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On efficient computation of equilibrium under social coalition structures

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Abstract: In game-theoretic settings the key notion of analysis is an equilibrium, which is a profile of agent strategies such that no viable coalition of agents can improve upon their coalitional welfare by jointly changing their strategies. A Nash equilibrium, where viable coalitions are only singletons, and a super strong equilibrium, where every coalition is deemed viable, are two extreme scenarios in regard to coalition formation. A recent trend in the literature is to consider equilibrium notions that allow for coalition formation in between these two extremes and which are suitable to model social coalition structures that arise in various real-life settings. The recent literature considered the question on the existence of equilibria under social coalition structures mainly in Resource Selection Games (RSGs), due to the simplicity of this game form and its wide range of application domains. We take the question on the existence of equilibria under social coalition structures from the perspective of computational complexity theory. We study the problem of deciding the existence of an equilibrium in RSGs with respect to a given social coalition structure. For an arbitrary coalition structure, we show that it is computationally intractable to decide whether an equilibrium exists even in very restricted settings of RSGs. In certain settings where an equilibrium is guaranteed to exist we give polynomial-time algorithms to find an equilibrium.

Key words: Algorithmic game theory, laminar equilibrium, contiguous equilibrium, resource selection games

1. Introduction

The research studies in game theory focus on how self-interested agents may coordinate their strategies so as to maximize their welfare. At the heart of these studies is the notion of an 'equilibrium', which is a profile of agent strategies such that no viable coalition of agents can improve upon their (coalitional) welfare (in the Pareto sense) by jointly changing their strategies. One may define various notions of equilibrium depending on what types of coalition formation are deemed *viable*. At one extreme is the notion of a *super strong equilibrium* (see [3]), in which any coalition is deemed viable: no coalition of agents can improve upon their welfare by jointly changing their strategies. Obviously, the notion of a super strong equilibrium is very appealing, but it is a very stringent one and hence a super strong equilibrium rarely exists in game-theoretic settings. At the other extreme is the notion of a *Nash equilibrium* (see [15]), in which viable coalitions are only singletons: no agent can improve its welfare by changing its strategy. Nash [15] showed that if mixed strategies are allowed every finite game admits a Nash equilibrium. There are also game forms in which a pure-strategy Nash equilibrium always exists (for instance, see [4, 17, 18]).

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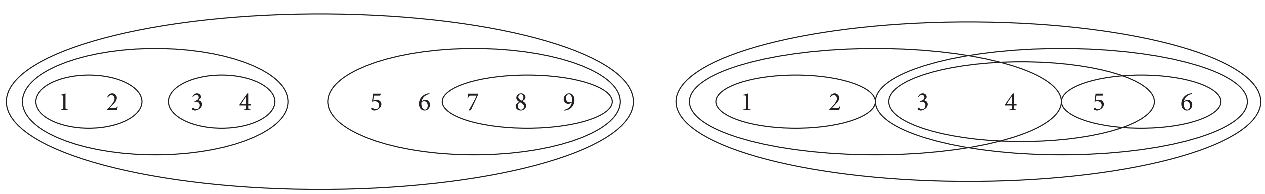
In the recent literature, there is a growing tendency to consider equilibrium notions under which coalition formation opportunities are restricted to 'social coalition structures'. Arguably, coalition structures with a social context naturally arise in certain real-life settings, and therefore, it is natural to consider equilibrium notions defined on the basis of such coalition structures. We mention below briefly the notions of equilibrium in the recent literature that are defined on the basis of social coalition structures and which are most relevant to our analysis.

