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Approximation of planar curves

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Abstract: In the present article, we have developed the G^2-approximation scheme for planar curves arising in science, engineering, computer-aided design, computer-aided manufacturing, and many other fields. The obtained results reveal that the proposed method is a significant addition to the approximation of planar curves. The method is illustrated using different numerical examples. The smaller absolute error confirms the applicability and efficiency of the proposed method.

Key words: Parabolic and elliptic arcs, G^2-constraints, parametric rational cubic function, absolute error

1. Introduction

A wide class of planar curves (parabola and ellipse) arise in various branches of pure and applied sciences, including astrophysics, medical imaging, structural engineering, biomedical engineering, chemical industry, wave propagation, objects motion, and optimization. Conics are extensively used in shape expression, mechanical accessories (tube benders, cutters, wrenches, clamp systems, inspection gauges), design of aircrafts, car headlights, rocket satellites, construction of roller coaster, suspension bridges, outline of fonts, and CAD/CAM systems [1]. In particular, ellipses are used in modern medicine where the reflection property is the basis for lithotripsy. Lithotripsy is a very useful medical treatment for kidney and gall stones without open surgery. The risks associated with this surgery are comparatively small. A parabolic dish (parabolic reflector) is just like the shape of a parabola, which is used to direct light or sound waves.

Approximation of planar curves is of great interest in CAD due to the inconsistency of parametric equations of conics with CAD. A lot of work has been done in the past few years in this regard. Some noticeable contributions include [2–16]. The existing approximation schemes have focused on the approximation of rational quadratic Bézier curves, which represent conic sections. The control points and weights of the approximating curves were defined in terms of a rational quadratic Bézier curve. The constraints on the weight functions of the approximating curves yielded a family of approximating curves for a given planar curve. Thus, the approximating curve was not unique and involved geometrical and computational complexity.

In this research paper a novel G^2-approximation scheme is presented for parabolic and elliptic arcs using a parametric rational cubic function. The control points of the parametric rational cubic function are calculated using G^2-approximation constraints. The proposed geometric approximation scheme is based on end tangents and curvatures of planar curves (parabolic and elliptic arcs), and the optimal values of free parameters are

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determined by optimization techniques. To the best of our knowledge, no similar work is found in the literature. The maximum absolute error for the developed approximation scheme is less than that of the prevailing schemes. The numerical experiments suggest the simplicity, feasibility, and efficiency of the presented scheme. A detailed comparison to the existing schemes is as follows:

- The existing schemes approximate conics in terms of control points and weights of the rational quadratic Bézier curve [2,6–9,15,16]. The proposed G^2-approximation scheme of this research paper is based on end tangents and curvatures of planar curves (parabola and ellipse). Therefore, it does not need the rational quadratic Bézier representation of planar curves and it is robust and simpler than the prevailing schemes.

- The existing approximation schemes of conics [2,6,7,8,15] provide a family of approximating curves of a given conic section due to the constraints on weight functions. In this research paper, the optimal value of the free parameter is calculated by an optimization technique. It provides the unique and optimal approximation of parabolic and elliptic arcs.

- It is clear from the Table that the maximum absolute error of the proposed G^2-approximation scheme for parabolic and elliptic arcs is less than that of the prevailing schemes [2,6,8,9,11]. Hence, the proposed approximation scheme of this research paper is more effective than prevailing schemes.

Table. Comparison of absolute errors of the proposed approximation schemes (parabolic and elliptic arcs) with the existing approximations schemes.

Approximating schemes	[2]	[7]	[8]	[11]	Proposed approximation scheme
Maximum absolute errors	2.8737×10^{-1}	1.4719×10^{-1}	1.3638×10^{-3}	4.5982×10^{-4}	3.4510×10^{-12}
Approximating schemes	[2]	Proposed approximation scheme			
Maximum absolute errors	4.3990×10^{-3}	2.8×10^{-3}			

- In [15], a rational cubic Bézier representation for conics was presented with the help of weights and vertices of the rational quadratic Bézier curve. However, in our proposed scheme, G^2-approximation is used to approximate parabolic and elliptic arcs by a rational cubic parametric function with two free parameters. The G^2-approximation scheme proposed in this research paper is more efficient than [15] as it provides unique and optimal approximation. Unlike [15], it does not constrain the geometry of vertices and does not require its rational quadratic Bézier representation. The parabolic and elliptic arcs are approximated in the first quadrant and then affine transformation is applied to obtain the complete parabola and ellipse. Therefore, the proposed scheme is simpler than [15].

The paper is organized as follows: in Section 2, approximation schemes for planar curves by parametric rational cubic function are introduced. In Section 3, the proposed schemes are demonstrated with the help of numerical examples, and concluding remarks are given in Section 4.

2. Approximation of planar curves by parametric rational cubic function

The parametric rational cubic function (PRCF) is given by:

$$p\,(t) = \sum_{i=0}^{3} B_i(t)p_i,\ t \in [0,1]\ p_i \in R^2 \tag{1}$$

Here, $B_i\,(t) = \frac{(1-t)^{3-i}t^i}{q(t)}$, $q\,(t) = \alpha + \beta\,(1-t)\,t$ $p_i^{'}s$ are the 2D control points and α, β are the positive real free parameters. For G^2-approximation of conics (parabolic and elliptic arcs) by the PRCF of Eq. (1), the following G^2-constraints are used:

$$p\,(0) = c_0\,, p\,(1) = c_1 \tag{2}$$

$$T_0 = t_0\,, T_1 = t_1 \tag{3}$$

$$\kappa_p\,(0) = \kappa_0\,\kappa_p\,(1) = \kappa_1 \tag{4}$$

Here, $p\,(0)$ and $p\,(1)$ are the end points, T_0 and T_1 are the end unit tangents, and $\kappa_p\,(0)$ and $\kappa_p\,(1)$ are the end curvatures of PRCF Eq. (1). c_k, $k = 0, 1$, are the initial and final points of the concerned conic and t_k, $k = 0, 1$, are its unit tangents at c_0 and c_1, respectively. Curvature of the concerned conic at c_0 and c_1 is denoted by κ_0 and κ_1, respectively. A simple calculation yields the following values of end points and end unit tangents of PRCF Eq. (1):

$$p\,(0) = \frac{p_0}{\alpha},\ p\,(1) = \frac{p_3}{\alpha}, T_0 = \frac{p^{'}\,(0)}{\|p^{'}\,(0)\|} = \frac{\alpha p_1 - (3\alpha+\beta)\,p_0}{g_1}\,, T_1 = \frac{p^{'}\,(1)}{\|p^{'}\,(1)\|} = \frac{(3\alpha+\beta)\,p_3 - \alpha p_2}{g_2} \tag{5}$$

Here, $g_1 = \|\alpha p_1 - (3\alpha+\beta)\,p_0\|$, $g_2 = \|(3\alpha+\beta)\,p_3 - \alpha p_2\|$. As $g_i^{\prime}s$ depend upon α and β these are also exploited as positive real free parameters in the construction of approximation schemes. Substituting the values from Eqs. (2) and (3) in Eq. (5), the values of control points of Eq. (1) are calculated, which are given in Eq. (6).

$$p_0 = \alpha c_0\,,\quad p_3 = \alpha c_1\,,\quad \alpha p_1 - (3\alpha+\beta)\,p_0 = g_1 t_0\,,\quad (3\alpha+\beta)\,p_3 - \alpha p_2 = g_2 t_1 \tag{6}$$

The scalar form of Eq. (1) was used in [17] for the shape-preservation of 2D data.

