

## Nonlinear analysis of hybrid phase-controlled systems in z-domain with convex LMI searches

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Received: 10.04.2017

Accepted/Published Online: 28.12.2017

Final Version: 30.03.2018

**Abstract:** Phase-controlled systems such as phase-locked loops (PLLs) have been used in numerous applications ranging from data communications to speed motor control. The hybrid case where only the phase detector is digital while others are analog has advantages over the classical PLLs in the sense that it provides a wider locking range and is more suitable when the input and output signals come in digital waveforms. Although such systems are inherently nonlinear due to the phase detector's characteristics, the nonlinearity is often bypassed in order to ease the analysis and design methods. This, however, will give erroneous results when the phase difference between input and output falls into the nonlinear range. Another source of inaccuracies in modeling PLLs is the continuous-time approximation, which is only useful if the operating frequencies of interest are much less than the incoming data transition rate. In this paper, we present a nonlinear analysis of a hybrid PLL in the z-domain where the stability is established via the discrete-time Lur'e–Postnikov Lyapunov function, and the performance is evaluated via the induced  $\ell_2$  norm objective. The results are formulated in the form of linear matrix inequality searches, which are computationally tractable. We also extend the result for analysis of a PLL-based frequency synthesizer and provide several numerical examples to illustrate the effectiveness of the results compared to the existing ones.

**Key words:** Lyapunov, phase-locked loop, nonlinear, linear matrix inequality

### 1. Introduction

For many decades, the study of phase-controlled systems has attracted many engineers, mainly from communications and control societies. One of the most common applications of such systems is phase-locked loops (PLLs), which have been used in various synchronization systems ranging from data communications to speed motor control [1, 2].

The standard modeling practice of a PLL in time space is shown in Figure 1, which consists of a phase detector (PD), a loop filter (LPF), and a voltage-controlled oscillator (VCO). The PD plays a very important role as it functions to detect the phase difference between the input signal  $f_i(t)$  and output signal  $f_o(t)$  and deliver appropriate signals to the LPF in order to minimize the error. Figure 1 can be restructured into an equivalent phase-domain model with four main blocks as shown in Figure 2; the blocks consisting of  $1/s$  and  $F(s)$  are directly translated from the LPF and integrator from the time-domain model, the parameter  $K_L$  represents the overall loop gain, and  $\phi$  denotes the nonlinearity resulting from the PD's characteristics [3]. The signals  $\theta_i$ ,  $\theta_o$ , and  $\theta_d$  represent the input phase, output phase, and phase difference, respectively.

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The structure shown in Figure 2 depicts the nonlinear model of a PLL in a continuous-time setting, which is typically used when the PD is analog and the input and output signals are of sinusoidal waveforms (see [3] and the references therein). A reasonable linear approximation, i.e. the nonlinearity effect from the PD, may be bypassed when the PLL is considered to be close to its lock state. This allows many well-developed linear control tools to be used for analysis and design (see, for example, [2, 4–6]). Nevertheless, this assumption is only valid when the phase deviation between the VCO’s output signal and the reference’s is sufficiently small [3]. When the deviation gets bigger, the system will be driven into the nonlinear region, which subsequently leads to performance degradation and in the worst case causes the system to fall out of lock. Hence, including the nonlinearities into PLL models is crucial for the accuracy of the analysis[7].

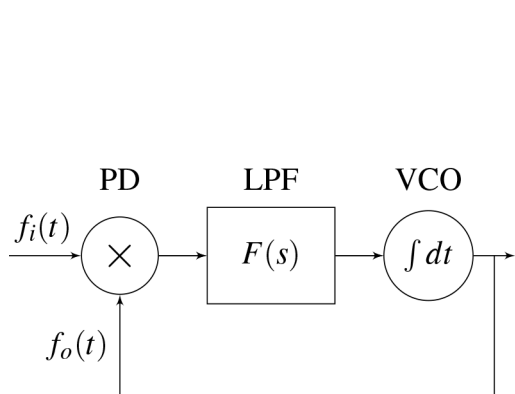


Figure 1. Basic PLL model in signal’s time space.

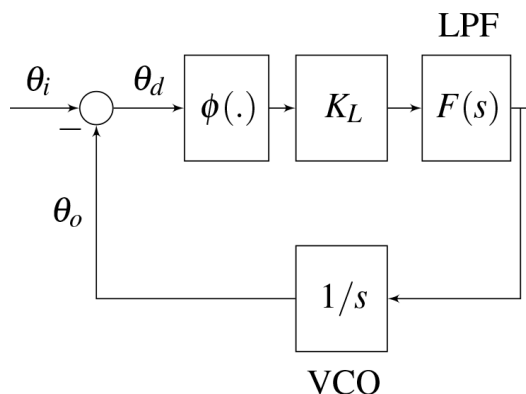


Figure 2. Equivalent model of Figure 1 in continuous-time domain.

Another source of inaccuracies in modeling PLLs is the continuous-time approximation, which is only useful if the operating frequencies of interest are much lower than the incoming data transition rate [8]. Although the continuous-time model is relatively easier to analyze as compared to its discrete counterparts, this may also not be accurate for some other cases, such as when a digital PD is used (i.e. hybrid PLLs) or when the input and output come in digital waveforms. The comparison between the s-domain and z-domain models is well explained in [8], where it was shown that the z-domain model of a PLL with digital PD predicts the loop performance of the system more accurately as compared to the s-domain model, particularly at higher jitter frequencies. The work in [8], nevertheless, only considers the two models when the nonlinearity is ignored (i.e. via linear approximation).

