Turkish Journal of Electrical Engineering & Computer Sciences

http://journals.tubitak.gov.tr/elektrik/

Research Article

Turk J Elec Eng & Comp Sci
(2019) 27: 4442 – 4455

doi:10.3906/elk-1807-347

On the output regulation for linear fractional systems

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Received: 31.07.2018 • **Accepted/Published Online:** 27.06.2019 • **Final Version:** 26.11.2019

Abstract: In this work, the regulation problem is extended to the field of fractional-order linear systems considering the Caputo fractional derivative. The regulation equations are obtained on the basis of the Francis equations. It is also shown that the linear fractional regulator exists at $t = 0$ only if the order of the plant is not greater than the order of the reference system.

Key words: Regulation theory, fractional ordinary differential equations, linear systems, tracking of references

1. Introduction

Within control theory, the imposing of reference signals to the outputs of dynamical systems is a very interesting problem because it finds application in many different fields as telecommunications, electronics, physics, chemistry, robotics, aeronautics, just to name a few. Consequently, there exist different ways to deal with it. The regulation theory is an elegant approach which provides the tools to guarantee asymptotic tracking of the desired signals while the rejection of the disturbances is also carried out.

Briefly, the regulation problem is solved by a feedback controller performing two tasks: 1) asymptotically stabilizes the equilibrium of the perturbation-free closed-loop system, and 2) the output of the plant converges to reference signal while the perturbations are rejected, where both the reference and perturbation signals are generated by an external system named exosystem [1].

The solution for the linear regulation problem was given in [2]. There, it was shown that such a solution is equivalent to the solution of a set of linear matrix equations named Francis equation. Later, in [3], this linear result was extended to the nonlinear field showing that the linear regulator can be considered as a special case of the nonlinear one. It is important to mention that the nonlinear regulator relies on the solution of a set of nonlinear partial differential equations, which, in some cases, are very difficult of obtain. Thus, in order to overcome such a drawback, in [4–9] the linear solution has been extended to the nonlinear area by means of Takagi–Sugeno fuzzy models, adaptive neuro-fuzzy inference system, and generic algorithm. In those works, the involvement of nonlinear partial differential equations is avoided.

On the other hand, fractional-order differential equations have been proven to be of great help during the modeling of real physics processes [10,11]. As consequence, the extension of integer-order results to the

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area of the fractional-order differential equations has been gaining interest among researchers, including those working on control topics. For example, the designing of observers of fractional order has been analyzed in [12]. An adaptive fuzzy controller capable of ensuring the convergence of the tracking error is developed in [13], while more conventional control approaches for systems described by differential equations of fractional order are meticulously explained in [14–16].

However, as was foreseen by Leibniz more than 300 years ago, the fractional-order calculus is founded on paradoxes [17]. Those paradoxes can be viewed as the different ways for obtaining the solutions of differential equations of fractional-order [18].

In the present work, the regulation problem for linear systems described by fractional-order differential equations is solved by considering the Caputo fractional derivative. Thus, assuming that both, the plant and the exosystem are modelled by linear equations of fractional-order, and that the fractional-order of the plant may be different from the fractional-order of the exosystem, the main contribution of this work is to provide an approach to design a linear controller capable of: 1) to stabilize the plant when it is not affected by the exosystem, and 2) to ensure that the tracking error converges to zero when the plant is under the influence of the exosystem.

The rest of the paper is organized as follows. The problem definition is given in Section 2, along with a brief reminder of the linear regulation, and of the fractional-order approaches considered. Then, in Section 3 the main results are presented. The numerical simulations are given in Section 4. Finally, in Section 5 some conclusions are drawn.

2. Preliminaries and problem formulation

In this section, the regulation problem for linear systems of integer-order is defined. Then, Cauchy's repeated integral, Riemann–Liouville fractional integral, Riemann–Liouville fractional derivative, and Caputo fractional derivative are briefly introduced. The problem formulation is given at the end of the section.

2.1. Linear regulation

Consider the linear plant described by:

$$\dot{x}(t) = Ax(t) + Bu(t) + Pw(t), \tag{1}$$

$$y(t) = Cx(t), \tag{2}$$

with $x(t) \in \mathbb{R}^n$ as the state vector of the linear plant, $u(t) \in \mathbb{R}^p$ as the input vector, $y(t) \in \mathbb{R}^m$ as the output vector, and $w(t) \in \mathbb{R}^\ell$ as the state vector of the linear exosystem, which is the generator of references and/or perturbation signals, i.e.