2.1. Parametric rational cubic approximation of parabolic arcs

This section presents a parametric rational cubic approximation scheme to approximate parabolic arcs. As a parabola is symmetric about the coordinate axes, a complete parabola is obtained through reflection transformations.

First, a numerical scheme is constructed to approximate the horizontal parabolic arc by the parametric rational cubic function given in Eq. (1). Suppose that the parabolic arc is part of the horizontal parabola $y^2 = 4ax$ with axis along x-axis, vertex at $(0,0)$, and focus at $(a,0)$ with $a > 0$. Any point S of this parabola has parametric representation $S\left(a\theta^2, 2a\theta\right)$ where θ is the positive real parameter. Take c_0 at origin and c_1 as an arbitrary point of parabola $y^2 = 4ax$ The hypothesis provides the following end points (c_0, c_1) end unit tangents (t_0, t_1), and the curvatures (κ_0, κ_1) of the parabolic arc c_0c_1:

$$c_0 = (0,0)\,,\quad c_1 = \left(a\theta^2, 2a\theta\right),\quad t_0 = (0,1)\,,\quad t_1 = \left(\frac{\theta}{\rho_1}, \frac{1}{\rho_1}\right),\quad \kappa_0 = -\frac{1}{2a}\quad \kappa_1 = -\frac{1}{2a\rho_1^3}\quad \rho_1 = \sqrt{\theta^2+1}$$

Substituting the above values into Eq. (6), the computed control points of PRCF Eq. (1) are written in Eq. (7).

$$p_0 = (x_0, y_0)\,, \quad p_1 = (x_1, y_1)\,, \quad p_2 = (x_2, y_2)\,, \quad p_3 = (x_3, y_3)\,, \tag{7}$$

$$x_0 = 0,\ y_0 = 0, \quad x_1 = 0,\ y_1 = \frac{g_1}{\alpha}, \quad x_2 = a\,(3\alpha + \beta)\,\theta^2 - \frac{g_2\theta}{\alpha\rho_1},$$

$$y_2 = 2a\,(3\alpha + \beta)\,\theta - \frac{g_2}{\alpha\rho_1}, \quad x_3 = \alpha a\theta^2,\ y_3 = 2a\alpha\theta.$$

Substituting the values of control points $p_i = (x_i, y_i)$ $i = 0, 1, 2, 3,$ from Eq. (7) into Eq. (1), the following parametric equations of Eq. (1) are obtained:

$$x\,(t) = \sum_{i=0}^{3} B_i(t)x_i,\ y\,(t) = \sum_{i=0}^{3} B_i(t)y_i \tag{8}$$

Curvature, $\kappa_p\,(t)$ of PRCF Eq. (1) at the end points of the domain of parameter $t \in [0, 1]$, is given by:

$$\kappa_p\,(0) = \frac{2\alpha^2}{g_1^2}\left(\frac{g_2\theta}{\rho_1} - a\alpha\,(3\alpha + \beta)\,\theta^2\right), \quad \kappa_p\,(1) = \frac{2\alpha^2}{g_2^2\rho_1}\left(g_1\theta - a\alpha\,(3\alpha + \beta)\,\theta^2\right)$$

The values of curvature $\kappa_p\,(t)$ are calculated by substituting the values of $x\,(t)$ and $y\,(t)$ from Eq. (8) in curvature formula $\kappa_p\,(t) = \left(\frac{dx}{dt}\cdot\frac{d^2y}{dt^2} - \frac{dy}{dt}\cdot\frac{d^2x}{dt^2}\right)\cdot\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)^{-\frac{3}{2}}$. By putting these values of curvature in Eq. (4), we have:

$$\rho_1 g_1^2 = -4a\alpha^2\left(g_2\theta - a\alpha\rho_1\,(3\alpha + \beta)\,\theta^2\right) \tag{9}$$

$$g_2^2 = -4a\alpha^2\rho_1^2\,(g_1\theta - a\alpha\,(3\alpha + \beta))\,\theta^2 \tag{10}$$

The set of Eqs. (9) and (10) has two solutions, $g_2 = \rho_1 g_1$ or $g_2 = 4a\alpha^2\rho_1\theta - \rho_1 g_1$ leading to the following two cases:

Case 1. If $g_2 = \rho_1 g_1$ then Eq. (10) can be written as follows:

$$g_1^2 + 4a\alpha^2\theta g_1 - 4\alpha^3 a^2\,(3\alpha + \beta)\,\theta^2 = 0. \tag{11}$$

Solutions of quadratic Eq. (11) are $g_1 = -2a\alpha^2\theta \pm 2\sqrt{(2a\alpha^2\theta)^2 + \alpha^3 a^2\theta^2\beta}$. As α, β, θ, and a are positive real entities, $g_1 = -2a\alpha^2\theta - 2\sqrt{(2a\alpha^2\theta)^2 + \alpha^3 a^2\theta^2\beta}$ gives a negative value of g_1. A negative value of g_1 is not acceptable as g_1 is a positive real unknown parameter. Therefore, $g_1 = -2a\alpha^2\theta + 2\sqrt{(2a\alpha^2\theta)^2 + \alpha^3 a^2\theta^2\beta}$ is the only acceptable value.

Case 2. If $g_2 = 4a\alpha^2\rho_1\theta - \rho_1 g_1$ then Eq. (10) can be rewritten as follows:

$$g_1^2 - 4a\alpha^2\theta g_1 + \left(4\alpha^4 a^2\theta^2 - 4\alpha^3 a^2\theta^2\beta\right) = 0. \tag{12}$$

Solutions of quadratic Eq. (12) are $g_1 = 2a\alpha^2\theta \pm 2\sqrt{\alpha^3 a^2\theta^2\beta}$. Recollecting that α, β, θ, and a are positive real entities, $g_1 = 2a\alpha^2\theta + 2\sqrt{\alpha^3 a^2\theta^2\beta}$ is therefore always positive but $g_1 = 2a\alpha^2\theta 2 - \sqrt{\alpha^3 a^2\theta^2\beta}$ is positive only for $\alpha > \beta$. As g_1 is a positive real unknown parameter, therefore the universally acceptable value of g_1 is $g_1 = 2a\alpha^2\theta + 2\sqrt{\alpha^3 a^2\theta^2\beta}$. Thus, the two solutions to the simultaneous Eqs. (9) and (10) are the following:

$$g_1 = -2a\alpha^2\theta + 2\sqrt{\left(2a\alpha^2\theta\right)^2 + \alpha^3 a^2\theta^2\beta} \text{ and } g_2 = \rho_1 g_1 \tag{13}$$

$$g_1 = 2a\alpha^2\theta + 2\sqrt{\alpha^3 a^2\theta^2\beta} \text{ and } g_2 = 4a\alpha^2\rho_1\theta - \rho_1 g_1 \tag{14}$$

By substituting the values of g_1 and g_2 from either Eq. (13) or (14) in Eq. (7), two sets of control points of PRCF Eq. (1) for the approximation of the horizontal parabolic arc are obtained. Now, by putting these values of control points in Eq. (8), two sets of parametric equations, $x(t)$ and $y\,(t)$ for the approximation of horizontal parabolic arc ($y^2 = 4ax$) are obtained. The absolute error function of the developed approximation scheme is $\gamma_1\,(\alpha,\beta,t) = \left|y^2(t) - 4ax(t)\right|$. To obtain the optimal approximation of the horizontal parabolic arc, the values of free parameters α and β are obtained by Optimization problem-I.