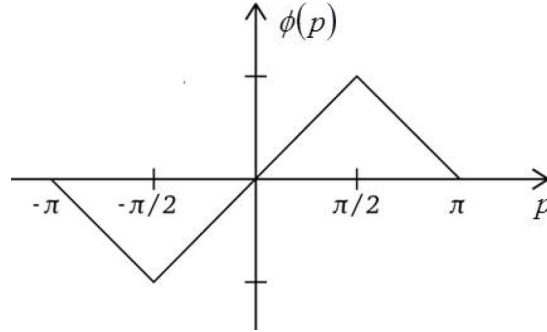
Recent works have shown interest in nonlinear analysis of analog PLLs, particularly in the study of operating frequency range and the existence of self-oscillations [6, 9–11]. For systems where the nonlinearities satisfy certain sector-bound conditions [12, 13], circle and Popov criteria [14] can provide sufficient stability conditions in the frequency domain. These are advantageous over the phase-plane portrait as they are applicable to systems of any order and can be restructured into a linear matrix inequality (LMI) search [15] via the IQC theory [16] and Lyapunov functions [17]. The modified LMI conditions for circle and Popov criteria are presented in [18] for systems with integrators, which retain the conditions of the original frequency-based conditions via encapsulation of the nonlinearity. This allows a direct application of these criteria to analog PLLs since the integrator always exists in the system due to the VCO. In the case of classical digital or hybrid PLLs, the stability condition can be established via the discrete-time counterpart of the circle criterion, namely the Tsytkin criterion [19]. The discrete-time counterparts of the Popov criterion are also available via the Jury–Lee

stability conditions [20], which were derived for nonlinearities with various slope conditions via passivity and the Lyapunov theory. The translation of the Jury–Lee criteria into LMI searches, however, requires further work due to the constraints on the properties of the associated Lyapunov function [21, 22].

The main contribution of this paper is the nonlinear stability analysis of hybrid PLLs in the  $z$ -domain, where the stability is established via the Lur’e–Postnikov Lyapunov function and the performance is evaluated via the induced  $\ell_2$  norm objective. Via a modified approach on the difference Lyapunov equation, the stability condition can be formulated in the form of an LMI search, which is computationally more attractive than the frequency-based condition. We also extend the search to the analysis of a PLL-based frequency synthesizer and include several numerical examples to illustrate the effectiveness of the results compared to the existing ones.

## 2. Preliminaries and methodology

In this work, we consider the PLL model as depicted in Figure 2, where  $\phi$  is the nonlinearity arising from the digital PD (XOR gate). The characteristic of  $\phi$  is shown in Figure 3 [12, 23].



**Figure 3.** Characteristic of  $\phi(\cdot)$  for PLL phase-domain model with XOR gate

The nonlinearity  $\phi : \mathbb{R}^v \rightarrow \mathbb{R}^v$  is static, memoryless, and bounded and satisfies  $\phi(0) = 0$ . It lies in the sector bound

$$0 \leq \frac{\phi_i(p_i)}{p_i} \leq k_i \quad \forall p_i \neq 0, \quad (1)$$

where  $k_i > 0$ , and in the slope restriction

$$-S \leq \frac{\phi_i(p_{i1}) - \phi_i(p_{i2})}{p_{i1} - p_{i2}} \leq S \quad \forall p_{i1} \neq p_{i2}, \quad (2)$$

where  $s_i > 0$ . Extensions of the sector and slope conditions in (1)–(2) to the multivariable case are given as

$$\phi(p)^T (Kp - \phi(p)) \geq 0 \quad (3)$$

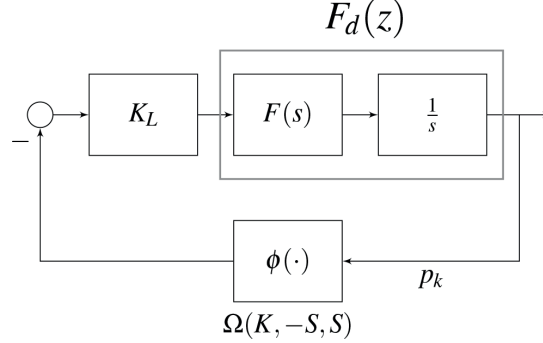
and

$$[S(p_1 - p_2) - (\phi(p_1) - \phi(p_2))][\phi(p_1) - \phi(p_2) + S(p_1 - p_2) + S(p_1 - p_2)] \geq 0, \quad (4)$$

respectively, where  $S, K \in \mathbb{R}^{v \times v}$  are diagonal and positive definite. Throughout the paper, the shorthand notation  $\phi \in \Omega(K, -S, S)$  is used to indicate that it lies in the sector and slope bounds (3)–(4).

For analysis purposes, the system in Figure 2 can be rearranged into Figure 4, which resembles a Lur’e structure (i.e. a LTI system in the forward path and a nonlinearity in the feedback path). Without loss of

generality, we set  $\lim_{s \rightarrow 0} sF(s) = 1$ , where  $F(s)$  contains the dynamic of the loop filter, and we represent  $K_L$  as the loop gain, which is obtained from the multiplication of the gains from the PD, LPF, and VCO.



**Figure 4.** Equivalent representation of Figure 2 in Lur'e structure with  $p_k = \theta_d$ .

In light of [8], the z-domain model of the PLL can be constructed by using an impulse invariant or zero-order hold methods where the discrete representation of the filter and VCO is obtained via transformation of  $F(s)/s$  into  $F_d(z)$ . This then allows the application of the discrete Lyapunov approach to ensure the stability of the system.

