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# New results on the global asymptotic stability of certain nonlinear RLC circuits

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**Abstract:** This paper deals with the global asymptotic stability (GAS) of certain nonlinear RLC circuit systems using the direct Lyapunov method. For each system a suitable Lyapunov function or energy-like function is constructed and the direct Lyapunov method is applied to the related system. Then the invariant equilibrium point of each system that makes the system solution to the global asymptotic stable is determined. Some new explicit GAS conditions of certain nonlinear RLC circuit systems are derived by Lyapunov's direct method. The presented simulations are compatible with the new results. The results are given with proofs.

**Key words:** Global stability, Lyapunov, nonlinear RLC circuit

## 1. Introduction

Lyapunov's direct method is still one of the most efficient ways to study asymptotic behavior of dynamical systems [1]. For a concise survey of Lyapunov stability, the reader is referred to various books [2–5], papers [1,6], and the references therein. In this paper, the method has been applied to certain nonlinear RLC circuit systems that are closely related to nonlinear oscillation [7], electronic theory (the differential equation of self-excited oscillation of an electronic triode [8]), and Lienard and Van der Pol equations. One can find some beautiful works in the literature discussing the stability (instability) behavior of circuit systems, such as Lyapunov stability for nonlinear descriptor systems [9], the global qualitative behavior of the double scroll system [10], chaos in the Colpitts oscillator due to positive Lyapunov exponents [11], unstable behavior of Hartley's oscillator because of the positive real parts of the eigenvalues of the Jacobian matrix of the system [12], and the global asymptotic stability (GAS) of the synchronization of Vilnius chaotic systems (using active and passive controls) determined by Lyapunov's direct method [13]. Moreover, some recent works have been done on the various behaviors of nonlinear RLC circuit systems: the existence of solutions [14], implicit solutions [15], power shaping [16], passivity and power-balance inequalities [17], and so on.

The main tool for tackling these systems will be the well-known direct method of Lyapunov. This method yields stability information directly, i.e., without solving differential equations [18]. In this regard, the most complete contribution to the stability analysis of dynamical systems was introduced by Lyapunov. Further authors [5,19,20] then emphasized the basic ideas of Lyapunov's functions as "energy-like functions", which empowered the method and made Lyapunov's functions more understandable for real applications. Furthermore, the Barbashin–Krasovskii–LaSalle invariance principle [20] states that the function's derivative along the related

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system vanishes. This implies that the system's invariant equilibrium points are globally asymptotically stable. This study also includes the application of the principle.

In view of the above explanation, we first obtain the systems from the related circuits and then investigate the GAS properties of the solutions.

## 2. Preliminaries

Before introducing our main results, we give some basic statements whose usage will guide us in the remainder of the paper.

Consider the nonlinear dynamical system

$$x'(t) = f(x(t)), \tag{1}$$

where $x(t) \in \mathcal{D} \subseteq \mathcal{R}^n$, $\mathcal{D}$ is an open set with $\in \mathcal{D}$, $f : \mathcal{D} \to \mathcal{R}^n$ continuous on $\mathcal{D}$, and $t \in [0, \infty)$. Let $f(0) = 0$ and $f(x) \neq 0$ for $x \neq 0$.

**Definition** *Consider the system of Eq. (1) with a point $x_e \in \mathcal{D}$ that is said to be an equilibrium point of Eq. (1) at time $t_e \in [0, \infty)$ if $f(x_e) = 0$.*

**Theorem 1** *Consider the nonlinear dynamical system of Eq. (1) and assume that there exists a continuously differentiable function $V : \mathcal{R}^n \to \mathcal{R}$ such that:*

(i) $V(0) = 0$,

(ii) $V(x) > 0, x \in \mathcal{R}^n, x \neq 0$,

(iii) $V'(x) f(x) \leq 0, x \in \mathcal{R}^n$

(iv) $V(x) \to \infty$ as $\|x\| \to \infty$.

*Furthermore, assume that the set $S = \{x \in R^n : V'(x) f(x) = 0\}$ contains no invariant set other than the set $\{0\}$. Then the zero solution$x(t) \equiv 0$ to Eq. (1) is globally asymptotically stable.*

**Proof** See [20].

Investigating the GAS properties of the zero solutions of the systems to the following RLC circuit systems is the main purpose of this paper. □

## 3. Main results

We shall be concerned here with the following circuits:

The circuit given in Figure 1 is supplied by $i_s(t)$ and constructed with a linear resistor $R$, a linear capacitor $C$, and a nonlinear inductor $L$. The inductor current is given by $i_L = I_0 \sin(k\Phi_L(t))$, where $\Phi_L$ is the magnetic flux, and $I_0$ and $k$ are constant numbers.