Optimization problem-I: Minimize (maximum $\gamma_1\,(\alpha,\beta,t) = \left|y^2(t) - 4ax(t)\right|$)

$$\text{subject to } \alpha \geq u,\ \beta \geq u, \text{ where, } u = 2.2204 \times 10^{-16}.$$

Here $u = 2.2204 \times 10^{-16}$ is a MATLAB special variable epsilon. It is the smallest difference between two values that can be represented by MATLAB. The parametric equations $x(t)$ and $y\,(t)$ are already defined in Eq. (8).

Theorem 1 *If the control points of the parametric rational cubic function in Eq. (1) are* $p_0 = (x_0, y_0), p_1 = (x_1, y_1), p_2 = (x_2, y_2), p_3 = (x_3, y_3), x_0 = 0,\ y_0 = 0, x_1 = 0,\ y_1 = \frac{g_1}{\alpha}, x_2 = a\,(3\alpha+\beta)\,\theta^2 - \frac{g_2\theta}{\alpha\rho_1},\ y_2 = 2a\,(3\alpha+\beta)\,\theta - \frac{g_2}{\alpha\rho_1}, x_3 = \alpha a\theta^2,\ y_3 = 2a\alpha\theta$ *and* $\rho_1 = \sqrt{\theta^2+1}$; g_1 *and* g_2 *are obtained from either Eq. (13) or (14); and the free parameters* $\alpha, \beta \in (0,\infty)$ *are obtained from Optimization problem-I, then PRCF Eq. (1) approximates the horizontal parabolic arc* c_0c_1.

In a similar fashion, we can approximate an arc of a vertical parabola. The equation of vertical parabola is $x^2 = 4ay$ with axis along the y-axis having vertex at $(0,0)$, focus at $(0,a), a > 0$. Consider an arc c_0c_1 of the vertical parabola where $c_0\,(0,0)$ and $c_1\left(2a\theta, a\theta^2\right)$. Here, θ is a positive real parameter. Following the same steps as detailed above for approximation of a horizontal parabolic arc, the G^2-approximation scheme is developed to approximate the vertical parabolic arc by PRCF Eq. (1). It is summarized in Theorem 2.

Theorem 2 *If the control points of the parametric rational cubic function of Eq. (1) are* $p_0 = (x_0, y_0), p_1 = (x_1, y_1), p_2 = (x_2, y_2)\, p_3 = (x_3, y_3), x_0 = 0,\ y_0 = 0, x_1 = \frac{g_1}{\alpha},\ y_1 = 0, x_2 = 2a\,(3\alpha+\beta)\,\theta - \frac{g_2}{\alpha\rho_2},\ y_2 = a\,(3\alpha+\beta)\,\theta^2 - \frac{g_2\theta}{\alpha\rho_2}, x_3 = 2a\alpha, \theta,\ y_3 = a\alpha\theta^2$, and $\rho_2 = \sqrt{\theta^2+1}$; *the values of* g_1 *and* g_2 *are obtained from either Eq. (13) or (14); and the free parameters,* $\alpha, \beta \in (0,\infty)$ *are obtained from Optimization problem-II, then PRCF Eq. (1) approximates the vertical parabolic arc* c_oc_1.

Optimization problem-II: Minimize (maximum $\gamma_2\,(\alpha,\beta,t) = \left|x^2(t) - 4ay(t)\right|$)

$$\text{subject to } \alpha \geq u,\ \beta \geq u, \text{ where, } u = 2.2204 \times 10^{-16}.$$

Here,

$$x(t) = B_0(t)\,x_0 + B_1(t)\,x_1 + B_2(t)\,x_2 + B_3(t)\,x_3$$

$$y(t) = B_0(t)\,y_0 + B_1(t)\,y_1 + B_2(t)\,y_2 + B_3(t)\,y_3$$

$$x_0 = 0,\ y_0 = 0, \quad x_1 = \frac{g_1}{\alpha},\ y_1 = 0, \quad x_2 = 2a(3\alpha+\beta)\,\theta - \frac{g_2}{\alpha\rho_2},\ y_2 = a(3\alpha+\beta)\,\theta^2 - \frac{g_2\theta}{\alpha\rho_2}, \quad x_3 = 2a\alpha\theta,\ y_3 = a\alpha\theta^2,$$

and $\rho_2 = \sqrt{\theta^2+1}$.

Remark 1 *The presented approximation scheme is useful for the approximation of all parabolic shapes (oblique parabolas) using affine transformations.*

2.2. Parametric rational cubic approximation of elliptic arcs

In this section, a parametric rational cubic approximation scheme is introduced to approximate an elliptic arc by using the parametric rational cubic function (PRCF) defined in Eq. (1). We can obtain a complete ellipse after applying affine transformations. Suppose that the concerned elliptic arc is part of the horizontal ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b > 0$, with major axis along the x-axis, center at origin, and focus at $(c, 0)$ where $c = \sqrt{a^2 - b^2}$. Any point S of the ellipse has representation $S(a\cos\theta, b\sin\theta)$, where θ is the angle that OS makes with the positive x-axis. Simplification suggests to choose $\theta \in \left[0, \frac{\pi}{2}\right]$. We take the initial point c_0 of the elliptic arc c_0c_1 on the horizontal x-axis and tangent t_0 as vertical tangent. Thus, the end points, end unit tangents, and end curvatures are given by:

$$c_0 = (a, 0), \quad c_1 = (a\cos\theta, b\sin\theta), \quad t_0 = (0, 1), \quad t_1 = \left(\frac{-a\sin\theta}{\rho_3}, \frac{b\cos\theta}{\rho_3}\right),$$

$$\kappa_0 = \frac{a}{b^2},\ \kappa_1 = \frac{ab}{\rho_3^3}, \quad \rho_3 = \sqrt{a^2 sin^2\theta + b^2 cos^2\theta}.$$

Substituting the values of end points and end unit tangents in Eq. (6), the following control points of the parametric rational cubic function Eq. (1) approximating the horizontal elliptic arc c_0c_1 are obtained.

$$p_0 = (x_0, y_0), \quad p_1 = (x_1, y_1), \quad p_2 = (x_2, y_2), \quad p_3 = (x_3, y_3), \tag{15}$$

$$x_0 = \alpha a,\ y_0 = 0, \quad x_1 = a(3\alpha+\beta),\ y_1 = \frac{g_1}{\alpha}, \quad x_2 = a(3\alpha+\beta)\cos\theta + \frac{ag_2\sin\theta}{\alpha\rho_3},$$

$$y_2 = b(3\alpha+\beta)\sin\theta - \frac{bg_2\cos\theta}{\alpha\rho_3}, \quad x_3 = \alpha a\cos\theta,\ y_3 = \alpha b\sin\theta.$$