### 3. Main results

Suppose the LTI system in the forward part of Figure 4,  $K_L F_d(z)$ , has a state-space

$$x_{k+1} = Ax_k - Bq_k; \quad p_k = Cx_k \quad (5)$$

with  $x_k \in \mathbb{R}^{n_a}$ ,  $q_k, p_k \in \mathbb{R}^v$ ; thus,  $A \in \mathbb{R}^{n_a \times n_a}$ ,  $B \in \mathbb{R}^{n_a \times v}$ , and  $C \in \mathbb{R}^{v \times n_a}$ . The system is always strictly proper due to the perfect integrator. The nonlinearity  $q_k = \phi(p_k)$  is memoryless and can be generalized into  $\phi \in \Omega(K, -S, S)$ . The following definition will be useful in the main result of this paper (Proposition 1).

**Definition 1** *The symmetric matrices  $M_P$ ,  $M_k$ ,  $M_1$ ,  $M_2$ , and  $M_3$  are defined as follows.*

$$M_P = \begin{bmatrix} A^T \mathbf{P} A - \mathbf{P} & -A^T \mathbf{P} B \\ -B^T \mathbf{P} A & B^T \mathbf{P} B \end{bmatrix} \quad (6)$$

$$M_k = \begin{bmatrix} 0 & C^T K \mathbf{W} \\ \mathbf{W} K C & -2\mathbf{W} \end{bmatrix} \quad (7)$$

$$M_1 = \begin{bmatrix} (A - I)^T C^T S | \mathbf{R}_1 | S C (A - I) & -(A - I)^T C^T S | \mathbf{R}_1 | S C B \\ -B^T C^T S | \mathbf{R}_1 | S C (A - I) & B^T C^T S | \mathbf{R}_1 | S C B \end{bmatrix} \quad (8)$$

$$M_2 = \begin{bmatrix} 0 & (A - I)^T C^T \mathbf{R}_1 \\ \mathbf{R}_1 C (A - I) & -\mathbf{R}_1 C B - B^T C^T \mathbf{R}_1 \end{bmatrix} \quad (9)$$

$$M_3 = \begin{bmatrix} A^T C^T | \mathbf{R}_1 | K C A - C^T | \mathbf{R}_1 | K C & -A^T C^T | \mathbf{R}_1 | K C B \\ -B^T C^T | \mathbf{R}_1 | K C A & B^T C^T | \mathbf{R}_1 | K C B \end{bmatrix} \quad (10)$$

Here,  $\mathbf{P} \in \mathbb{R}^{n_a \times n_a}$  is positive semidefinite,  $\mathbf{W} \in \mathbb{R}^{v \times v}$  is diagonal and positive definite, and  $\mathbf{R}_1 \in \mathbb{R}^{v \times v}$  is diagonal and indefinite.

The result below presents the stability analysis of the system under consideration:

**Proposition 1** Consider the system in Figure 4 whose state space is as described in Eq. (5) and the nonlinearity  $\phi \in \Omega(K, -S, S)$ . Define  $M_P$ ,  $M_1$ ,  $M_2$ ,  $M_3$ , and  $M_k$  as in Definition 1. The system is stable if there exist  $P \geq 0$ ,  $R_1 \in \mathbb{R}$ , and  $W > 0$  such that the following LMI is satisfied:

$$M_P + M_k + M_1 + M_2 + M_3 \leq 0. \quad (11)$$

The corresponding Lur'e-Postnikov Lyapunov function for the system is constructed as follows:

$$V(x_k) = x_k^T P x_k + 2R_{11} \int_0^{p_k} \phi(\sigma) d\sigma + 2R_{12} \int_0^{p_k} (K\sigma - \phi(\sigma)) d\sigma, \quad (12)$$

where  $R_{11} \geq 0$ ,  $R_{12} \geq 0$ ,  $R_{11}R_{12} = 0$ , and  $R_1 = R_{11} - R_{12}$ .

**Proof** To prove stability via the Lyapunov method, the Lyapunov function of Eq. (12) needs to satisfy all the conditions as in the Lyapunov theory in [24]. First, it is straightforward that  $V(0) = 0$  as each term on the right-hand side of Eq. (12) depends on  $x_k$ , and we have  $\phi(0) = 0$ . Since  $\phi \in \Omega(K, -S, S)$ , it is clear that the second and third terms of the right-hand side of (12) are nonnegative. Together with  $P \geq 0$ , we have  $V(x_k) \geq 0$  for all  $x_k \neq 0$  and  $V \rightarrow \infty$  as  $\|x_k\| \rightarrow \infty$ . In order to prove that the difference Lyapunov equation is negative definite for all  $x_k \neq 0$ , we have

$$\begin{aligned} \Delta V &= V(x_{k+1}) - V(x_k) \\ &= x_{k+1}^T P x_{k+1} - x_k^T P x_k + 2R_{11} \int_{p_k}^{p_{k+1}} \phi(\sigma) d\sigma + 2R_{12} \int_{p_k}^{p_{k+1}} (K\sigma - \phi(\sigma)) d\sigma. \end{aligned} \quad (13)$$

Choosing  $\Lambda_1 = [x_k^T \ q_k^T]^T$ , the first and second terms of the right-hand side of (13) lead to  $\Lambda_1^T M_P \Lambda_1$ . It is also straightforward that

$$2R_{12} \int_{p_k}^{p_{k+1}} K\sigma d\sigma = 2|R_1| \int_{p_k}^{p_{k+1}} K\sigma d\sigma.$$