Under the notion of a *laminar equilibrium* (see [5]), the coalition structure is assumed to be laminar: i.e., for any pair of viable coalitions, either they are disjoint, or one of them is a subset of the other. The laminar equilibrium notion is motivated by real-life hierarchical communities. In hierarchical communities, one may consider institutional constraints that restrict coalition formation possibilities to the units of the hierarchy, such as a corps, a legion or a brigade in the military; or a faculty or a department at some university. (It is worth mentioning that in a related study and with similar motivation, Kamihigashi et al. [13] introduced the notion of an *organizational Nash equilibrium*, under which the coalition structure is assumed to be laminar and every singleton is assumed to be a viable coalition. Notice that the family of coalition structures considered under their equilibrium notion is a strict subset of the larger family of laminar coalition structures.)

Under the notion of a *contiguous equilibrium* (see [5]), it is assumed that there exists a permutation of agents such that each viable coalition consists of a number of agents that are subsequently ordered under this permutation. For instance, imagine the residents of a street or some people in a queue. In such 'contiguous communities', constraints on communication or familiarship may restrict coalition formation possibilities to neighbors on the street or people that are subsequently ordered in the queue. This equilibrium notion turns out to be a generalization of the laminar equilibrium notion (see [5]). Below, Figure 1 illustrates a coalition structure that is laminar, and hence also contiguous; and Figure 1 illustrates a coalition structure that is contiguous but not laminar.



(a) An example of a laminar coalition structure.

(b) An example of a contiguous coalition structure.

Figure 1. Examples of laminar & contiguous coalition structures.

Several papers in the literature studied the existence of equilibria under the above-mentioned social coalition structures. In all of these papers the game form considered was *resource selection games* (RSGs): In an RSG, there is a number of agents which are to utilize from a set of resources, and agents try to avoid 'congested resources' since an agent using some resource incurs a 'congestion cost' which is strictly increasing in the number of agents which use the same resource. There are two reasons as to why earlier studies considered the above-mentioned equilibrium notions in the context of RSGs. First, this class of games is useful in modeling a wide range of real-life problems such as in computer networking, task allocation, server farms, scheduling, transportation, and evolutionary and ecological biology (see, for instance, [6, 8, 9, 14, 16]). Second, in RSGs a super strong equilibrium does not always exist (see [10]) yet a (pure strategy) Nash equilibrium always does, which renders this class of games a perfect setting to study the existence of equilibrium under social coalition structures.[1]

In our paper, we take the question on the existence of equilibria under social coalition structures from the perspective of computational complexity theory. Before proceeding to our results we briefly mention below the results in the literature that are most relevant to our findings.

Feldman & Tennenholtz [10] introduced the notion of a *partition equilibrium* where the coalition structure is assumed to be a partition of the set of agents, and they proved that in an RSG a partition equilibrium is guaranteed to exist under the following restrictions: (i) if the size of each viable coalition is at most two; or (ii) if there are only two resources; or (iii) if the resources are identical. Anshelevich et al. [1] generalized this result by proving that in an RSG a partition equilibrium always exists (i.e., without any stipulations). Caskurlu et al. [5] showed via an intricate counterexample that in an RSG a laminar equilibrium (and hence, a contiguous equilibrium) may not exist in general. Nonetheless, they proved that in an RSG a laminar equilibrium always exists under the following restrictions: (i) if the resources are identical, or (ii) if there are only two resources.[2] Caskurlu et al. [5] also showed that a contiguous equilibrium is guaranteed to exist in an RSG if the resources are identical; however, they also showed that unlike a laminar equilibrium, a contiguous equilibrium is not guaranteed to exist in an RSG even if there are only two resources. Table 1 below summarizes these results.[3]

Equilibrium notions	Resources		
	general	two	identical
Partition equilibrium	$+^{\star}$	$+^{*}$	$+^{*}$
Laminar equilibrium	$-^{\dagger}$	$+^{\dagger}$	$+^{\dagger}$
Contiguous equilibrium	$-^{\dagger}$	$-^{\dagger}$	$+^{\dagger}$

Table 1. The existence and non-existence results in the literature (* by [10], * by [1], † by [5])

We take the question on the existence of equilibria under social coalition structures from the perspective of computational complexity theory. We first consider the problem of deciding the existence of an equilibrium in RSGs with respect to a given coalition structure. We show that this problem is NP-HARD even in the very restricted setting where each coalition consists of exactly two agents and there are only two identical resources *(Theorem 1)*.