$$\dot{w}(t) = Sw(t), \tag{3}$$

$$y_{ref}(t) = Qw(t), \tag{4}$$

where $y_{ref}(t) \in \mathbb{R}^m$ is the desired reference signal. With this in mind, the tracking error can be defined by $e(t) = y(t) - y_{ref}(t)$, and the linear regulation problem consists of finding a control vector $u(t)$ such that:

$$\lim_{t\to\infty} e(t) = 0. \tag{5}$$

The control vector $u(t)$ capable of solving the linear regulation problem is [2–3]:

$$u(t) = -K(x(t) - x_{ss}(t)) + u_{ss}(t), \tag{6}$$

where $x_{ss}(t)$ is the steady-state manifold, which becomes invariant by means of the steady-state input $u_{ss}(t)$. However, the solution for the linear regulation problem has been proven to be equivalent to the solution of a set of linear matrix equations, named Francis equations:

$$\Pi S = A\Pi + B\Psi + P, \tag{7}$$

$$C\Pi = Q, \tag{8}$$

and the linear regulator can be written as:

$$u(t) = -K(x(t) - \Pi w(t)) + \Psi w(t), \tag{9}$$

where $x_{ss}(t) = \Pi w(t)$, $u_{ss}(t) = \Psi w(t)$, with $\Pi \in \mathbb{R}^{n\times\ell}$ and $\Psi \in \mathbb{R}^{p\times\ell}$ as the solution of the Francis equations (7) and (8).

The main result for the linear regulation problem is given in the following theorem.

Theorem 1 *Assume that the linear plant is described by equations (1) and (2), the exosystem is defined by equations (3) snd (4) and*

H1.- The pair (A, B) is stabilizable,

H2.- There exist matrices $\Pi \in \mathbb{R}^{n\times\ell}$ and $\Psi \in \mathbb{R}^{p\times\ell}$ solving Francis equations (7) and (8),

then the linear regulation problem defined by (1)–(4) is solvable by means of (9).

Proof [3] Define the steady-state error as $e_{ss}(t) = x(t) - \Pi w(t)$. By taking the time derivate of first order of $e_{ss}(t)$, one gets:

$$\dot{e}_{ss}(t) = \dot{x}(t) - \dot{w}(t), \tag{10}$$

$$\dot{e}_{ss}(t) = Ax(t) + Bu(t) + Pw(t) - Sw(t). \tag{11}$$

Using the definition of $e_{ss}(t)$ and the substitution of (9) in (11) produces:

$$\begin{aligned} \dot{e}_{ss}(t) &= A(e_{ss}(t) + \Pi w(t)) + B(-K(e_{ss}(t) + \Pi w(t) \\ &- \Pi w(t)) + \Psi w(t)) + Pw(t) - Sw(t), \end{aligned} \tag{12}$$

$$\begin{aligned} \dot{e}_{ss}(t) &= (A - BK)e_{ss}(t) + A\Pi w(t)) + B\Psi w(t) \\ &+ Pw(t) - Sw(t). \end{aligned} \tag{13}$$

From (13), it can be readily observed that $e_{ss}(t)$ tends to zero if: 1) all the eigenvalues of matrix $A-BK$ are in the left side of the complex plane, and 2) $A\Pi w(t)) + B\Psi w(t) + Pw(t) = Sw(t)$, which can be rewritten as (7). However, notice that (7) is a matrix equation involving $n\times\ell + p\times\ell$ unknowns in $n\times\ell$ equations. The missing equations are obtained by considering the tracking error $e(t) = y(t) - y_{ref}(t)$, which in terms of $e_{ss}(t)$ and $w(t)$ is:

$$e(t) = C(e_{ss}(t) + \Pi w(t)) - Qw(t), \tag{14}$$

which, in steady-state coincides with (8).

Thus, $e(t)$ goes to zero as e_{ss} tends to zero, and (8) is fulfilled. Finally, notice that if equations (7) and (8) can be solved, then either the solution is unique when $m = p$, or there are an infinite of solutions when $p > m$. On the other hand, equations (7) and (8) may not have a solution, in general, when $p < m$ [2]. □

2.2. Fractional derivatives and important functions

Defining

$$ {}_0\mathcal{I}_t f(t) = \int_0^t f(\tau) d\tau, \tag{15} $$

the formula for Cauchy's repeated or iterated integral is [16]:

$$ {}_0\mathcal{I}_t^n f(t) = \frac{1}{(n-1)!} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-n}} d\tau, \tag{16} $$

where n is a positive integer and the order of the integral.