Expanding this, we get

$$\begin{aligned} 2|R_1| \int_{p_k}^{p_{k+1}} K\sigma d\sigma &= [\sigma^T |R_1| K\sigma]_{p_k}^{p_{k+1}} \\ &= p_{k+1}^T |R_1| K p_{k+1} - p_k^T |R_1| K p_k \\ &= x_{k+1}^T C^T |R_1| K C x_{k+1} - x_k^T C^T |R_1| K C x_k \\ &= (Ax_k - Bq_k)^T C^T |R_1| K C (Ax_k - Bq_k) - x_k^T C^T |R_1| K C x_k \\ &= x_k^T A^T C^T |R_1| K C A - x_k^T A^T C^T |R_1| K C B q_k - q_k^T B^T C^T |R_1| K C A x_k \\ &\quad + q_k^T B^T C^T |R_1| K C B q_k - x_k^T C^T |R_1| K C x_k \\ &= \Lambda_1^T M_3 \Lambda_1. \end{aligned}$$

The remaining terms become

$$2(R_{11} - R_{12}) \int_{p_k}^{p_{k+1}} \phi(\sigma) d\sigma = 2R_1 \int_{p_k}^{p_{k+1}} \phi(p_k) d\sigma + 2R_1 \int_{p_k}^{p_{k+1}} (\phi(\sigma) - \phi(p_k)) d\sigma,$$

where  $2R_1 \int_{p_k}^{p_{k+1}} \phi(p_k) d\sigma = \Lambda_1^T M_2 \Lambda_1$ . From the slope conditions of  $\phi$  in Eq. (2), we have the property

$$-S(p_1 - p_2) \leq \phi(p_1) - \phi(p_2) \leq S(p_1 - p_2) \quad \forall p_1 \neq p_2. \quad (14)$$

It follows that

$$-\int_{p_k}^{p_{k+1}} (\sigma - p_k)^T S d\sigma \leq \int_{p_k}^{p_{k+1}} (\phi(\sigma) - \phi(p_k)) d\sigma \leq \int_{p_k}^{p_{k+1}} (\sigma - p_k)^T S d\sigma. \quad (15)$$

Letting  $R_{12} = 0$  so that  $R_1 = R_{11} > 0$ , we have

$$2R_{11} \int_{p_k}^{p_{k+1}} (\phi(\sigma) - \phi(p_k)) d\sigma \leq 2R_1 \int_{p_k}^{p_{k+1}} (\sigma - p_k)^T S d\sigma.$$

Now let  $R_{11} = 0$  so that  $R_1 = -R_{12} < 0$ , and then we also have, from Eq. (15),

$$-2R_{12} \int_{p_k}^{p_{k+1}} (\phi(\sigma) - \phi(p_k)) d\sigma \leq 2R_{12} \int_{p_k}^{p_{k+1}} (\sigma - p_k)^T S d\sigma = 2|R_1| \int_{p_k}^{p_{k+1}} (\sigma - p_k)^T S d\sigma.$$

We can therefore conclude that

$$2R_1 \int_{p_k}^{p_{k+1}} (\phi(\sigma) - \phi(p_k)) d\sigma \leq 2|R_1| \int_{p_k}^{p_{k+1}} (\sigma - p_k)^T S d\sigma = \Lambda_1^T M_1 \Lambda_1$$

and, combining these, we get

$$\Delta V \leq \Lambda_1^T (M_P + M_1 + M_2 + M_3) \Lambda_1. \quad (16)$$

The sector condition of Eq. (3) is also equivalent to

$$\Lambda_1^T M_k \Lambda_1 \geq 0. \quad (17)$$

If the LMI in Eq. (11) holds, then, via the S-procedure,  $\Lambda_1^T (M_P + M_1 + M_2 + M_3) \Lambda_1 < 0$ , and consequently  $\Delta V < 0$  for all  $x_k \neq 0$ . Hence the result.  $\square$

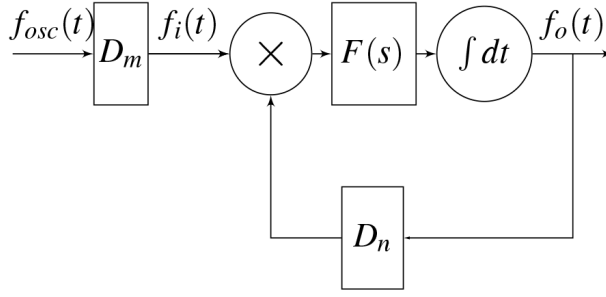
**Remark 1** Via the discrete KYP lemma [25], the LMI in Proposition 1 implies

$$\text{Re} \left\{ K^{-1} + (I + (z-1)R_1)F_d(z) - \frac{S|R_1|}{2} |(z-1)F_d(z)|^2 \right\} > 0, \quad (18)$$

where  $K_L F_d \sim (A, B, C, 0)$ . This is also one of the frequency conditions of the Jury-Lee criteria [26] derived via the area inequality method for  $\phi \in \Omega(K, -S, S)$ . The reverse implication, however, is not true as we need the requirement  $A$  to be Schur stable. This condition is not satisfied due to the integrator in  $F_d$ , but it can be removed via the Lyapunov method as proposed in the result.