Substituting the values of control points $p_i = (x_i, y_i),\ i = 0, 1, 2, 3,$ from Eq. (15) into Eq. (1), the following parametric equations of Eq. (1) are obtained:

$$x(t) = \frac{\sum_{i=0}^{3} (1-t)^{3-i}\, t^i x_i}{\alpha + \beta(1-t)\,t}, \quad y(t) = \frac{\sum_{i=0}^{3} (1-t)^{3-i}\, t^i y_i}{\alpha + \beta(1-t)\,t} \tag{16}$$

The curvatures $\kappa_p(t)$ of the PRCF at the end points of the domain of the parameter $t \in [0,1]$ are given by:

$$\kappa_p(0) = \frac{2a\alpha^2}{g_1^2}\left(\alpha(3\alpha+\beta)(1-\cos\theta) - \frac{g_2\sin\theta}{\rho_3}\right) \tag{17}$$

$$\kappa_p(1) = \frac{2a\alpha^2}{g_2^2\rho_3}\left(b\alpha(3\alpha+\beta)(1-\cos\theta) - g_1\sin\theta\right) \tag{18}$$

By putting the values of curvatures in Eq. (4), we have

$$\rho_3 g_1^2 = 2\alpha^2 b^2\left(\alpha\rho_3(3\alpha+\beta)(1-\cos\theta) - g_2\sin\theta\right) \tag{19}$$

$$b g_2^2 = 2\alpha^2\rho_3^2\left(b\alpha(3\alpha+\beta)(1-\cos\theta) - g_1\sin\theta\right) \tag{20}$$

Solutions of simultaneous Eqs. (19) and (20) are $bg_2 = \rho_3 g_1$ or $bg_2 + \rho_3 g_1 = 2b\alpha^2\rho_3\sin\theta$ which leads to the following two cases.

Case 1. If $g_2 = \frac{\rho_3 g_1}{b}$ then Eq. (20) can be written as follows:

$$g_1 = -\alpha^2 b\sin\theta \pm g_{1,0}, \quad g_{1,0} = \sqrt{h},\ h. \tag{21}$$

Solutions of quadratic Eq. (21) are $g_1 = -\alpha^2 b\sin\theta \pm g_{1,0} \pm g_{1,0} = \sqrt{h}, h = \left(\alpha^2 b\sin\theta\right)^2 + 2\alpha^3 b^2(3\alpha+\beta)(1-\cos\theta)$ As $\alpha, \beta, a,\ b$ are positive real entities and $\theta \in \left[0, \frac{\pi}{2}\right]$, $g_1 = -\alpha^2 b\sin\theta - g_{1,0}$ is always negative, but by the definition of g_1 it is a positive real parameter. Therefore, the only acceptable value of g_1 is given in Eq. (22).

$$g_1 = -\alpha^2 b\sin\theta + \sqrt{\left(\alpha^2 b\sin\theta\right)^2 + 2\alpha^3 b^2(3\alpha+\beta)(1-\cos\theta)} \tag{22}$$

Case 2. If $g_2 = 2\alpha^2\rho_3\sin\theta - \frac{\rho_3 g_1}{b}$ then Eq. (20) can be rewritten as follows:

$$g_1^2 - 2b\alpha^2\sin\theta g_1 + \left(4\alpha^4 b^2 sin^2\theta - 2\alpha^3 b^2(3\alpha+\beta)(1-\cos\theta)\right) = 0 \tag{23}$$

Solutions of quadratic Eq. (23) are $g_1 = \alpha^2 b\sin\theta \pm g_{1,1}, g_{1,1} = \sqrt{\tilde{h}}, \tilde{h} = -3\alpha^4 b^2\sin^2\theta + 2\alpha^3 b^2(3\alpha+\beta)(1-\cos\theta)$. These solutions become imaginary for different choices of α, β, a, b , and θ. Therefore, the acceptable solution to simultaneous Eqs. (19) and (20) is given in Eq. (24):

$$g_1 = -\alpha^2 b\sin\theta + \sqrt{\left(\alpha^2 b\sin\theta\right)^2 + 2\alpha^3 b^2(3\alpha+\beta)(1-\cos\theta)}, g_2 = \frac{\rho_3 g_1}{b} \tag{24}$$

Substituting the values of g_1 and g_2 from Eq. (24) in Eq. (15), we get a unique set of control points of PRCF Eq. (1) approximating the horizontal elliptic arc. Putting these values of control points in Eq. (16), a unique set of parametric equations, $x(t)$ and $y(t)$ for the approximation of the horizontal elliptic arc is obtained. The absolute error function of the concerned approximation scheme is $\gamma_3(\alpha,\beta,t) = \left|b^2x^2(t) + a^2y^2(t) - a^2b^2\right|$. To obtain the optimal approximation of the horizontal elliptic arc, the values of free parameters α and β are obtained by Optimization problem-III.

Optimization problem-III:

$$\text{Minimize (maximum } \gamma_3(\alpha,\beta,t) = \left| b^2x^2(t) + a^2y^2(t) - a^2b^2 \right|)$$

$$\text{subject to } \alpha \geq u,\ \beta \geq u, \text{ where, } u = 2.2204 \times 10^{-16}$$

Here, $x(t)$ and $y(t)$ are already defined in Eq. (16).

Theorem 3 *If the control points of the parametric rational cubic function of Eq. (1) approximating the horizontal elliptic arc are* $p_0 = (x_0, y_0)$, $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$, $p_3 = (x_3, y_3)$, $x_0 = \alpha a$, $y_0 = 0, x_1 = a(3\alpha+\beta)$, $y_1 = \frac{g_1}{\alpha}, x_2 = a(3\alpha+\beta)\cos\theta + \frac{a g_2 \sin\theta}{\alpha\rho_3}, y_2 = b(3\alpha+\beta)\sin\theta - \frac{b g_2 \cos\theta}{\alpha\rho_3}$, $x_3 = \alpha a \cos\theta$, $y_3 = \alpha b \sin\theta, g_1 = -\alpha^2 b\sin\theta +\ g_{10}$, $g_{10} = \sqrt{(\alpha^2 b \sin\theta)^2 + 2\alpha^3 b^2 (3\alpha+\beta)(1-\cos\theta)}$, $g_2 = \frac{\rho_3 g_1}{b}$,
$\rho_3 = \sqrt{a^2 sin^2\theta + b^2 cos^2\theta}$, $0 \leq \theta \leq \frac{\pi}{2}$ *and the free parameters* $\alpha,\beta \in (0,\infty)$ *are obtained from Optimization problem-III, then PRCF Eq. (1) approximates the horizontal elliptic arc* $c_o c_1$.

In a similar way, we can approximate an arc of a vertical ellipse. Suppose that the concerned arc is a part of the vertical ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ b > a > 0$, with major axis along the y-axis, center at origin, and focus at $(0,c)$ where $c = \sqrt{b^2 - a^2}$. The computed control points of PRCF Eq. (1) approximating the vertical elliptic arc are the same as given in Eq. (15). Therefore, Theorem 3 can be used to approximate the vertical elliptic arc $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ b > a$

Remark 2 *The proposed approximation scheme can also be used to get oblique ellipses using affine transformations.*

Remark 3 *In Optimization problem-I, -II, and -III, the values of* α *and* β *are determined by minimizing the maximum value of absolute error* $(\gamma_2, \gamma_2, \gamma_3)$ *using MATLAB's built-in function 'fminimax.'*

The implementation of the proposed approximation schemes for parabolic and elliptic arcs by the parametric rational cubic function of Eq. (1) is outlined in the following algorithm.