To prove (16), consider the double integral

$$ {}_0\mathcal{I}_t^2 f(t) = \int_0^t \int_0^{\tau_1} f(\tau) d\tau d\tau_1, \tag{17} $$

where, without loss of generality, the order of the integrals can be switched, taking care that the limits are changed accordingly, i.e.:

$$ {}_0\mathcal{I}_t^2 f(t) = \int_0^t \int_\tau^t f(\tau) d\tau_1 d\tau. \tag{18} $$

However, $f(\tau)$ is a constant in the inner integral, and the solution of such an integral is $(t-\tau)f(\tau)$, resulting:

$$ {}_0\mathcal{I}_t^2 f(t) = \int_0^t (t-\tau) f(\tau) d\tau. \tag{19} $$

In a similar way:

$$ {}_0\mathcal{I}_t^3 f(t) = \frac{1}{2} \int_0^t (t-\tau)^2 f(\tau) d\tau, \tag{20} $$

and

$$ \begin{aligned} {}_0\mathcal{I}_t^n f(t) &= \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} f(\tau) d\tau d\tau_1 \ldots d\tau_{n-1} \\ &= \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau \qquad (21) \\ &= \frac{1}{(n-1)!} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-n}} d\tau, \qquad (22) \end{aligned} $$

where, as before, n is a positive integer and, at the same time, it is the order of the integral.

On the other hand, the Gamma function defined by [18]:

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} \exp(-y) dy, \tag{23}$$

which is convergent in the right semi-plane $\mathbb{R}(\alpha) > 0$, has (among others) the following properties:

$$\Gamma(1) = 1, \tag{24}$$

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \tag{25}$$

and when α is a positive integer, it can be proven that

$$\begin{aligned} \Gamma(\alpha+1) &= \alpha\Gamma(\alpha) = \alpha(\alpha-1)\Gamma(\alpha-1) && (26)\\ &= \alpha(\alpha-1)\ldots\cdot 2\cdot 1 = \alpha!. && (27)\end{aligned}$$

Therefore, as the Gamma function (23) is not restricted to integer arguments, it can be viewed as an extension of the factorial. The interested reader is referred to [18], where a thorough analysis of the Gamma function is carried out.

With this in mind, the Cauchy's repeated integral was extended to the fractional domain by Riemann–Liouville through the inclusion of the Gamma function in (16) as follows:

$$_0\mathcal{I}_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \tag{28}$$

where $\alpha > 0$ is the fractional order of the so-called Riemann–Liouville fractional integral (28).

The previous results were considered by Riemann–Liouville to propose their fractional-order derivative. This procedure can be described as follows [10]. In order to obtain the α-order derivative of $f(t)$, where $\alpha \in \mathbb{R}^+$, first, a fractional-order integration, of adequate order, must be performed using (28), and afterwards a derivative of an adequate integer order must be taken, i.e.

$$^{RL}_{\;\;0}\mathcal{D}_t^\alpha f(t) = \frac{d^q}{dt^q}\left[\frac{1}{\Gamma(q-\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-q+\alpha}} d\tau\right], \tag{29}$$

with $\alpha \in \mathbb{R}^+$, $q-1 < \alpha < q$ and $q \in \mathbb{N}$, where (29) is known as the Riemann–Liouville fractional-order derivative.

A different approach to obtain derivatives of fractional order was proposed by Caputo [10]. In such a procedure, it is suggested to take a derivative of integer order prior the application of a fractional-order integral, i.e.

$$^{C}_{0}\mathcal{D}_t^\alpha f(t) = \frac{1}{\Gamma(q-\alpha)} \int_0^t \frac{f^{(q)}(\tau)}{(t-\tau)^{1-q+\alpha}} d\tau, \tag{30}$$

with $\alpha \in \mathbb{R}^+$, $q-1 < \alpha < q$, and $q \in \mathbb{N}$, where (30) is known as the Caputo fractional-order derivative. Notice that the Caputo fractional derivative of a constant is zero, besides, this fractional-order derivative allows to consider the initial conditions in the usual way, which makes it more suitable during the modeling of physical systems.

Before proceeding, some useful Laplace transforms are introduced [16]:

$$\mathcal{L}\left\{{}_0\mathcal{I}_t^{\alpha} f(t)\right\} = s^{-\alpha}F(s), \tag{31}$$

$$\mathcal{L}\left\{{}_0^{RL}\mathcal{D}_t^{\alpha} f(t)\right\} = s^{\alpha} - \sum_{k=0}^{q-1} s^k \left[{}_0^{RL}\mathcal{D}_t^{\alpha-k-1} f(t)\right]\Big|_{t=0}, \tag{32}$$

$$\mathcal{L}\left\{{}_0^{C}\mathcal{D}_t^{\alpha} f(t)\right\} = s^{\alpha} - \sum_{k=0}^{q-1} s^{\alpha-k-1} f^{(k)}(0), \tag{33}$$

with $\alpha \in \mathbb{R}^+$, $q-1<\alpha<q$ and $q \in \mathbb{N}$.

It can be easily verified, that the solution, $x(t)$, of

$$ {}_0^{C}\mathcal{D}_t^{\alpha} x(t) = Ax(t), \text{ subject to initial conditions} \tag{34}$$

$$x(0) = x_0, \tag{35}$$

with $x(t) \in \mathbb{R}^n$ as the state vector of the linear plant and $A \in \mathbb{R}^{n\times n}$, can be expressed as:

$$x(t) = \mathcal{L}^{-1}\left\{(s^{\alpha}I - A)^{-1}s^{\alpha-1}\right\} x_0, \tag{36}$$

where I is the identity matrix of appropriate dimension $(n \times n)$.

