)^{n_1} - A^{n_2} SBB^T \left(A^T\right)^{n_2} - A^{n_2} BB^T S^H \left(A^T\right)^{n_2} \right) \left(A^T\right)^i \\ &= \sum_{i=0}^{\infty} A^i X_{T,\Omega} \left(A^T\right)^i \end{aligned} \tag{23}$$

Therefore, the summation in Eq. (15) can be computed using Eq. (19). Similarly, it can be proven that $Q_{T,\Omega}$ satisfies Eq. (20). □

Remark 2 *When the specified frequency interval is $\Omega = [0, \pi]$ rad/s, $S = \frac{1}{2} I$, $X_{zw} = BB^T$, $Y_{zw} = C^T C$, $X_{T,\Omega} = A^{n_1} BB^T \left(A^T\right)^{n_1} - A^{n_2} BB^T \left(A^T\right)^{n_2}$, $Y_{T,\Omega} = \left(A^T\right)^{n_1} C^T C A^{n_1} - \left(A^T\right)^{n_2} C^T C A^{n_2}$, $P_{T,\Omega} = P_T$ and $Q_{T,\Omega} = Q_T$, where $P_T = \sum_{i=n_1}^{n_2-1} A^i BB^T \left(A^T\right)^i$ and $Q_T = \sum_{i=n_1}^{n_2-1} \left(A^T\right)^i C^T C A^i$ are the time-limited controllability and observability Gramians, respectively.*

Theorem 3 *$P_{T,\Omega}$ and $Q_{T,\Omega}$ can be computed from P_T and Q_T, respectively, as*

$$P_{T,\Omega} = SP_T + P_T S^H$$

$$Q_{T,\Omega} = S^H Q_T + Q_T S$$

Proof First we will show that $SA^n = A^n S$.

$$SA^n = -\frac{\omega_2 - \omega_1}{4\pi} A^n + \frac{1}{2\pi}\int_{\delta\omega} \left(I - Ae^{-j\,\omega}\right)^{-1} A^n \, d\omega$$

Since $\left(I - Ae^{-j\,\omega}\right)^{-1}$ is the discrete fourier transform of A^n,

$$SA^n = -\frac{\omega_2 - \omega_1}{4\pi} A^n + \frac{1}{2\pi}\int_{\delta\omega} A^n \left(I - Ae^{-j\,\omega}\right)^{-1} d\omega = A^n S$$

By putting Eq. (21) in Eq. (23), we get

$$\begin{aligned} P_{T,\Omega} &= \sum_{i=0}^{\infty} A^i \Big(SA^{n_1}BB^T\left(A^T\right)^{n_1} - SA^{n_2}BB^T\left(A^T\right)^{n_2} + A^{n_1}BB^T\left(A^T\right)^{n_1}S^H - A^{n_2}BB^T\left(A^T\right)^{n_2}S^H\Big)\left(A^T\right)^i \\ &= S\sum_{i=0}^{\infty} A^i \Big(A^{n_1}BB^T\left(A^T\right)^{n_1} - A^{n_2}BB^T\left(A^T\right)^{n_2}\Big)\left(A^T\right)^i \\ &\quad + \sum_{i=0}^{\infty} A^i \Big(A^{n_1}BB^T\left(A^T\right)^{n_1} - A^{n_2}BB^T\left(A^T\right)^{n_2}\Big)\left(A^T\right)^i S^H \\ &= SP_T + P_T S^H \end{aligned}$$

Similarly, it can be shown that $Q_{T,\Omega}{=}S^H Q_T{+}Q_T S$. □

Algorithm 1 If the original system is close to nonminimal, the computation of the transformation matrix in the balancing methods is often ill-conditioned. Several algorithms have been presented in the literature to address this issue [18,19]. Among these, the most promising is the balancing free algorithm [18]. Therefore, to calculate ROM, the balancing free algorithm [18] is adapted.

Given the original system $H(z)$, ROM $H_r(z)$ in the desired time and frequency intervals can be obtained with the following steps:

1. Calculate the Schur decomposition of $P_{T,\Omega}Q_{T,\Omega}$ with the eigenvalues of $P_{T,\Omega}Q_{T,\Omega}$ in ascending and descending order, i.e. $V_a^T P_{T,\Omega}Q_{T,\Omega}V_a{=}S_a$ and $V_d^T P_{T,\Omega}Q_{T,\Omega}V_d{=}S_d$, respectively. $V_a{=}[V_{a_1} \;\; V_{a_2}]$ and $V_d{=}\,[V_{d_1} \;\; V_{d_2}]$ are orthogonal matrices, where $V_{a_1}{\in}\, R^{n\times(n-r)}$, $V_{a_2}{\in}R^{n\times r}$ $\;V_{d_1}{\in}\, R^{n\times r}$, and $V_{d_2}{\in}R^{n\times(n-r)}$. S_a and S_d are upper triangular matrices.