Algorithm

Step 1. Input the values of a and parameter θ for the parabolic arc and the values of a, b, and θ for the elliptic arc.
Step 2. Calculate the values of g_1 and g_2.
Step 3. Compute the control points of the PRCF of Eq. (1) by putting the values of g_1 and g_2.
Step 4. Calculate the parametric equations of the PRCF in Eq. (1).
Step 5. Use the concerned optimization problem to obtain the optimal values of α and β.
Step 6. Put the optimal values of α and β to obtain parametric rational cubic approximation of the parabolic and elliptic arcs.

2.3. Approximation order of parabolic and elliptic aγcs' approximation schemes

Here, we will present the approximation order of parabolic and elliptic aγcs' approximation schemes.

Theorem 4 *If* $\alpha \in (0,\infty)$, *and* $t \in [0,1]$ *then*

$$\gamma_1(\alpha,\beta,t) = \begin{cases} 0, & \theta \in [0,1],\ \beta \to 0 \\ O(\theta^2), & \theta \in [0,1],\ \beta \to \infty \end{cases}.$$

Proof The absolute error of the parabolic arc approximation scheme in Optimization problem-I is the following:

$$\gamma_1(\alpha,\beta,t) = \left|y^2(t) - 4ax(t)\right| \tag{25}$$

where $\alpha,\beta \in (0,\infty), t \in [0,1]$. Here, $x(t) = \sum_{i=0}^{3} B_i(t)x_i$ and $y(t) = \sum_{i=0}^{3} B_i(t)y_i$ with $x_0 = 0,\ y_0 = 0, x_1 = 0,\ y_1 = \frac{g_1}{\alpha}, x_2 = a(3\alpha+\beta)\theta^2 - \frac{g_2\theta}{\alpha\rho_1},\ y_2 = 2a(3\alpha+\beta)\theta - \frac{g_2}{\alpha\rho_1},\ x_3 = \alpha a\theta^2,\ y_3 = 2a\alpha\theta, \rho_1 = \sqrt{\theta^2+1}$. Without loss of generality, we shall consider only positive values of $\gamma_1(\alpha,\beta,t)$ to calculate the approximation order of the concerned scheme, i.e.

$$\gamma_1(\alpha,\beta,t) = y^2(t) - 4ax(t)$$

As the maximum and minimum values of $\gamma_1(\alpha,\beta,t)$ lie at the end points of the domain of parameters α,β, and t and at the critical points of the function $\gamma_1(\alpha,\beta,t)$, therefore, to find out the approximation order of the horizontal parabolic arc approximation scheme, $\gamma_1(\alpha,\beta,t)$ is evaluated at these points. First, the behavior of $\gamma_1(\alpha,\beta,t)$ at the extremes of interval $\beta \in (0,\infty)$ is observed.

(i) When $\beta \to 0$

$$U_1(t) = \lim_{\beta\to 0}\gamma_1(\alpha,\beta,t) = 4a^2\theta^2\left[t^6 + (1-t)^4t^2 - t^2\right] + 24a^2\theta^2(1-t)^2t^4 + 16a^2\theta^2\left[(1-t)^3t^3 + (1-t)t^5\right] \tag{26}$$

Here, $U_1(t)$ is independent of α. Moreover, the function $U_1(t)$ is zero at its critical points, and at the extreme points of its domain $t \in [0,1]$. It follows from the above observation that

$$\max_{0\le t\le 1,\beta\to 0}\gamma_1(\alpha,\beta,t) = \max_{0\le t\le 1}U_1(t) = 0. \tag{27}$$

(ii) When $\beta \to \infty$

$$\lim_{\beta\to\infty}\gamma_1(\alpha,\beta,t) = U_2(t) = 4a^2\theta^2\left(t^2 - t\right)$$

$U_2(t)$ is independent of α, so its extreme values are only dependent upon $t \in [0,1]$ It can be seen that $U_2(0) = 0$ and $U_2(1) = 1$. Differentiating $U_2(t)$ with respect to t, we have $\frac{dU_2}{dt} = 4a^2\theta^2(2t-1)$ It is observed that $\frac{dU_2}{dt} = 0$ for $t = 0.5$ and $\frac{dU_2}{dt} \neq \infty$ for $t \in [0,1]$. It follows from the above observations that

$$\max_{0\le t\le 1,\beta\to\infty}\gamma_1(\alpha,\beta,t) = U_2(t) = \max_{0\le t\le 1}U_2(0.5) = r_1(\theta) \tag{28}$$

Here,

$$r_1(\theta) = -a^2\theta^2 \tag{29}$$

(iii) When $\frac{\partial \gamma_1(\alpha,\beta,t)}{\partial \beta} = 0$ or $\frac{\partial \gamma_1(\alpha,\beta,t)}{\partial \beta} = \infty$.

Differentiating $\gamma_1(\alpha,\beta,t)$ with respect to β, we have $\frac{\partial \gamma_1}{\partial \beta} = \frac{S_2(\alpha,\beta,t)}{(\alpha+\beta(1-t)t)^3}$ where $S_2(\alpha,\beta,t) = (\alpha+\beta(1-t)t)\frac{\partial S_1}{\partial \beta} - 2S_1(1-t)t,\ S_1 = \left(\sum_{i=0}^{3} B_i(t)y_i\right)^2 - 4a(\alpha+\beta(1-t)t)\sum_{i=0}^{3} B_i(t)x_i$. It can be observed that $\frac{\partial \gamma_1}{\partial \beta} = 0$ for negative and imaginary values of β, which is not acceptable as $\beta \in (0,\infty)$. It is clear that $\frac{\partial \gamma_1}{\partial \beta} = \infty$ when $\alpha + \beta(1-t)t = 0$ or $\beta = \frac{-\alpha}{(1-t)t}$. Since $t \in [0,1]$ and $\alpha \in (0,\infty)$, $\beta = \frac{-\alpha}{(1-t)t}$ is either negative or undefined, which is not acceptable as β is a positive real number. Hence, the order of approximation of the parametric rational cubic approximation scheme for the horizontal parabolic arc depends on $r_1(\theta)$, which is given in Eq. (29). The Taylor expansion of $r_1(\theta)$ near $\theta = 0$ is the following:

$$r_1(\theta) = \frac{\theta^2}{2!}\left(-2a^2\right) = O\left(\theta^2\right) \tag{30}$$

This shows that the approximation order of the developed parametric rational cubic approximation scheme for horizontal parabolic arcs is $O\left(\theta^2\right)$. The above discussion can be summarized as follows:

$$\gamma_1(\alpha,\beta,t) = \begin{cases} 0 & \theta \in [0,1],\ \beta \to 0 \\ O(\theta^2), & \theta \in [0,1],\ \beta \to \infty \end{cases}. \tag{31}$$

□

Remark 4 *The approximation order of the developed parametric rational cubic approximation scheme for a vertical parabolic arc is also calculated and it is observed that it is the same as given in Eq. (31).*