**Theorem 2** *Assume that $\Phi_L$ and $v_C$ are the state variables of the circuit given in Figure 1, such that $(\Phi_L, v_C) = (\Phi_L, 0)$ are the equilibrium points of the system, when $i_s(t) = I_0 \sin(k\Phi_L(t))$, for $t \geq 0$, and*

$$\Phi_L(t) \neq \left(\frac{1+2n}{2k}\right)\pi, n = 0, \pm 1, \pm 2, \ldots \quad .$$

*Then the origin $(0, 0)$ of the system is globally asymptotically stable.*

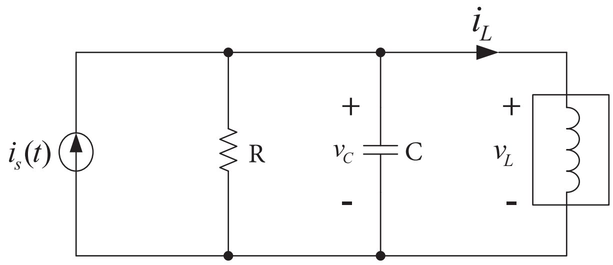


**Figure 1**. RLC circuit with nonlinear inductor.

**Proof** From Figure 1, and $L\frac{di_L}{dt} = v_C = \frac{1}{C}q$, we get the system

$$\begin{aligned} {\Phi_L}' &= \tfrac{v_C}{kLI_0\cos(k\Phi_L)}, \\ {v_C}' &= \tfrac{1}{C}\left(i_s\left(t\right) - I_0\sin(k\Phi_L) - \tfrac{kLI_0\cos(k\Phi_L){\Phi_L}'}{R}\right). \end{aligned} \tag{2}$$

Since $\frac{dW}{dt} = P = iv$, let the total energy function of Figure 1 or the Lyapunov function be

$$W\left(t\right) = V\left(t\right) = \frac{1}{2}Li_L^2 + \frac{1}{2C}q^2 + \frac{1}{R}\int_0^t v_c^2\left(s\right)ds,$$

$$V = V\left(\Phi_L, v_C\right) = \frac{1}{2}LI_0^2 sin^2(k\Phi_L) + \frac{1}{2}Cv_c^2\left(t\right) + \frac{1}{R}\int_0^t v_c^2\left(s\right)ds.$$

Taking the time derivative of $V$ along the trajectory of Eq. (2), we obtain

$$V' = kLI_0^2 sin\left(k\Phi_L\right) cos\left(k\Phi_L\right){\Phi_L}' + Cv_C{v_C}' + \frac{1}{R}v_c^2.$$

Then an elementary computation yields

$$V' = i_s v_C.$$

The equilibrium points $(\Phi_L, v_C) = (\Phi_L, 0)$ of the system of Eq. (2) yield

$$V' = 0.$$

Thus, $V'\left(\Phi_L, 0\right) = 0$ at all points $\left(\Phi_L, 0\right) \in \mathcal{R}^2$ and $V\left(\Phi_L, v_C\right) \to \infty$ as $\Phi_L^2 + v_C^2 \to \infty$. Hence, all the solutions of Eq. (2) are bounded as $t \to \infty$. Moreover, $S$ is the set of all points on the $\Phi_L$-axis. Clearly, $(0,0)$ is the only invariant subset of $S$. The application of Theorem 2 shows that $(0,0)$ of Eq. (2) is globally asymptotically stable. This explanation implies Figure 2. □

**Theorem 3** *Assume that the charge on the nonlinear capacitor of the circuit given in Figure 3 is* $G\left(q\right)$ *such that:*