2. Calculate the singular value decomposition $V_{a_2}V_{d_1}{=}U_{bf}\Sigma V_{bf}$, where $\Sigma = diag\sigma_1\sigma_2,\cdots,\sigma_n$.

3. The ROM is then given by $A_rB_rC_rD{=}\,\{\Sigma^{-\frac{1}{2}}U_{bf}^TV_{a_2}^TAV_{d_1}V_{bf}\Sigma^{-\frac{1}{2}},\, \Sigma^{-\frac{1}{2}}U_{bf}^TV_{a_2}^TB,\, CV_{d_1}V_{bf}\Sigma^{-\frac{1}{2}},\, D$.

Remark 3 *Since $X_{T,\Omega}$ and $Y_{T,\Omega}$ may be indefinite, the ROMs obtained using Algorithm 1 may be unstable.*

3.1. Stability preservation and error bound

To ensure stability in the limited scenarios, the symmetric indefinite matrices are replaced with their positive semidefinite approximations. As pointed out in [20], the nearest positive semidefinite matrix to an indefinite matrix is obtained by replacing its negative eigenvalues with zeros. Moreover, this is an optimal solution in the Frobenius norm sense [20]. Therefore, the approximation used by Ghafoor and Sreeram [16] is adapted to ensure stability and error bound expression.

Let the fictitious input and output matrices be $B_u B_u^T$ and $C_u^T C_u$, respectively, where $B_u{=}K_u \bar{M}_u^{\frac{1}{2}}$ and $C_u{=}\bar{N}_u^{\frac{1}{2}} L_u^T$. Matrices $K_u L_u \bar{M}_u$ and $\bar{N}_u$ are obtained from the orthogonal eigenvalue decompositions of $X_{T,\Omega}$ and $Y_{T,\Omega}$, i.e. $X_{T,\Omega}{=}K_u M_u K_u^T$ and $Y_{T,\Omega}{=}L_u N_u L_u^T$. M_u , and N_u can be partitioned as

$$X_{T,\Omega}{=}K_u M_u K_u^T = \begin{bmatrix} K_{u_1} & K_{u_2} \end{bmatrix} \begin{bmatrix} M_{u_1} & 0 \\ 0 & M_{u_2} \end{bmatrix} \begin{bmatrix} K_{u_1}^T \\ K_{u_2}^T \end{bmatrix}$$

$$Y_{T,\Omega}{=}L_u N_u L_u^T = \begin{bmatrix} L_{u_1} & L_{u_2} \end{bmatrix} \begin{bmatrix} N_{u_1} & 0 \\ 0 & N_{u_2} \end{bmatrix} \begin{bmatrix} L_{u_1}^T \\ L_{u_2}^T \end{bmatrix},$$

where $M_{u_1}{=}diag m_{u_1}, m_{u_2}, \cdots, m_{u_l}$, $M_{u_2}{=}diag m_{u_{l+1}}, m_{u_{l+2}}, \cdots, m_{u_n}$, $N_{u_1}{=}diag n_{u_1}, n_{u_2}, \cdots, n_{u_q}$, $N_{u_2}{=}diag n_{u_{q+1}}, n_{u_{q+2}}, \cdots, n_{u_n}$, $m_{u_1} \geq m_{u_2} \geq \cdots \geq m_{u_l} > 0$, $m_{u_{l+1}} \leq m_{u_{l+2}} \leq \cdots \leq m_{u_n} \leq 0$, $n_{u_1} \geq n_{u_2} \geq \cdots \geq n_{u_l} > 0$, and $n_{u_{q+1}} \leq n_{u_{q+2}} \leq \cdots \leq n_{u_n} \leq 0$.