**Remark 2** The LMI via the difference equation of the Lyapunov function of Eq. (12) is derived via exploitation of both the nonlinearity's sector and slope conditions. This technique makes the stability condition less conservative than the classical  $s$ -domain method, namely the Popov criterion, which is derived based only on the nonlinearity's sector restriction. The comparison between the two methods is illustrated in the next section.

Based on the analysis in Proposition 1, we extend the result to find the stability condition of a PLL-based frequency synthesizer, which is presented in the following corollary.



**Figure 5.** PLL-based frequency synthesizer.

**Corollary 1** Consider the PLL-based frequency synthesizer as shown in Figure 5, where  $D_m$  and  $D_n$  represent the frequency dividers and  $f_{osc}(t)$ ,  $f_r(t)$ , and  $f_o(t)$  denote the oscillator's, reference, and synthesized frequency signals, respectively.

Write  $F_d(z) \sim (A_f, B_f, C_f, 0)$  as the  $z$ -domain model of  $F_L(s)/s$ , where  $K_L F_d(z) \sim (A, B, C, 0)$  with  $A = A_f$ ,  $B = B_f$ , and  $C = K_L C_f$ . Let  $M_P, M_k, M_1, M_2$ , and  $M_3$  be the matrices as defined in Definition 1. If the optimization problem

$$\begin{aligned} & \text{maximize } K_L \\ & \text{subject to } M_P + M_k + M_1 + M_2 + M_3 \leq 0 \end{aligned} \quad (19)$$

is feasible, where  $P \geq 0$ ,  $R_1 \in \mathbb{R}$ , and  $W > 0$ , then  $K_L$  is the loop gain that can stabilize the system.

**Proof** Let  $K_{fs}$  be the overall loop gain of the system when  $D_m = D_n = 1$ . Figure 5 can also be transformed into the Lur'e structure as shown in Figure 4 with  $K_L = K_{fs} D_m / D_n$ . The relationship between the oscillator frequency,  $f_{osc}$ , and the synthesized frequency,  $f_o$ , is then  $f_o = \frac{D_n}{D_m} \times f_{osc}$ . The rest of the proof then follows directly from that of Proposition 1.  $\square$

The existence of the Lyapunov function for the system under consideration paves the way for  $\ell_2$  gain performance analysis [15, 27], where the goal is to suppress the error with respect to the exogenous input. Let  $z_k$  and  $w_k$  represent the error and the exogenous input, respectively, in the standard  $\ell_2$  gain performance analysis. The smallest  $\gamma > 0$  is sought such that, for all  $\phi \in \Omega(K, -S, S)$ , we have

$$\sup_{\|w_k\|_2 \neq 0} \frac{\|z_k\|_2}{\|w_k\|_2} \leq \gamma. \quad (20)$$

To extend the previous result to the  $\ell_2$  gain performance analysis, consider the state-space

$$x_{k+1} = Ax_k - Bq_k + Bw_k, \quad p_k = Cx_k, \quad z_k = w_k - q_k, \quad (21)$$

where  $z_k = \theta_d$  and  $w_k = \theta_i$  are the phase error and input phase signals, respectively. The corollary below presents the  $\ell_2$  gain performance analysis of the system.

**Corollary 2** Consider the system of Eq. (21) and the Lyapunov function as described in Proposition 1. Let  $M_P, M_1, M_2, M_3$ , and  $M_k$  be defined as in Definition 1. The system has performance with  $\ell_2$  gain less than

$\gamma = \sqrt{\mu}$  if there exist  $P \geq 0$ ,  $R_1 \in \mathbb{R}$ ,  $T > 0$  such that the following LMI is satisfied:

$$Q < 0 \quad \text{where}$$

$$Q = \begin{bmatrix} M_P + M_k + M_1 + M_2 + M_3 & A^T P B \\ B^T P A & -B^T P B \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 - \mu \end{bmatrix} \quad (22)$$

**Proof** Introducing a new vector  $\Lambda_2 = [x_k^T \quad q_k^T \quad w_k^T]^T$ , the LMI of Eq. (22) is also equivalent to  $\Lambda_2^T Q \Lambda_2 < 0$ , or

$$\Delta V + 2q_k^T W(p_k - q_k) + z_k^T z_k - \gamma^2 w_k^T w_k < 0 \quad (23)$$

where  $\Delta V = V(x_{k+1}) - V(x_k)$  and  $z_k^T z_k - \gamma^2 w_k^T w_k = w_k^T(1 - \mu)w_k - w_k^T q_k - q_k^T w_k$  with  $\mu = \gamma^2$ . Integrating both sides of Eq. (23) gives

$$V(x_\infty) - V(x_0) + 2 \sum_{k=0}^{\infty} q_k^T W(p_k - q_k) + \sum_{k=0}^{\infty} (z_k^T z_k - \gamma^2 w_k^T w_k) \leq 0. \quad (24)$$

Since  $V(x_\infty) - V(x_0) \geq 0$ , and the third term of the LHS of Eq. (24) is positive, then  $\sum_{k=0}^{\infty} (z_k^T z_k - \gamma^2 w_k^T w_k) \leq 0$  and consequently Eq. (20) is satisfied. Hence the result.  $\square$

#### 4. Numerical examples

In this section, simple applications of the results to hybrid PLLs with XOR phase detector are presented. The example considers the PLL as shown in Figure 2 consisting of a second order RLC low pass filter given by

$$F(s) = \frac{1/LC}{s^2 + R/Ls + 1/LC}, \quad (25)$$

where  $R = 200 \Omega$ ,  $C = 100 \mu F$ , and  $L = 0.2 H$ . With these values, we get

$$F(s) = \frac{50000}{s^2 + 1000s + 50000}.$$

The nonlinearity is described by  $\phi \in \Omega(1, -1, 1)$ . The z-domain model of the loop can then be obtained by transforming  $K_L F_L(s)/s$  into  $K_L F_d(z)$  via the impulse invariant method with a sampling period of 0.03 s. The objective is to find the loop gain  $K_L$  that can stabilize the system via the search in Corollary 1 with  $D_m$  and  $D_n$  set to unity, while minimizing the induced  $\ell_2$ -gain as presented in Corollary 2.