On the other hand, an important function for the analysis of linear fractional dynamic systems is introduced. Such a function is [10]:

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+\beta)}, \ \alpha, \beta > 0, \tag{37}$$

which is known as the generalized Mittag-Leffler function. When $\beta = 1$, expression (37) becomes:

$$E_{\alpha,1}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+1)} \equiv E_{\alpha}(t), \ \alpha > 0, \tag{38}$$

which was first studied by Mittag-Leffler. Some properties of Mittag-Leffler functions are:

$$E_{1,1}(t) = \exp(t), \tag{39}$$

$$E_{1,2}(t) = \frac{1}{t}(\exp(t)-1) \tag{40}$$

$$E_{2,1}(t) = \cosh(\sqrt{z}), \tag{41}$$

among others. However, for the present work, the most important properties of Mittag-Leffler functions are the following:

$$\mathcal{L}\left\{E_{\alpha,1}(\lambda t^{\alpha})\right\} = (s^{\alpha}-\lambda)^{-1}s^{\alpha-1}, \tag{42}$$

$$\mathcal{L}\left\{t^{\beta-1}E_{\alpha,\beta}^{\gamma}(\lambda t^{\alpha})\right\} = \left((s^{\alpha}-\lambda)\,\gamma\right)^{-1} s^{\alpha\gamma-\beta}, \tag{43}$$

with

$$E_{\alpha,\beta}^{\gamma}(t) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{t^k}{k!},\ \alpha,\beta,\gamma > 0, \tag{44}$$

as the three parameter Mittag-Leffler function, where $(\gamma)_n = \gamma(\gamma+1)\ldots(\gamma+n-1)$ is the Pochhammer symbol [16]. Notice that $E_{\alpha,\beta}^{1}(t)$ coincides with $E_{\alpha,\beta}(t)$.

Thus, considering the linear fractional system (34) and (35) and from (42), the solution $x(t)$ can be expressed in terms of (38) as:

$$x(t) \quad = \quad E_{\alpha}(At^{\alpha})x_0. \tag{45}$$

The interested reader is referred to [10], where a thoroughly analysis of Mittag-Leffler functions in the solution of linear fractional systems is given.

2.3. Problem formulation

On the basis of the definitions given above, in the next section the linear fractional regulation problem will be solved. Such a problem is defined as follows. Considering a linear plant of fractional order of the form:

$$x^{(\alpha)}(t) \quad = \quad Ax(t) + Bu(t) + Pw(t), \tag{46}$$

$$y(t) \quad = \quad Cx(t), \tag{47}$$

with $x(t) \in \mathbb{R}^n$ as the state vector of the linear plant, $u(t) \in \mathbb{R}^p$ as the input vector, $y(t) \in \mathbb{R}^m$ as the output vector, $0 < \alpha \leq 1$ and $w(t) \in \mathbb{R}^{\ell}$ as the state vector of the following linear exosystem:

$$w^{(\beta)}(t) \quad = \quad Sw(t), \tag{48}$$

$$y_{ref}(t) \quad = \quad Qw(t), \tag{49}$$

where $y_{ref}(t) \in \mathbb{R}^m$ is the desired reference signal, $0 < \beta \leq 1$ and $v^{(\gamma)}(t) \equiv_0^C \mathcal{D}_t^{\gamma} v(t)$, with $v(t)$ as any vector function and $\gamma \in \mathbb{R}^+$; the linear fractional regulation problem consists of finding, if possible, a control $u(t)$ such that the tracking error, defined by $e(t) = y(t) - y_{ref}(t)$, tends to zero as time goes to infinity, i.e.

$$\lim_{t\to\infty} e(t) \quad = \quad 0. \tag{50}$$

3. The linear fractional regulator

Considering that the steady-state manifold and the steady-state input can be expressed as $x_{ss}(t) = \Pi(t)w(t)$ and $u_{ss}(t) = \Psi(t)w(t)$, respectively, the following theorem arises directly from previous section.

Theorem 2 *Assume the plant and the exosystem are linear fractional systems described by equations (46) and (47) and (48) and (49), respectively, and*

H_f1.- The pair (A, B) is stabilizable,

H_f2.- There exist time-dependent matrices $\Pi(t) \in \mathbb{R}^{n\times\ell}$ and $\Psi(t) \in \mathbb{R}^{p\times\ell}$ solving:

$$\begin{aligned} \Pi(t)w^{(\alpha)}(t) + \Pi^{(\alpha)}(t)w(t) \quad &= \quad (A\Pi(t) + B\Psi(t) \\ &+ \quad P)\, w(t), \end{aligned} \tag{51}$$

$$C\Pi(t) \quad = \quad Q, \tag{52}$$

then, the linear regulation problem defined by (46)–(49) is solvable by means of

$$u(t) = -K(x(t) - \Pi(t)w(t)) + \Psi(t)w(t). \tag{53}$$

Proof As in [2], suppose that the controller capable of solving the regulation problem defined by equations (46)–(49) has the form of (53); thus, the steady-state error can be defined as:

$$\begin{aligned} e_{ss}(t) &= x(t) - x_{ss}(t), & (54)\\ &= x(t) - \Pi(t)w(t), & (55) \end{aligned}$$

because $x_{ss}(t) = \Pi(t)w(t)$.