$\bar{M}_u$ and $\bar{N}_u$ are defined as

$$\bar{M}_u = \begin{bmatrix} M_{u_1} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \bar{N}_u = \begin{bmatrix} N_{u_1} & 0 \\ 0 & 0 \end{bmatrix}$$

The new time- and frequency-limited controllability P_u and observability Q_u Gramians are calculated from the following Lyapunov equations:

$$AP_u A^T - P_u + B_u B_u^T = 0$$

$$A^T Q_u A - Q_u + C_u^T C_u = 0$$

$P_{T,\Omega}$ and $Q_{T,\Omega}$ are then replaced by P_u and Q_u in Algorithm 1, respectively.

Proposition 1 *The following a priori error bound holds if* $rank[B_u\ B] = rank[B_u]$ *and* $rank \begin{bmatrix} C_u \\ C \end{bmatrix} = rank[C_u]$*:*

$$||H(z) - H_r(z)||_\infty \leq 2 \left|\left|\tilde{L}\right|\right|_\infty \left|\left|\tilde{K}\right|\right|_\infty \sum_{k=r+1}^{n} \sigma_k,$$

where

$$\tilde{L} = CL_u diag\left\{ |n_{u_1}|^{-\frac{1}{2}}, |n_{u_2}|^{-\frac{1}{2}}, \cdots, |n_{u_{\bar{j}}}|^{-\frac{1}{2}}, 0, \cdots, 0 \right\}$$

$$\tilde{K} = diag\left\{ |m_{u_1}|^{-\frac{1}{2}}, |m_{u_2}|^{-\frac{1}{2}}, \cdots, |m_{u_{\bar{i}}}|^{-\frac{1}{2}}, 0, \cdots, 0 \right\} K_u^T B$$

$\bar{i} = rank[X_{T,\Omega}]$ *and* $\bar{j} = rank[Y_{T,\Omega}]$.

Proof The proof is similar to the error bound expressions in [16] and is hence omitted. □

3.2. Time- and frequency-limited Gramians for unstable systems

The Gramian definitions in Eqs. (15) and (16) are only valid if A is Hurwitz. When $H(z)$ is unstable, the summation in the above equations becomes unbounded. However, if the pair (A, B) is stabilizable and (A, C) is detectable, the Gramians can be defined for unstable systems using certain mathematical manipulation (see [21] for details). Consider the following right and left coprime factorization:

$$(zI - A)^{-1} B = N(z) M^{-1}(z)$$

$$C(zI - A)^{-1} = \tilde{M}^{-1}(z) \tilde{N}(z),$$

where $M(z)$ and $\tilde{M}(z)$ are the inner transfer function matrix. These coprime factors can be calculated accordingly in the following form:

$$\begin{bmatrix} N(z) \\ M(z) \end{bmatrix} = \begin{bmatrix} A+BF & BW \\ I & 0 \\ F & W \end{bmatrix},$$

where

$$F = -WW^T B^T XA$$

$$W^T (I + B^T XB) W = I$$

X is the stabilizing solution of the following discrete time Riccati equation:

$$A^T X \left(I + BB^T X\right)^{-1} A - X = 0$$

Similarly,

$$\begin{bmatrix} \tilde{N}(z) & \tilde{M}(z) \end{bmatrix} = \begin{bmatrix} A+GC & I & G \\ \tilde{W}C & 0 & \tilde{W} \end{bmatrix},$$

where

$$G = -AYC^T \tilde{W}^T \tilde{W}$$

$$\tilde{W}(I + CYC^T)\bar{W}\lambda T = IY$$

is the stabilizing solution of the following discrete time Riccati equation:

$$AY \left(I + YC^T C\right)^{-1} A^T - Y = 0$$

Definition 2 *The time- and frequency-limited controllability and observability Gramians for unstable systems are defined as*

$$\bar{P}_{T,\Omega} = \sum_{i=n_1}^{n_2-1} A_F^i \left(S_F BB^T + BB^T S_F^H\right) \left(A_F^T\right)^i \; \bar{Q}_{T,\Omega} = \sum_{i=n_1}^{n_2-1} \left(A_G^T\right)^i \left(S_G^H C^T C + C^T C S_G\right) A_G^i,$$

where

$$S_F = -\frac{\omega_2 - \omega_1}{4\pi} I + \frac{1}{2\pi} \int_{\delta\omega} \left(I - A_F e^{-j\omega}\right)^{-1} d\omega$$

$$S_G = -\frac{\omega_2 - \omega_1}{4\pi} I + \frac{1}{2\pi} \int_{\delta\omega} \left(I - A_G e^{-j\omega}\right)^{-1} d\omega$$