Theorem 5 *If* $\alpha \in (0,\infty)$ *and* $t \in [0,1]$ *then*

$$\gamma_3(\alpha,\beta,t) = \begin{cases} -\frac{a^2b^2}{32} + O\left(\theta^2\right), & \theta \in \left[0,\frac{\pi}{2}\right],\ \beta \to 0 \\ \frac{a^2b^2\theta^2}{4} + O\left(\theta^4\right), & \theta \in \left[0,\frac{\pi}{2}\right],\ \beta \to \infty \end{cases}.$$

Proof As discussed in Optimization problem-III, the absolute error of the approximation scheme for a horizontal elliptic arc is given by Eq. (32):

$$\gamma_3(\alpha,\beta,t) = \left|b^2x^2(t) + a^2y^2(t) - a^2b^2\right| \tag{32}$$

where $\alpha,\beta \in (0,\infty),\ t \in [0,1],\ \ \theta \in \left[0,\frac{\pi}{2}\right]$. From Eq. (16), we have

$$x^2(t) = \frac{\tilde{h}_1(t)}{(\alpha+\beta(1-t)t)^2}\ \ y^2(t) = \frac{\tilde{h}_2(t)}{(\alpha+\beta(1-t)t)^2}, \tag{33}$$

where

$$\begin{aligned}\tilde{h}_1(t) =& x_0^2(1-t)^6 + 2x_0x_1(1-t)^5t + \left(x_1^2 + 2x_0x_2\right)(1-t)^4t^2 + (2x_0x_3 + 2x_1x_2) \\ &\times (1-t)^3t^3 + \left(x_2^2 + 2x_1x_3\right)(1-t)^2t^4 + 2x_2x_3(1-t)t^5 + x_3^2t^6, \\ \tilde{h}_2(t) =& y_0^2(1-t)^6 + 2y_0y_1(1-t)^5t + \left(y_1^2 + 2y_0y_2\right)(1-t)^4t^2 + (2y_0y_3 + 2y_1y_2)(1-t)^3t^3 \\ &+ \left(y_2^2 + 2y_1y_3\right)(1-t)^2t^4 + 2y_2y_3(1-t)t^5 + y_3^2t^6.\end{aligned}$$

$$x_0 = \alpha a, \quad x_1 = a\left(3\alpha+\beta\right), \quad x_2 = a\left(3\alpha+\beta\right)\cos\theta + \frac{ag_2\sin\theta}{\alpha\rho_3}, \quad x_3 = \alpha a\cos\theta, \quad y_0 = 0, \quad y_1 = \frac{g_1}{\alpha},$$

$$y_2 = b\left(3\alpha+\beta\right)\sin\theta - \frac{bg_2\cos\theta}{\alpha\rho_3}, \quad y_3 = \alpha b\sin\theta.$$

Without loss of generality we shall consider only the positive value of $\gamma_3\left(\alpha,\beta,t\right)$ to calculate the approximation order of the concerned scheme, i.e. $\gamma_3\left(\alpha,\beta,t\right) = b^2x^2\left(t\right) + a^2y^2\left(t\right) - a^2b^2$. Using Eq. (33), we have $\gamma_3\left(\alpha,\beta,t\right) = \frac{H_1(\alpha,\beta,t)}{(\alpha+\beta(1-t)t)^2} - a^2b^2$ where $H_1\left(\alpha,\beta,t\right) = b^2\tilde{h}_1^2(t) + a^2\tilde{h}_2^2(t)$. The behavior of $\gamma_3\left(\alpha,\beta,t\right)$ is observed at its critical end points with respect to parameters α,β, and t to find the approximation order of the rational cubic approximation scheme for a horizontal elliptic arc.

First, the behavior of $\gamma_3\left(\alpha,\beta,t\right)$ at extremes of the interval $\beta\in(0,\infty)$ is observed.

(iv) When $\beta\to 0$

$$M_1(t) = \lim_{\beta\to 0}\gamma_3\left(\alpha,\beta,t\right) = \sum_{i=0}^{6}\left(1-t\right)^{6-i}t^i m_i - a^2b^2 \tag{34}$$

where $m_0 = a^2b^2$, $m_1 = 6a^2b^2$, $m_2 = 15a^2b^2$, $m_3 = 6a^2b^2\cos\theta - 12a^2b^2sin^2\theta - 4a^2b^2\cos\theta sin^2\theta + 12a^2b^2cos^2\theta + \sqrt{sin^2\theta + 6\left(1-\cos\theta\right)} \times \left(12a^2b^2\sin\theta + 4a^2b^2\sin\theta\cos\theta\right)$, $m_4 = m_2$, $m_5 = m_1$, $m_6 = m_0$.

Hence, $M_1\left(t\right)$ is independent of α. Therefore, the extreme values of $M_1\left(t\right)$ only depend on t. Differentiating $M_1\left(t\right)$ with respect to t we have $\frac{dM_1}{dt} = \left(60a^2b^2 - 3m_3\right)\left(\left(1-t\right)^2t^3 - \left(1-t\right)^3t^2\right)$. It is observed that $\frac{dM_1}{dt} = 0$ for $t=0$, $t=0.5$, and $t=1$. Substituting these values of t in Eq. (34), we have $M_1\left(0\right) = 0$, $M_1\left(0.5\right) = q_1(\theta)$, and $M_1\left(1\right) = 0$, and also $\frac{dM_1}{dt} \neq \infty$ for $t\in[0,1]$. It follows from the above observation that

$$\max_{0\le t\le 1,\beta\to 0}\gamma_3\left(\alpha,\beta,t\right) = \max_{0\le t\le 1} M_1\left(t\right) = M_1\left(0.5\right) = q_1(\theta)$$

Here,

$$q_1\left(\theta\right) = \frac{-5}{16}a^2b^2 + \frac{1}{32}\Big[3a^2b^2\cos\theta - 6a^2b^2sin^2\theta - 2a^2b^2\cos\theta sin^2\theta + 6a^2b^2cos^2\theta + \left(6a^2b^2\sin\theta + 2a^2b^2\sin\theta\cos\theta\right)\sqrt{sin^2\theta + 6\left(1-\cos\theta\right)}\Big]. \tag{35}$$

(v) When $\beta\to\infty$

$$\gamma_3\left(\alpha,\beta,t\right) = M_2\left(t\right) = 2a^2b^2\left(1-\cos\theta\right)\left(t^2-t\right)$$

$M_2\left(t\right)$ is independent of α, so its extreme values are only dependent on $t\in[0,1]$. It can be seen that $M_2\left(0\right) = 0$ and $M_2\left(1\right) = 1$. Differentiating $M_2\left(t\right)$ with respect to t, we have $\frac{dM_2}{dt} = 2a^2b^2\left(1-\cos\theta\right)\left(2t-1\right)$. It is observed that $\frac{dM_2}{dt} = 0$ for $t = 0.5$ and $\frac{dM_2}{dt} \neq \infty$ for $t\in[0,1]$. It follows from the above observation that

$$\max_{0\le t\le 1,\beta\to 0}\gamma_3\left(\alpha,\beta,t\right) = \max_{0\le t\le 1} M_2\left(t\right) = M_2\left(0.5\right) = q_2\left(\theta\right),$$

Here,

$$q_2(\theta) = \frac{a^2b^2}{2}(\cos\theta - 1) \tag{36}$$

(vi) When $\frac{\partial\gamma_3(\alpha,\beta,t)}{\partial\beta} = 0$ or $\frac{\partial\gamma_3(\alpha,\beta,t)}{\partial\beta} = \infty$.