Now, by taking the fractional derivate of order α of e_{ss} and by considering (46) and (9), one gets:

$$\begin{aligned} e_{ss}^{(\alpha)}(t) &= x^{(\alpha)}(t) - x_{ss}^{(\alpha)}(t), & (56)\\ &= x^{(\alpha)}(t) - \Pi(t)w^{(\alpha)}(t) - \Pi^{(\alpha)}(t)w(t), & (57)\\ &= Ax(t) + Bu(t) + Pw(t) \\ &- \Pi(t)w^{(\alpha)}(t) - \Pi^{(\alpha)}(t)w(t), & (58)\\ &= Ax(t) - BKx(t) + BK\Pi(t)w(t) + B\Psi(t)w(t) \\ &+ Pw(t) - \Pi(t)w^{(\alpha)}(t) - \Pi^{(\alpha)}(t)w(t). & (59) \end{aligned}$$

At this point, according to (11), $x(t)$ can be replaced by $x(t) = e_{ss}(t) + \Pi(t)w(t)$, resulting:

$$\begin{aligned} e_{ss}^{(\alpha)}(t) &= Ae_{ss}(t) + A\Pi(t)w(t) - BKe_{ss}(t) \\ &- BK\Pi(t)w(t) + BK\Pi(t)w(t) + B\Psi(t)w(t) \\ &+ Pw(t) - \Pi(t)w^{(\alpha)}(t) - \Pi^{(\alpha)}(t)w(t), & (60)\\ e_{ss}^{(\alpha)}(t) &= (A - BK)e_{ss}(t) \\ &+ A\Pi(t)w(t) + B\Psi(t)w(t) + Pw(t) \\ &- \Pi(t)w^{(\alpha)}(t) - \Pi^{(\alpha)}(t)w(t). & (61) \end{aligned}$$

Notice that $(A - BK)e_{ss}$ tends to zero by choosing an adequate matrix K, whose existence is guaranteed because the pair (A, B) is stabilizable. Thus, in order to make (61) asymptotically stable the following equation must be fulfilled:

$$\Pi(t)w^{(\alpha)}(t) + \Pi^{(\alpha)}(t)w(t) = (A\Pi + B\Psi + P)w(t), \tag{62}$$

where $w^{(\alpha)}(t) \equiv_0^C \mathcal{D}_t^\alpha w(t)$ and $\Pi^{(\alpha)}(t) \equiv_0^C \mathcal{D}_t^\alpha \Pi(t)$. However, notice that (62) is a matrix equation involving $n \times \ell + p \times \ell$ unknowns in $n \times \ell$ equations. The missing equations are obtained by considering the tracking error $e(t) = y(t) - y_{ref}(t)$, which in terms of $e_{ss}(t)$ and $w(t)$ is:

$$e(t) = C(e_{ss}(t) + \Pi w(t)) - Qw(t), \tag{63}$$

which, in steady-state and by means of (36), turns into:

$$\begin{aligned} 0 &= C\Pi(t)w(t) - Qw(t), & (64)\\ C\Pi(t)w(t) &= Qw(t), & (65)\\ C\Pi(t) &= Q. & (66) \end{aligned}$$

Notice that $w(t)$ can be obtained either from (36) or (45). □

It is important to mention that Theorem 2 is a generalization of Theorem 1. The following corollary is intended to clarify this assessment.

Corollary 1 *Assume that the plant and the exosystem are linear fractional systems described by equations (46) and (47) and (48) and (49), respectively, with $\alpha = \beta$, also assume that $H_f 1$ is fulfilled, and that*

H_{f1} 2.- There exist time-dependent matrices $\Pi(t) \in \mathbb{R}^{n\times\ell}$ and $\Psi(t) \in \mathbb{R}^{p\times\ell}$ solving:

$$\begin{aligned} \Pi(t)S + \Pi^{(\alpha)}(t) &= A\Pi(t) + B\Psi(t) + P, && (67)\\ C\Pi(t) &= Q, && (68) \end{aligned}$$

then, the linear regulation problem defined by (46)–(49) is solvable by means of (53)