$$A_F = A + BF$$
$$A_G = A + GC$$

$\bar{P}_{T,\Omega}$ and $\bar{Q}_{T,\Omega}$ are the solutions of the following Lyapunov equations:

$$A\bar{P}_{T,\Omega}A^T + A^{n_1}\left(S_F BB^T + BB^T S_F^H\right)\left(A^T\right)^{n_1} - A^{n_2}\left(S_F BB^T + BB^T S_F^H\right)\left(A^T\right)^{n_2} = \bar{P}_{T,\Omega}$$

$$A^T\bar{Q}_{T,\Omega}A + \left(A_G^T\right)^{n_1}\left(S_G^H C^T C + C^T C S_G\right)A_G^{n_1} - \left(A_G^T\right)^{n_2}\left(S_G^H C^T C + C^T C S_G\right)A_G^{n_2} = \bar{Q}_{T,\Omega}$$

ROM is then obtained by replacing $P_{T,\Omega}$ and $Q_{T,\Omega}$ with $\bar{P}_{T,\Omega}$ and $\bar{Q}_{T,\Omega}$ in Algorithm 1, respectively.

Remark 4 *When A is Hurwitz, $X = Y = 0$, $\bar{P}_{T,\Omega} = P_{T,\Omega}$, and $\bar{Q}_{T,\Omega} = Q_{T,\Omega}$.*

3.3. Time- and frequency-limited cross Gramian

Since A is Hurwitz, the Sylvester Eq. (10) can be written in summation form as follows:

$$R_a = \sum_{i=0}^{\infty} A^i\left(SBC + BCS\right)A^i$$

Definition 3 *The limited time and frequency interval cross Gramians in the specified time and frequency intervals $T = [n_1 n_2]$ and $\Omega = [\omega_1\omega_2]$ rad/s, respectively, are defined as*

$$R_{T,\Omega} = \sum_{i=n_1}^{n_2-1} A^i\left(SBC + BCS\right)A^i \tag{24}$$

Proposition 2 *$R_{T,\Omega}$ can be computed from R_a as*

$$R_{T,\Omega} = A^{n_1}R_a A^{n_1} - A^{n_2}R_a A^{n_2}$$

Proof The proof is like Theorem 1 and hence omitted. □

Proposition 3 *$R_{T,\Omega}$ satisfies the following Sylvester equation:*

$$AR_{T,\Omega}A - R_{T,\Omega} + A^{n_1}SBCA^{n_1} + A^{n_1}BCSA^{n_1} - A^{n_2}SBCA^{n_2} - A^{n_2}BCSA^{n_2} = 0$$

Proof The proof is like Theorem 2 and hence omitted. □

Remark 5 *When the specified frequency interval is $\Omega = [0, \pi]$ rad/s, $R_{T,\Omega} = R_j$.*

Proposition 4 *$R_{T,\Omega}$ can be computed from R_j as*

$$R_{T,\Omega} = SR_j + R_jS$$

Proof The proof is like Theorem 3 and hence omitted. □

Remark 6 *Similar to the standard cross Gramian definition, when $H(z)$ is symmetric, i.e. $H^T(z) = H(z)$, the following property holds:*

$$R_{T,\Omega}^2 = P_{T,\Omega} Q_{T,\Omega} \tag{25}$$

Algorithm 2: The truncating projection matrices that obtain ROM are calculated by adapting the algorithm given in [22]. We choose this algorithm because it gives a similar ROM (i.e. ROM with the same transfer function), which is obtained from the balanced transformation due to the property of cross Gramian in [23]. Therefore, for symmetric systems, $H_r(z)$ yielded by Algorithms 1 and 2 is the same. Moreover, Algorithm 2 is less computational, as only one Gramian is to be computed, and less prone to ill-conditioning. It can even be applied to nonminimal systems, as discussed in [8]. There is some abuse in the usage of mathematical variables in the following algorithm and the remainder of this subsection; however, the context clearly shows the difference. ROM can be obtained with the following steps:

1. Calculate real ordered Schur form of $R_{T,\Omega}$, i.e. $T^T R_{T,\Omega} T = R_S = \begin{bmatrix} R_{S11} & R_{S12} \\ 0 & R_{S22} \end{bmatrix}$, where $R_{S11} \in R^{r\times r}$ and $R_{S22} \in R^{(n-r)\times(n-r)}$.