Differentiating $\gamma_3(\alpha,\beta,t)$ with respect to β, we have

$$\frac{\partial\gamma_3}{\partial\beta} = \frac{H_2(\alpha,\beta,t)}{(\alpha+\beta(1-t)t)^3},$$

where

$$H_2(\alpha,\beta,t) = (\alpha+\beta(1-t)t)\frac{dH_1}{d\beta} - 2H_1(1-t)t.$$

It can be observed that $\frac{\partial\gamma_3}{\partial\beta} = 0$ for negative and imaginary values of β which is not acceptable as $\beta \in (0,\infty)$. It is clear that $\frac{\partial\gamma_3}{\partial\beta} = \infty$ when $\alpha+\beta(1-t)t = 0$ or $\beta = \frac{-\alpha}{(1-t)t}$. Since $t \in [0,1]$ and $\alpha \in (0,\infty)$, $\beta = \frac{-\alpha}{(1-t)t}$ is either negative or undefined, which is not acceptable as β is a positive real number. It follows from the above discussion that the order of approximation of the parametric rational cubic horizontal elliptic arc approximation scheme depends on $q_1(\theta)$ and $q_2(\theta)$ already defined in Eqs. (35) and (36). The Taylor expansion of $q_1(\theta)$ near $\theta = 0$ is given in Eq. (37):

$$q_1(\theta) = -\frac{a^2b^2}{32} + \frac{\theta^2}{2!}\left(\frac{a^2b^2}{32}\right) - \frac{\theta^4}{4!}\left(\frac{a^2b^2}{32}\right) - \frac{\theta^6}{6!}\left(\frac{37a^2b^2}{256}\right)\ldots = -\frac{a^2b^2}{32} + O\left(\theta^2\right) \tag{37}$$

The Taylor expansion of $q_2(\theta)$ near $\theta = 0$ is given in Eq. (38):

$$q_2(\theta) = -\frac{\theta^2}{2!}\left(\frac{a^2b^2}{2}\right) + \frac{\theta^4}{4!}\left(\frac{a^2b^2}{2}\right) - \frac{\theta^6}{6!}\left(\frac{a^2b^2}{2}\right) + \ldots = -\frac{a^2b^2\theta^2}{4} + O(\theta^4). \tag{38}$$

Eqs. (37) and (38) show that the approximation order of the developed parametric rational cubic approximation scheme for horizontal elliptic arcs is either $O\left(\theta^2\right)$ or $O\left(\theta^4\right)$ The above discussion can be summarized as follows:

$$\gamma_3(\alpha,\beta,t) = \begin{cases} -\frac{a^2b^2}{32} + O\left(\theta^2\right), & \theta \in \left[0,\frac{\pi}{2}\right], \beta \to 0 \\ \frac{a^2b^2\theta^2}{4} + O(\theta^4), & \theta \in \left[0,\frac{\pi}{2}\right], \beta \to \infty \end{cases}. \tag{39}$$

□

Remark 5 *The approximation order of the developed parametric rational cubic approximation scheme for a vertical elliptic arc is also calculated and it is the same as given in Eq. (39).*

3. Numerical examples

In this section, we shall illustrate the proposed scheme through some numerical examples.

Example 1 *Take horizontal parabolic arc* c_0c_1 *for approximation with end points* $c_0\,(0,0)\,, c_1\,(0.75,3)$ *and* $\theta = 0.5, a = 3, t_0\,(0,1)\,, t_1\,(0.4472, 0.8945)$. *Figure 1 is the plot of a horizontal parabolic arc approximated by the approximation scheme presented in Section 2.1 and the complete horizontal parabola in Figure 2 is obtained by reflection of Figure 1 about the* x*-axis.*

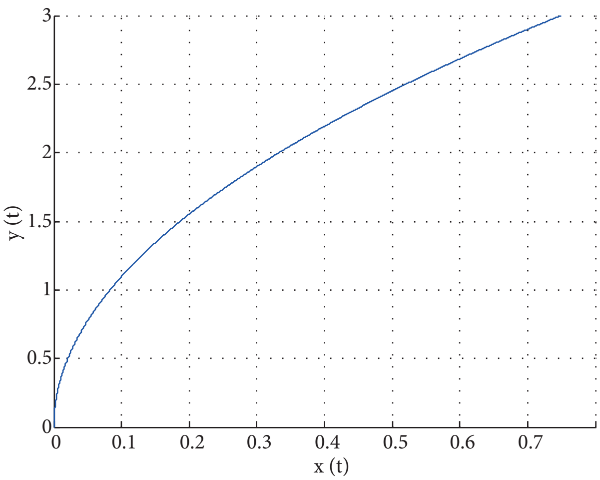

Figure 1. Plot of the approximated horizontal parabolic arc.

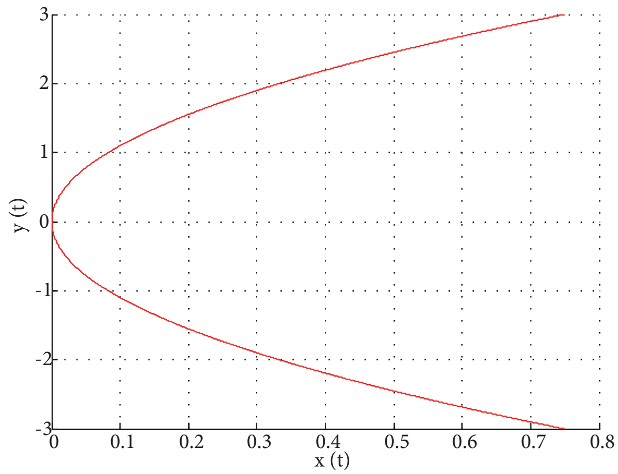

Figure 2. Plot of the horizontal parabola.

Example 2 *Here, a numerical example is presented to illustrate that the proposed parametric rational cubic approximation scheme of parabolic arcs is applicable to oblique parabolas. Consider the oblique parabola* $16x^2 + 24xy + 9y^2 - 5x - 10y + 1 = 0$ *for approximation by PRCF Eq. (1). It is transformed into standard vertical parabola* $X^2 = \frac{1}{5}Y$ *in the* XY*-plane by applying rotation through angle* $\theta = \tan^{-1}\left(\frac{3}{4}\right)$. *The vertical parabola* $X^2 = \frac{1}{5}Y$ *is approximated by the approximation scheme developed in Section 2.1 and its plot is shown in Figure 3. The required oblique parabola in Figure 4 is obtained from Figure 3 by applying inverse rotation.*

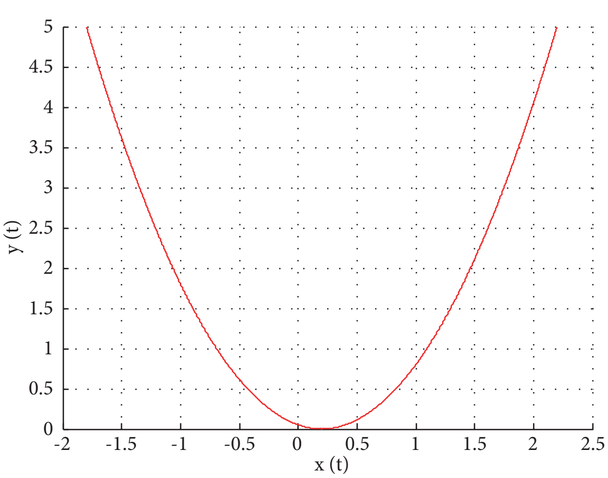

Figure 3. Plot of the approximated vertical parabolic arc.