Proof If $\alpha = \beta$ then, by (48) and (49), equations (51) and (52) can be rewritten as

$$\begin{aligned} \left(\Pi(t)S + \Pi^{(\alpha)}(t)\right) w(t) &= (A\Pi(t) + B\Psi(t) \\ &+ P)\, w(t), && (69)\\ C\Pi(t) &= Q, && (70) \end{aligned}$$

or as in (67) and (68), which clearly coincide with equations (7) and (8) when Π and Ψ are constants. The rest of the proof follows directly from Theorem 2. □

Another approach to solve the linear fractional regulation problem consists of solving an adequate set of simultaneous equations for $\Pi(t)$ and $\Psi(t)$ at every instant t, avoiding, in this way, the problem of solving the set of fractional time-variant differential equations given in Theorem 2 and Corollary 1. To do this, the explicit expressions for $w(t)$ and $w^{(\alpha)}(t)$ are needed, such that $\Pi(t)$ and $\Psi(t)$ can be expressed in terms of the elements of $w(t)$ and $w^{(\alpha)}(t)$. This proposal is summarized in the following theorem.

Theorem 3 *Assume that the plant and the exosystem are linear fractional systems described by equations (46) and (47) and (48) and (49), respectively, with $0 < \alpha \leq 1$, $0 < \beta \leq 1$, and $\alpha \leq \beta$, also assume $H_f 1$ is fulfilled, and that*

H_{f2} 2.- There exist time-dependent matrices $\Pi(t) \in \mathbb{R}^{n\times\ell}$ and $\Psi(t) \in \mathbb{R}^{p\times\ell}$ solving:

$$\begin{aligned} \Pi(t)M(t) &= (A\Pi(t) + B\Psi(t) + P)N(t), && (71)\\ C\Pi(t) &= Q, && (72) \end{aligned}$$

where $M(t) = \left(t^{-\alpha}E_{\beta,1-\alpha}(St^{\beta}) - \frac{t^{-\alpha}I}{\Gamma(1-\alpha)}\right)$ and $N(t) = E_{\beta}(St^{\beta})$ are time-dependent matrices of dimension $\ell \times \ell$, then, the linear regulation problem defined by (46)–(49) is solvable by means of (53).

Proof At first, notice that $0 \leq \|\alpha - \beta\| \leq 1$ because $0 < \alpha \leq 1$ and $0 < \beta \leq 1$. Thus, from (36) and (45), the solution $w(t)$ for (48) can be expressed as:

$$w(t) = E_{\beta}(St^{\beta})w(0) = N(t)w(0), \tag{73}$$

where its Laplace transform is $W(s) = (s^{\beta}I - S)^{-1}s^{\beta-1}$, and from (33), (36) the fractional derivative $w^{(\alpha)}(t)$ can be expressed as:

$$w^{(\alpha)}(t) = \mathcal{L}^{-1}\left\{(s^{\beta}I - S)^{-1}s^{\alpha+\beta-1}w(0) - s^{\alpha-1}w(0)\right\}, \tag{74}$$

$$w^{(\alpha)}(t) = \mathcal{L}^{-1}\left\{(s^{\beta}I - S)^{-1}s^{\alpha+\beta-1} - s^{\alpha-1}I\right\}w(0), \tag{75}$$

$$w^{(\alpha)}(t) = \mathcal{L}^{-1}\left\{(s^{\beta}I - S)^{-1}s^{\beta-(1-\alpha)} - s^{-(1-\alpha)}I\right\}w(0), \tag{76}$$

because $E_{\beta}(St^{\beta})\big|_{t=0} = I$.

From (43) and $\mathcal{L}\left\{t^{\alpha-1}\right\} = \Gamma(\alpha)/s^{\alpha}$:

$$w^{(\alpha)}(t) = M(t)w(0), \tag{77}$$

with $M(t) = \left(t^{-\alpha}E_{\beta,1-\alpha}(St^{\beta}) - \frac{t^{-\alpha}I}{\Gamma(1-\alpha)}\right)$ and from (37):

$$\begin{aligned}
E_{\beta,1-\alpha}(St^{\alpha}) &= \sum_{k=0}^{\infty}\frac{\left(St^{\beta}\right)^{k}}{\Gamma(k\beta+1-\alpha)}, && (78)\\
E_{\beta,1-\alpha}(St^{\alpha}) &= \frac{I}{\Gamma(1-\alpha)} + \frac{St^{\beta}}{\Gamma(\beta+1-\alpha)} \\
&+ \frac{S^{2}t^{2\beta}}{\Gamma(2\beta+1-\alpha)} + \ldots \\
&+ \frac{S^{k}t^{k\beta}}{\Gamma(k\beta+1-\alpha)} + \ldots. && (79)
\end{aligned}$$