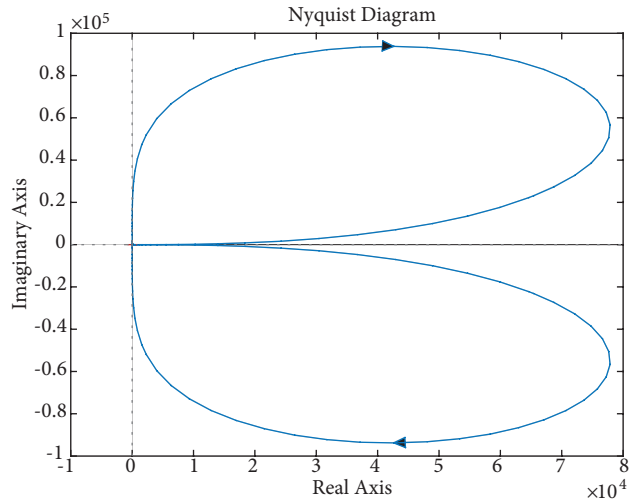
We use the SeDumi and Yalmip MATLAB toolboxes for the LMI searches. Since the simulation is done in the phase domain, the nonlinearity characteristic of the digital phase detector needs to be described precisely as a MATLAB function. In this regard, we follow the nonlinear approximation previously proposed in [12], i.e.

$$\phi(p) = \frac{4}{\pi} \sum_{n=0}^5 \frac{(-1)^n}{(2n+1)^2} \sin((2n+1)p).$$

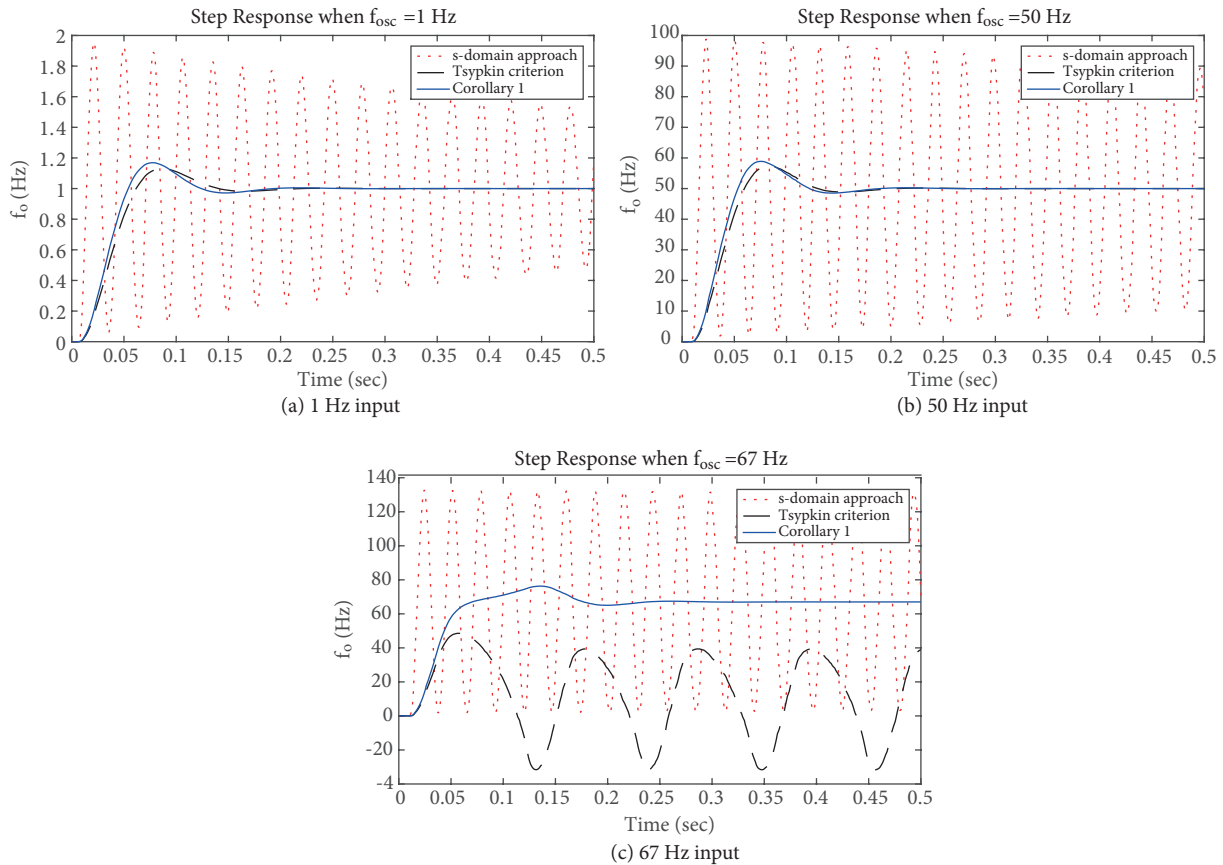
Applying Corollary 1, the optimal value of  $K_L$  obtained is 51.4846 with  $R_1 = 0.5557$ , and the induced  $\ell_2$ -gain via Corollary 2 is 1.9496. The corresponding frequency condition in Eq. (18) is also verified via the Nyquist plot in Figure 6, which shows that all the real parts are strictly on the right-hand plane. A linear approximation, which can be obtained via the Nyquist criterion in the s-domain, gives a maximum  $K_L$  of 1000, which is also the value obtained via the Popov criterion. We also compare the result when the multiplier  $R_1$