2. Compute matrix Q from the Sylvester equation $R_{S11}Q - QR_{S22} + R_{S12} = 0$.

3. Partition T as $[T_{(n\times r)}\ T_{(n\times n-r)}]$. Then $V_{11} = T_{(n\times r)}$ and $U_{11} = T^T_{(n\times n-r)} - QT^T_{(n\times n-r)}$.

4. Then $A_rB_rC_rD_r = \{U_{11}AV_{11}, U_{11}B, CV_{11}, D\}$.

Remark 7 *The cross Gramian definition in Eq. (24) can be extended to a wider class of orthogonally symmetric systems, i.e. if $AP = PA^T$ and $= PCU^T$; if $p \leq m$ or $C = PBU^T$; and if $m \leq p$, where P and U are symmetric and orthogonal matrices, respectively [24]. The definition in Eq. (24) for such systems becomes*

$$R_{T,\Omega} = \sum_{i=n_1}^{n_2-1} A^i \left(SBUC + BUCS\right) A^i$$

such that the property in Eq. (25) holds [24], and Algorithm 2 yields the same $H_r(z)$ as Algorithm 1.

Remark 8 *For general nonsymmetric systems, only an averaged cross Gramian can be calculated. Using the concept in [25], the definition in Eq. (24) can be extended to nonsymmetric systems as follows. The matrices B and C can be decomposed column-wise and row-wise, respectively, as*

$$B = \begin{bmatrix} b_1 & \cdots & b_m \end{bmatrix},\ b_k \in R^{n\times 1}, \qquad C = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}, \qquad c_l \in R^{1\times n}$$

where $k \in \{1, 2, \cdots, m$ and $l \in \{1, 2, \cdots, p$.

Let $\tilde{B} = \sum_{i=1}^{m} b_k$ and $\tilde{C} = \sum_{i=1}^{p} c_l$ be matrices of order $n \times 1$ and $1 \times n$, respectively. Then the limited time and frequency intervals cross Gramian is defined as

$$\tilde{R}_{T,\Omega} = \sum_{i=n_1}^{n_2-1} A^i \left(S\tilde{B}\tilde{C} + \tilde{B}\tilde{C}S \right) A^i$$

Neither the property in Eq. (25) holds for $\tilde{R}_{T,\Omega}$, nor does Algorithm 2 yield the same $H_r(z)$ as Algorithm 1.

4. Numerical example

Example 1 *Consider the 200th-order discrete heat equation model in [26] (MATLAB files of the model can be downloaded from [http://slicot.org/20-site/126-benchmark-examples-for-model-reduction]). A 13th-order ROM is obtained using balanced truncation, Algorithm 1 (proposed I), stability preserving modification of Algorithm 1 (proposed II), and Algorithm 2 (proposed III). The specified time and frequency intervals chosen for the experiment are* $[1, 3]$ *s and* $[0.30\pi, 0.50\pi]$ *rad/s. Figure 1 shows the singular value plot of the error function* $(H(z) - H_r(z))$ *in the specified frequency interval. Figure 2 shows the impulse response of the error function* $(H(z) - H_r(z))$. *It can be observed from Figures 1 and 2 that ROMs yielded by the proposed algorithms are more accurate within the desired time and frequency intervals than the balanced truncation [5].*

Example 2 *Consider the 256th-order advection equation model (MATLAB files of the model can be downloaded from [http://gramian.de]). A 17th-order ROM is obtained at the specified time and frequency intervals chosen* $[2, 4]$ *s and* $[0.70\pi, 0.80\pi]$ *rad/s, respectively. Figures 3 and 4 show the singular value plot and impulse response of the error function* $(H(z) - H_r(z))$ *in the specified frequency and time intervals.*

5. Discussion and conclusion

It can be observed from the figures that the error yielded by the proposed algorithms is least within the specified time and frequency intervals. The ROM yielded by the proposed Algorithm II is slightly inferior in accuracy.