Thus,

$$\begin{aligned}
t^{-\alpha}E_{\beta,1-\alpha}(St^{\beta}) &= \sum_{k=0}^{\infty}\frac{t^{-\alpha}\left(St^{\beta}\right)^{k}}{\Gamma(k\beta+1-\alpha)}, && (80)\\
t^{-\alpha}E_{\beta,1-\alpha}(St^{\beta}) &= \frac{t^{-\alpha}I}{\Gamma(1-\alpha)} + \frac{St^{\beta-\alpha}}{\Gamma(\beta+1-\alpha)} \\
&+ \frac{S^{2}t^{2\beta-\alpha}}{\Gamma(2\beta+1-\alpha)} + \ldots \\
&+ \frac{S^{k}t^{k\beta-\alpha}}{\Gamma(k\beta+1-\alpha)} + \ldots, && (81)
\end{aligned}$$

and

$$\begin{aligned} t^{-\alpha}E_{\beta,1-\alpha}(St^{\beta}) - \frac{t^{-\alpha}I}{\Gamma(1-\alpha)} &= \frac{St^{\beta-\alpha}}{\Gamma(\beta+1-\alpha)} \\ &+ \frac{S^2t^{2\beta-\alpha}}{\Gamma(2\beta+1-\alpha)} + \dots \\ &+ \frac{S^kt^{k\beta-\alpha}}{\Gamma(k\beta+1-\alpha)} + \dots . \end{aligned} \tag{82}$$

Notice that equation (82) goes to infinity at $t = 0$ when $\alpha > \beta$. For that reason, condition $\alpha \leq \beta$ is needed. The rest of the proof follows directly from Theorems 1 and 2. □

It is important to mention that, as in the classical linear result, the linear fractional regulation problem is independent of initial conditions.

Remark 1 *From the previous analysis, in practical cases when $\alpha > \beta$, it is suggested to apply the regulator from $t > 0$. Also, notice that if the fractional order of the exosytem is greater that the fractional order of the plant, then matrix $M(t)$ may diverge.*

As previously mentioned, in the following section, matrices $\Pi(t)$ and $\Psi(t)$ will be solved in terms of the elements of matrices $M(t)$ and $N(t)$.

4. Numerical simulations

4.1. Example 1

Consider the fractional regulation problem defined by equations (46) and (47) and (48) and (49), with $A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and $Q = \begin{bmatrix} 1 & 0 \end{bmatrix}$; where, due to the form of C and Q the output is $y = x_1$ and the reference signal is $y_{ref} = w_1$.

It can be easily verified that the pair (A, B) is not only stabilizable but controllable, and for this example the Ackermann's formula has been used to place the eigenvalues of the closed-loop matrix $A - BK$ in $[-1\ -2]$, resulting $K = [4\ 3]$. On the other hand, it can be deduced that $\Pi(t)$ and $\Psi(t)$ are matrices of dimension (2×2) and (1×2), respectively, and $M(t) = \begin{bmatrix} m_{11}(t) & m_{12}(t) \\ m_{21}(t) & m_{22}(t) \end{bmatrix}$, $N(t) = \begin{bmatrix} n_{11}(t) & n_{12}(t) \\ n_{21}(t) & n_{22}(t) \end{bmatrix}$. Thus, after solving (71) and (72) in terms of the elements of $M(t)$ and $N(t)$, one gets:

$$\Pi(t) = \begin{bmatrix} 1 & 0 \\ \frac{m_{11}n_{22}-m_{12}n_{21}}{n_{11}n_{22}-n_{12}n_{21}} & \frac{m_{11}n_{12}-m_{12}n_{11}}{n_{11}n_{22}-n_{12}n_{21}} \end{bmatrix}, \tag{83}$$

and

$$\Psi(t) = \begin{bmatrix} \psi_{11} & \psi_{12} \end{bmatrix}, \tag{84}$$

with $\psi_{11} = (m_{11}^2n_{22}^2 - 2m_{11}m_{12}n_{21}n_{22} + m_{22}m_{11}n_{12}n_{21} - m_{21}m_{11}n_{12}n_{22} + m_{12}^2n_{21}^2 - m_{22}m_{12}n_{11}n_{21} + m_{21}m_{12}n_{11}n_{22} - 2n_{11}^2n_{22}^2 + 4n_{11}n_{12}n_{21}n_{22} - 2n_{12}^2n_{21}^2)/(n_{11}n_{22} - n_{12}n_{21})^2$, and $\psi_{12} = (-n_{22}m_{11}^2n_{12} + n_{22}m_{11}m_{12}n_{11} + n_{21}m_{11}m_{12}n_{12} -$

$m_{22}m_{11}n_{11}n_{12} + m_{21}m_{11}n_{12}^2 - n_{21}m_{12}^2n_{11} + m_{22}m_{12}n_{11}^2 - m_{21}m_{12}n_{11}n_{12})/(n_{11}n_{22} - n_{12}n_{21})^2$, where argument t has been omitted for the sake of space. In what follows, $M(t)$ and $N(t)$ will be approximated by their first 100 terms.