(i) $G\left(0\right) = 0, G\left(q\right) > 0, q \neq 0,$

(ii) $F'\left(0\right) = 0, F'\left(I\right) > 0, I \neq 0.$

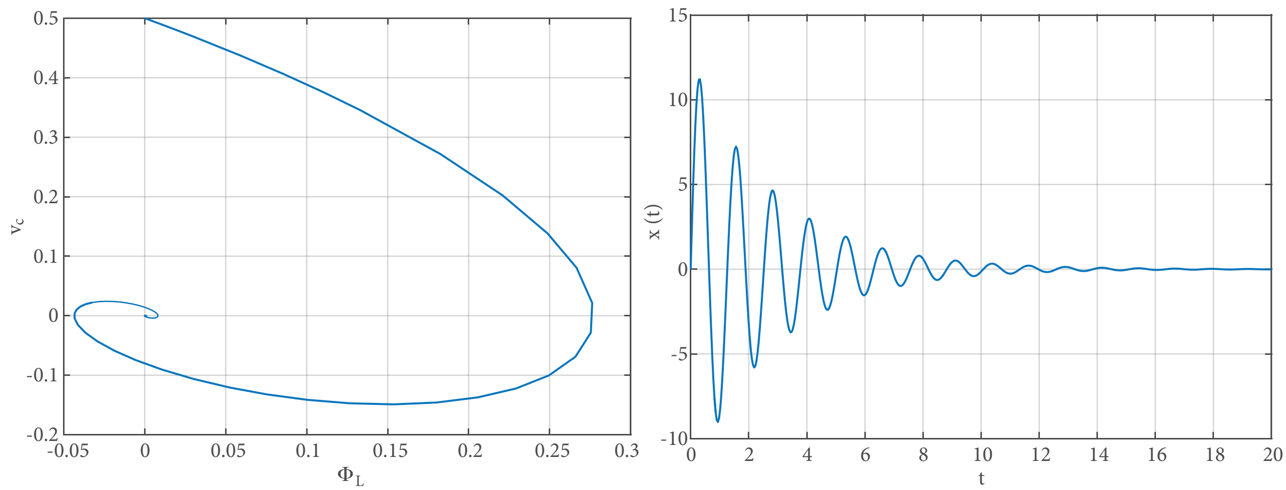


**Figure 2**. a) Phase plane plot of the system of Eq. (2); b) solution of the system of Eq. (2) $(k = I_0 = 1, L = 1H, R = 1\Omega, C = 1F, i_s = 1mA, t \in [0, 30])$.

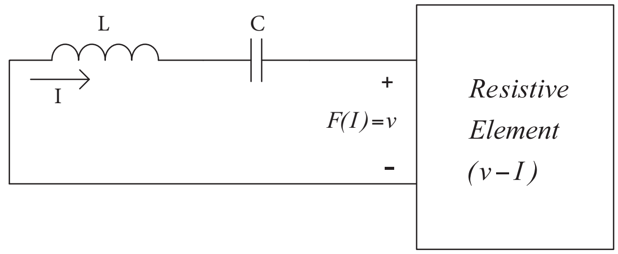


**Figure 3**. RLC circuit with nonlinear resistive element.

*Then the zero solution of the Lienard equation that will be derived from the circuit of Figure 3 is globally asymptotically stable.*

**Proof** From the circuit given in Figure 3, we have

$$LI' + F(I) + \frac{1}{C}G(q) = 0. \tag{3}$$

Taking the time derivative of Eq. (3) yields the following:

$$I'' + f(I)\,I' + g(I) = 0,$$

where

$$f(I) = \frac{1}{L}F'(I),$$

and

$$g(I) = \frac{1}{LC}G'(q)\,I.$$

Let

$$x = I.$$

Then we have

$$x'' + f(x)x' + g(x) = 0.$$

From Figure 3, we obtain the Lienard equation, and it is equivalent to the following system:

$$\begin{aligned} x' &= y, \\ y' &= -f(x)y - g(x). \end{aligned} \tag{4}$$

Hence, the origin $(x, y) = (0, 0)$ is the only equilibrium point of Eq. (4).

For Eq. (4), we can construct a Lyapunov function:

$$V(x, y) = \frac{1}{2}y^2 + G(x) = \frac{1}{2}y^2 + \int_0^x g(s)ds. \tag{5}$$

Taking the time derivative of Eq. (5) along the trajectory of Eq. (4), we have

$$V' = -f(x)y^2 \leq 0.$$

Thus, $V'(x, y) \leq 0$ at all points $(x, y) \in \mathcal{R}^2$ and $V(x, y) \to \infty$ as $x^2 + y^2 \to \infty$. Hence, all the solutions of Eq. (4) are bounded as $t \to \infty$. The set $S$ where $V' = 0$ is either the $x$-axis or the union of the $x$-axis and the $y$-axis (if $f(0) = 0$). Clearly, $(0, 0)$ is the only invariant subset of $S$. Thus, the application of Theorem 3 shows that $(0, 0)$ of Eq. (4) is globally asymptotically stable. This explanation is compatible with Figure 4.