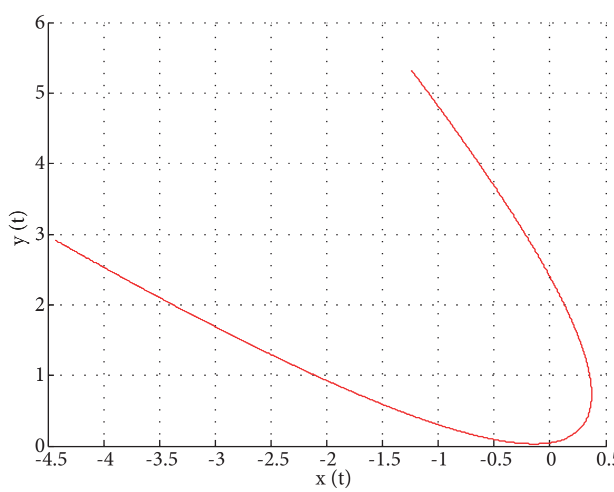

Figure 4. Plot of the oblique parabola.

Example 3 *The oblique ellipse to be approximated is* $29x^2 - 24xy + 36y^2 + 118x - 24y - 55 = 0$*. The oblique ellipse is transformed into standard horizontal ellipse* $\frac{X^2}{9} + \frac{Y^2}{4} = 1$ *in the* XY*-plane with center* $(0,0)$ *by applying rotation through angle* $\theta = \tan^{-1}\left(\frac{3}{4}\right)$*. The horizontal ellipse* $\frac{X^2}{9} + \frac{Y^2}{4} = 1$ *is approximated by the approximation scheme developed in Section 2.2, and its plot is shown in Figure 5. The required oblique ellipse in Figure 6 is obtained from Figure 5 by inverse rotation transformation.*

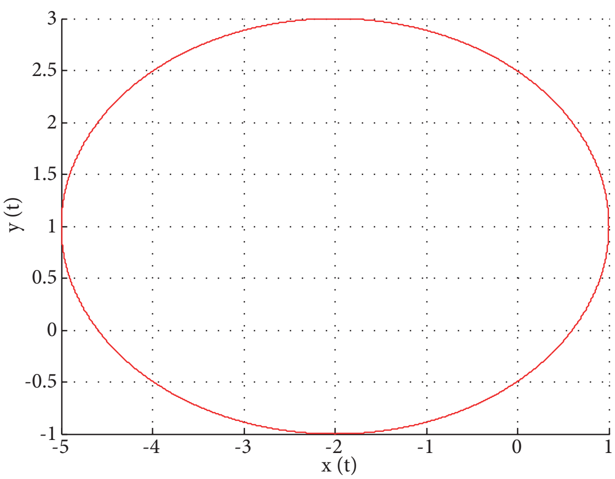

Figure 5. Plot of the approximated horizontal elliptic arc.

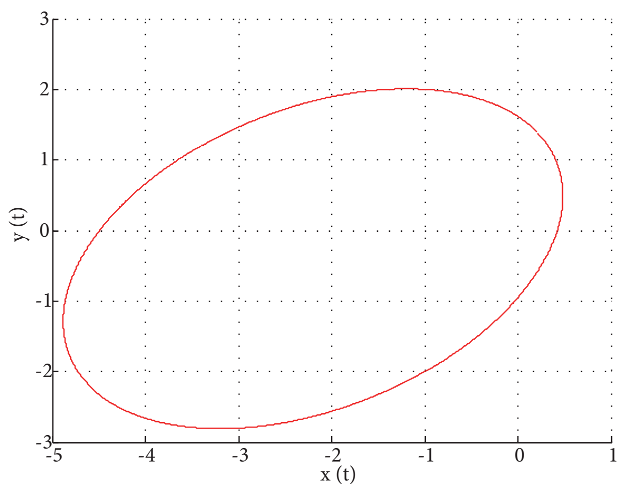

Figure 6. Plot of the oblique ellipse.

Example 4 *Let* $c_0(1,0)$ *and* $c_1(0,3)$ *be the end points of a vertical elliptic arc for approximations. The other initial conditions are* $\theta = \frac{\pi}{2}, a = 1, b = 3, t_0(0,1)$*, and* $t_1(-1,0)$*. Figure 7 shows the plot of the vertical elliptical arc approximated by the approximation scheme presented in Section 2.2. The vertical ellipse in Figure 8 is obtained by reflection of Figure 7 about the* x*-axis.*

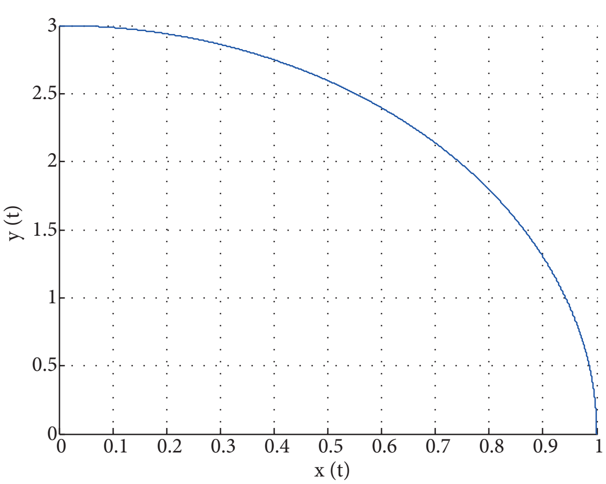

Figure 7. Plot of the approximated vertical elliptic arc.

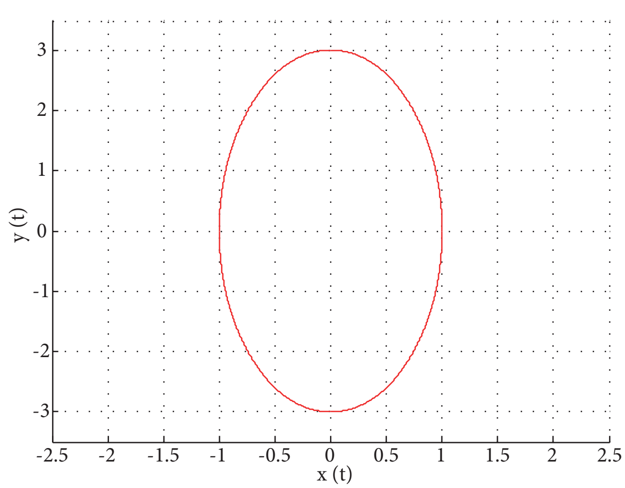

Figure 8. Plot of the vertical ellipse.

4. Conclusion

In this research paper, a dynamic G^2-approximation scheme is developed using PRCF Eq. (1). The rational cubic parametric curve with two free parameters in the control point form is used for G^2-approximation. A unique approximation of a conic is obtained by finding the optimal value of free parameters. The choice of these approximation schemes serves the purpose of favorable approximations with minimized errors. The proposed scheme is simple and efficient.

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