is set to zero, which reduces to the Tsytkin criterion [19]. The maximum loop gain obtained via this method is 43.1 with induced  $\ell_2$ -gain of 11.8380. The responses of the system to these values of  $K_L$  are compared in Figure 7 for different input frequency steps.



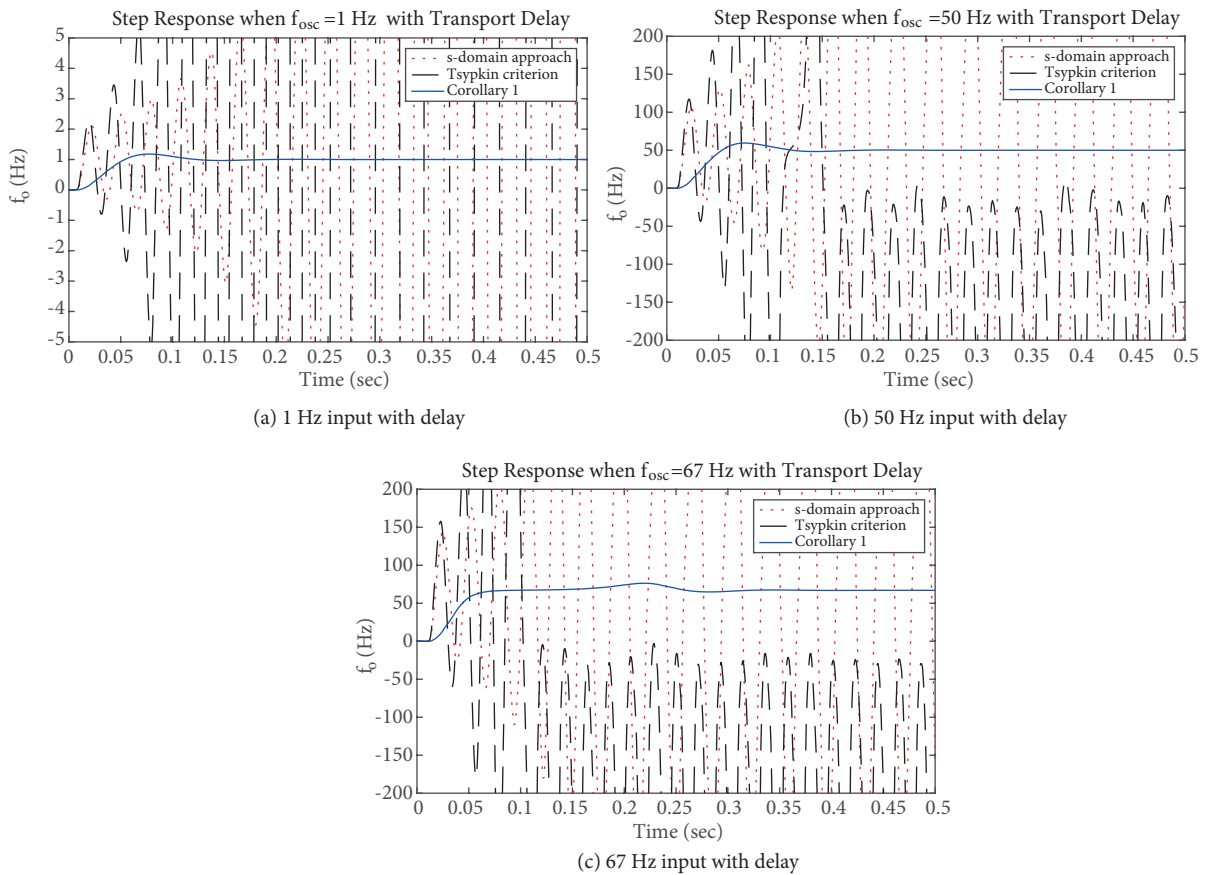
**Figure 6.** The Nyquist plot of the resulting frequency condition for the first example via Corollary 1.