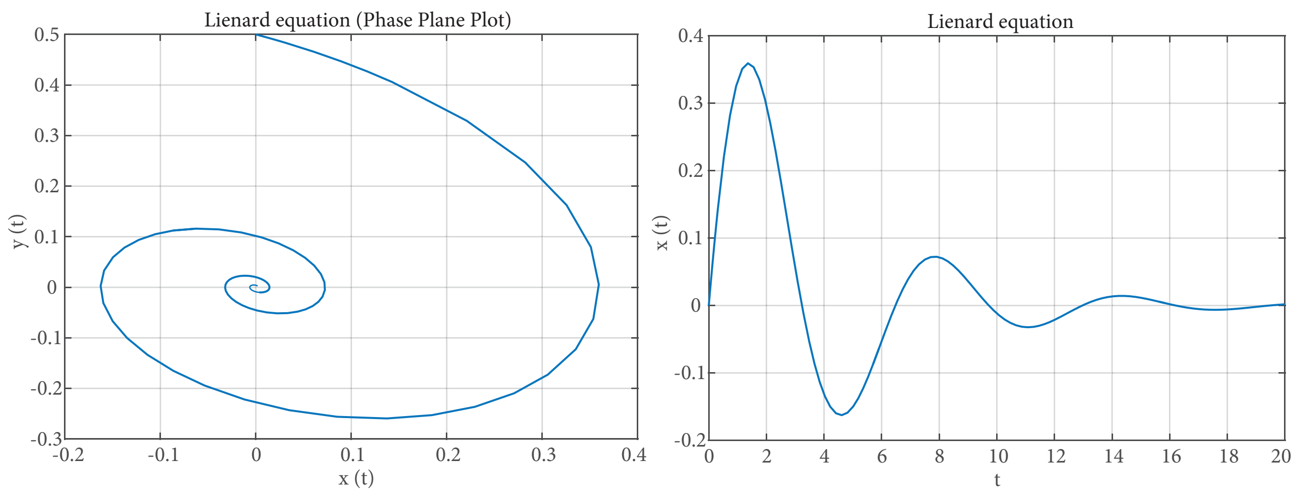


**Figure 4**. a) Phase plane plot of the system of Eq. (4); b) solution of the system of (4) ($f(x) = 0.5(1 - x^2), g(x) = x, t \in [0, 20]$).

The state variables $x$ and $y$ of the circuit given in Figure 5 are

$$x = \frac{C_1 v_1 + C_2 v_2}{C_1 + C_2},$$

$$y = v_2.$$

□

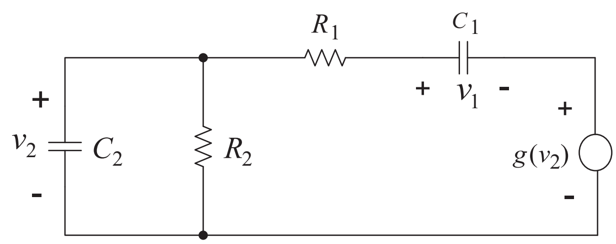


**Figure 5**. RLC circuit with nonlinear dependent voltage source.

**Theorem 4** *The zero solution $x(t) \equiv 0$ to the system with respect to the state variables $x$ and $y$ is globally asymptotically stable if:*

(i) $g(0) = 0$,

(ii) $3v_2^2 \le v_2 g(v_2)$.

**Proof** From Figure 5 and the state variables $x$ and $y$, we have the following system:

$$\begin{aligned} x' &= \tfrac{1}{R_2(C_1+C_2)} y \\ y' &= -\tfrac{C_1+C_2}{R_1C_1C_2} x + \left( \tfrac{1}{R_1C_1} + \tfrac{1}{R_1C_2} + \tfrac{1}{R_2C_2} \right) y - \tfrac{1}{R_1C_2} g(y) \end{aligned}.$$

Let $C_1 = C_2 = C$ and $R_1 = R_2 = R$; then we have

$$\begin{aligned} x' &= \tfrac{1}{2RC} y, \\ y' &= \tfrac{1}{RC} \left[ -2x + 3y - g(y) \right]. \end{aligned} \tag{6}$$

Hence, the origin $(x, y) = (0, 0)$ is the only equilibrium point of Eq. (6). □

Now consider the function $V = V(x, y)$ defined by:

$$V(x, y) = 2x^2 + \frac{1}{2} y^2. \tag{7}$$

It is clear that

$$V(0, 0) = 0, \tag{8}$$

and $V(x, y) \ge 0$, for all

$$(x, y) \in R^2. \tag{9}$$

Eqs. (8) and (9) imply that $V$ is a positive definite function.