The first simulation has been performed considering $\alpha = 1$, $\beta = 0.99$. The results are given in Figure 1. The second and third simulations are obtained with $\beta = 0.90$ and $\beta = 0.85$, while the rest of the parameters remain unchanged. The results are depicted in Figures 2 and 3, respectively.

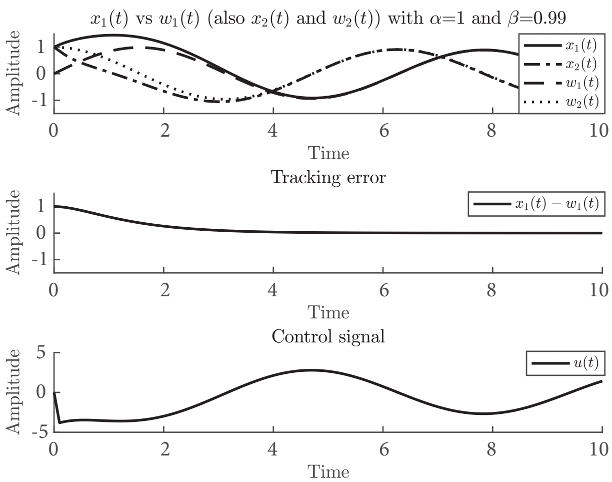

Figure 1. Simulation results for $x(0) = [1\ 1]^T$ and $w(0) = [0\ 1]^T$.

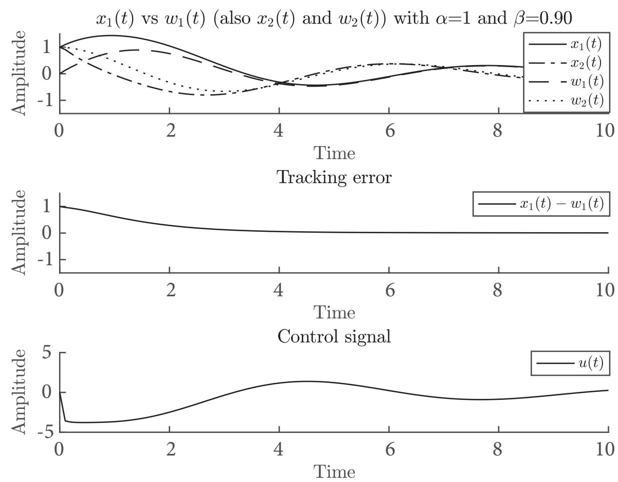

Figure 2. Simulation results for $x(0) = [1\ 1]^T$ and $w(0) = [0\ 1]^T$.

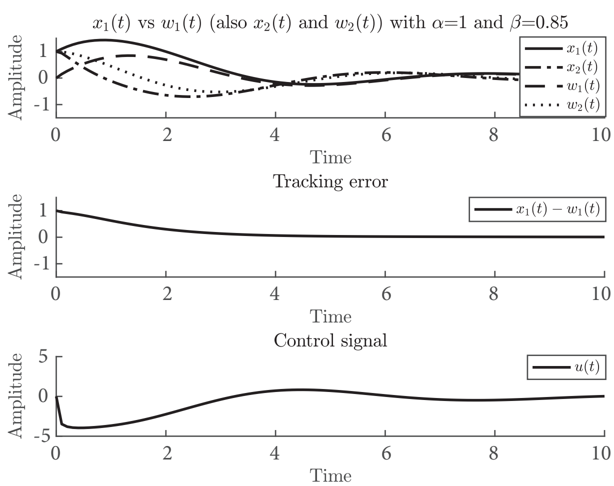

Figure 3. Simulation results for $x(0) = [1\ 1]^T$ and $w(0) = [0\ 1]^T$.

It is important to notice that according to Theorem 3 and Remark 1, in both cases, the regulator has been applied from $t > 0$.

5. Conclusions

In this work, the regulation problem has been extended to the field of fractional-order linear systems considering the Caputo fractional derivative. The regulation equations were obtained on the basis of the Francis equations.

It was also shown that the linear fractional regulator exists at $t = 0$ only if the order of the plant is not greater than the order of the reference system. On the basis of a set of simultaneous equations, an alternative way to solve the fractional regulation problem has been given also. Finally, some numerical simulations have been considered to illustrate the validity of the proposed approach.

Acknowledgment

This work was partially supported by CONACYT (Consejo Nacional de Ciencia y Tecnología) through scholarship SNI (Sistema Nacional de Investigadores) and by IPN (Instituto Politécnico Nacional) through research projects 20180095 and 20190083; and scholarships EDI (Estímulo al Desempeño de los Investigadores), COFAA (Comisión de Operación y Fomento de Actividades Académicas) and BEIFI (Beca de Estímulo Institucional de Formación de Investigadores).

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