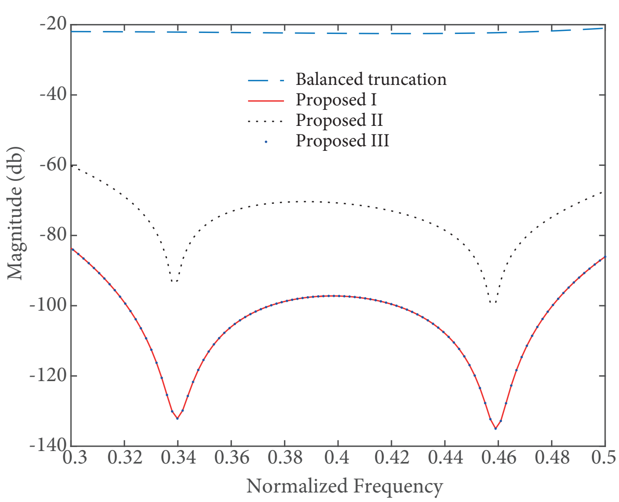

Figure 1. Error plot in the frequency interval $[0.30\pi, 0.50\pi]$ rad/s.

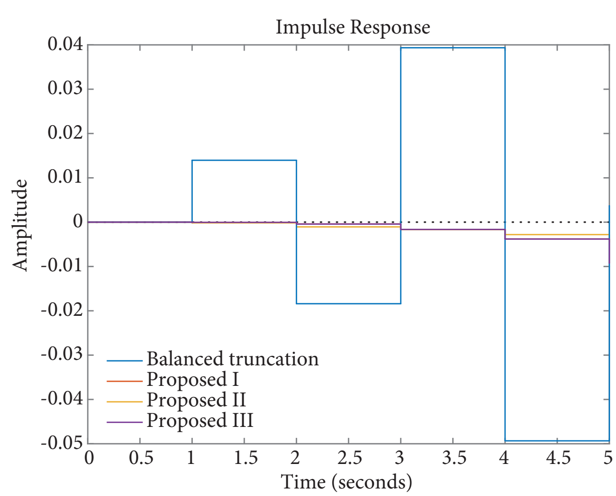

Figure 2. Error plot in the time interval [0, 5] s.

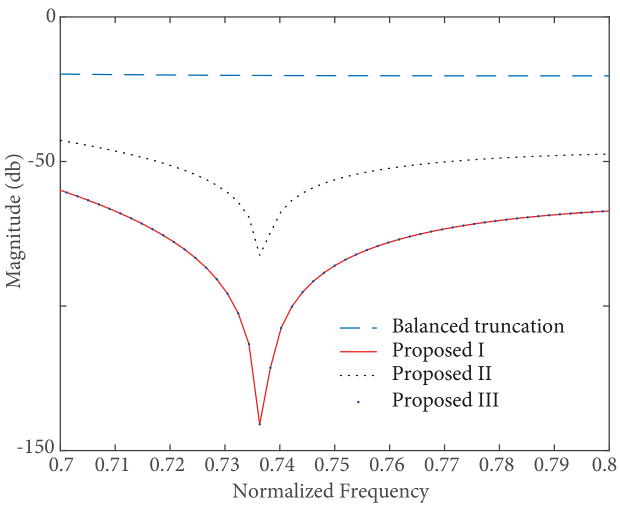

Figure 3. Error plot in the frequency interval [0.70π, 0.80π] rad/s.

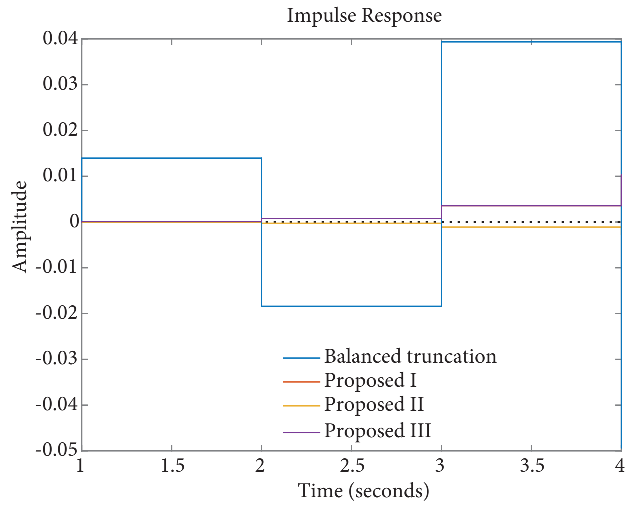

Figure 4. Error plot in the time interval [1, 4] s.

However, the stability preservation and the availability of the easily computable error-bound expression are the main advantages of this algorithm. ROM yielded by the proposed Algorithm III has the same accuracy as in Algorithm I. However, it is less computational and less prone to ill-conditioning. In conclusion, all the proposed algorithms performed well and ensured accuracy within the desired intervals.

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