Taking the time derivative of Eq. (7) along the trajectory of Eq. (6) and using (ii), we obtain

$$V' = 3y^2 - yg(y) \le 0.$$

Thus, $V'(x, y) \le 0$ at all points $(x, y) \in \mathcal{R}^2$ and $V(x, y) \to \infty$ as $x^2 + y^2 \to \infty$. Hence, all the solutions of Eq. (6) are bounded as $t \to \infty$. Moreover, $S$ is the set of all points on the $x$-axis because $V'(x, 0) = 0$. Clearly, $(0, 0)$ is the only invariant subset of $S$. Thus, the application of Theorem 4 shows that $(0, 0)$ of Eq. (6) is globally asymptotically stable. This explanation implies Figure 6.

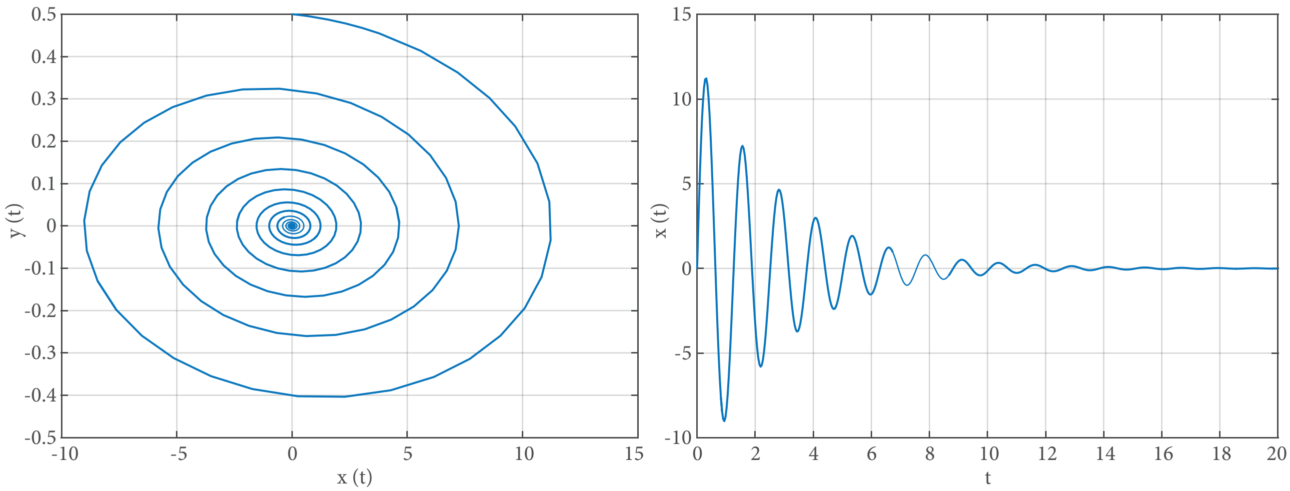


**Figure 6**. a) Phase plane plot of the system of Eq. (6); b) solution of the system of Eq. (6) ($R = 50\Omega$, $C = 5F$, $g(y) = 10y, t \in [0, 20]$).

## 4. Conclusion

As we have seen in the works mentioned earlier, the problem of stability analysis of systems is still one of the most burning problems of control theory, because of the absence of its complete solution.

This work is an elementary and excellent account of Lyapunov's direct method on the GAS properties of certain nonlinear RLC circuit systems. The reader can learn some essentials of Lyapunov's direct method as well as how efficient it can be. Readers will find in the paper some interesting comments, instructive interpretations, and the application of Lyapunov's direct method to the Lienard equation. The previous studies and this work imply that study of the stability behavior of solutions of nonlinear oscillation is especially fruitful. In this direction some good works can be done on the stability properties of nonlinear circuit systems by Lyapunov's direct method in the future.

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