**Figure 7.** Step responses of the first example with different input frequency steps.

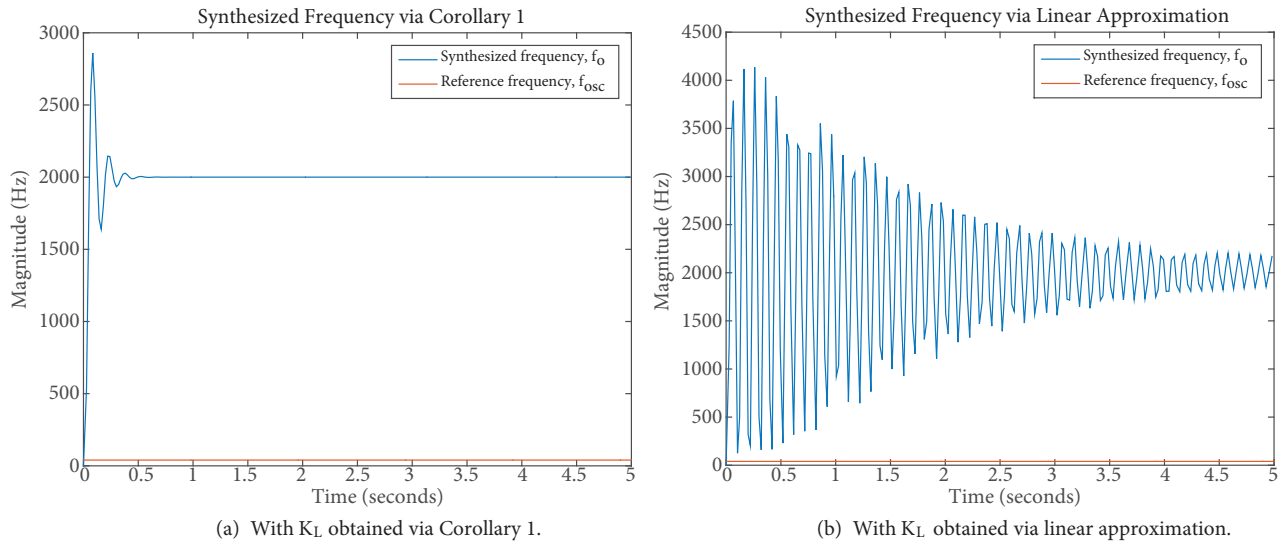
It can be observed that when 1 Hz step frequency is fed in, the PLL locks when Corollary 1 and the Tsytkin criterion are applied. The worst response is seen from the z-domain linear approximation method where it does not drive the system to the lock state at all. When the input frequency is increased to 50 Hz and 67 Hz, only the PLL with the loop gain obtained from Corollary 1 can track the input while others go out of lock.

The stability conditions from Corollary 1, the Tsytkin criterion, and the s-domain method are also applied in the case when there is a transport delay of 0.5 ms in the loop. In this scenario, both the Tsytkin criterion and s-domain method lead to instability when the same frequency steps are applied, as illustrated in Figure 8. Corollary 1 nevertheless still provides a sufficient condition to guarantee that the PLL stays in a locked state.



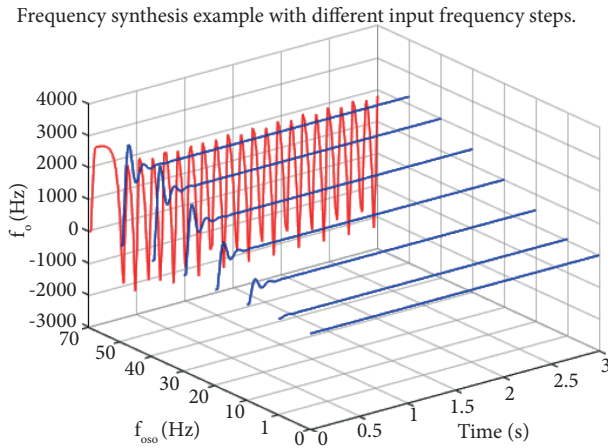
**Figure 8.** Step responses of the first example with a transport delay in the loop with different input frequency steps.

The second example considers a PLL-based frequency synthesizer consisting of the same RLC low pass filter as in Eq. (25), but the discrete-time model  $F_d(z)$  is obtained via the zero-order-hold method with a sampling time of  $T = 0.02$  s. Suppose  $D_m$  is set to unity, the maximum scaling factor  $D_n$  for this design is set to 50, and each available output frequency is separated by 10 Hz. The gain  $K_L$  obtained by the optimization method in Corollary 1 is 44.7, whereas via the z-domain linear approximation, the value obtained is 104. The VCO's output from the application of Corollary 1 is shown in Figure 9a, which shows stability with a smooth response. The linear approximation, however, is not suitable for this case as it leads to a much higher oscillation as shown in Figure 9b.



**Figure 9.** Frequency synthesizer’s responses for the second example with  $D_m = 1$  and  $D_n = 50$ .

The system is also tested when the input frequency step is increased from 0 Hz to 52 Hz, as illustrated in Figure 10. Via observation, for this case, the lock-in range, or the range of input frequency such that the system can track the input, is  $[0, 52)$  Hz. This is also in line with the classical considerations in [28, 29] where the maximum lock-in frequency for a PLL with triangular wave output PD is less than or equal to  $K_L\pi/2$ .



**Figure 10.** Frequency synthesizer’s responses for the second example with  $K_L$  obtained via Corollary 1.

### 5. Conclusion

This work focuses on the analysis of one type of hybrid phase-controlled systems, namely PLLs with XOR gate phase detector. The z-domain model of the system is chosen due to its advantage of having frequency response closer to the experimental results, and the nonlinear characteristic of the phase detector is taken into account in order to get a more accurate analysis. We propose the algorithms in the form of convex LMI searches where the stability is guaranteed via a discrete-time Lur’e–Postnikov Lyapunov function, and the performance is evaluated based on the induced  $\ell_2$  norm objective. The result is also extended for analysis of PLL-based

frequency synthesizers. From the simulation results, it is clearly shown how the proposed method can be used to predict the loop gain that can stabilize the system, which is also related to the range of frequencies where the system can be driven to the locked state. A comparison is also made with the classical Tsypkin criterion and s-domain approach, where the two methods result in poor performance and instability for the cases under consideration.

### Acknowledgment

We would like to thank the Malaysia MoE for financial support under the FRGS Scheme (203/PELECT/6071347).

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