All the positive existence results in the literature in RSGs, except for the existence of a laminar equilibrium in the two-resource setting in [5], were shown by presenting efficient (polynomial-time) algorithms. Although the proof in [5] is a constructive one, the algorithm implied by the proof to find such an equilibrium has exponential running time. We show that the problem of finding a laminar equilibrium in an RSG with two resources is in P by presenting a polynomial-time algorithm that finds the desired equilibrium *(Theorem 2)*.

As shown in [5] a contiguous equilibrium may not exist in an RSG with two resources. We find that in this setting the cause of instability is those coalitions that consist of exactly two agents. More precisely, we show

[1]RSGs fall into the class of congestion games for which the existence of a Nash equilibrium is always guaranteed; for studies on congestion games and its applications, see [2, 7, 18]

[2]In the counterexample in Caskurlu et al., singletons are deemed as viable coalitions. Therefore, their counterexample also shows that in an RSG an organizational Nash equilibrium may not exist. Also, note that an organizational Nash equilibrium is by definition also a laminar equilibrium. Therefore, the existence result in Caskurlu et al. in the identical- and two-resource settings for a laminar equilibrium also hold for an organizational Nash equilibrium.

[3]Several other relevant works need mentioning. In the literature on RSGs, in addition to the above-mentioned equilibrium notions, several other notions of equilibrium have been studied: the notions of a 'centralized equilibrium', a 'collision equilibrium' and a 'considerate equilibrium'. For existence studies pertaining to these equilibrium notions, see [5, 11, 12], respectively.

that in an RSG with two resources there always exists a contiguous equilibrium if no viable coalition consists of exactly two agents; and under this stipulation we also show that an equilibrium can be computed efficiently *(Theorem 3)*.

In Section 2, we introduce the model. In Section 3, we establish the computational complexity of deciding the existence of a C-*stable* allocation in RSGs. Section 4 is devoted to presenting efficient algorithms to compute a laminar equilibrium and a contiguous equilibrium in two-resource RSGs.

2. The model

In a game in strategic-form, we have a set of agents $N = \{1, \ldots, n\}$ such that each agent $i \in N$ is associated with a set of possible strategies S_i. An outcome of the game is a vector of strategies $s = (s_1, \ldots, s_n)$ where $s_i \in S_i$ is the strategy selected by agent $i \in N$. A vector of strategies is also referred to as a *strategy profile*. The set of all strategy profiles is referred to as the *strategy space* which we denote by S, i.e., $S = S_1 \times \ldots \times S_n$. For an agent $i \in N$ and a strategy profile $s \in S$, the utility of agent i at strategy profile s is denoted by $U_i(s)$. Each agent tries to maximize its utility.

A *coalition* c is a nonempty subset of the set of agents N. The domain of coalitions is $\mathcal{P}(N) \setminus \{\emptyset\}$, where $\mathcal{P}(N)$ is the power set of N. We use $\mathcal{P}_{\geq 1}(N)$ to denote this domain. A *coalition structure* C is a set of viable coalitions; i.e., $C \subseteq \mathcal{P}_{\geq 1}(N)$. Let S_c denote the restriction of the strategy space S for coalition c, and let s_c denote the restriction of the strategy profile s for coalition c. That is, $S_c = \bigtimes_{i \in c} S_i$ and $s_c = (s_i)_{i \in c}$. Note that the strategy space S and the strategy profile s can be written as $(S_c, S_{N \smallsetminus c})$ and $(s_c, s_{N \smallsetminus c})$, respectively.

The space S_c represents the domain of *deviations* for coalition c. At strategy profile s if coalition c takes a deviation $\widetilde{s}_c \in S_c$, then the resulting strategy profile is $(\widetilde{s}_c, s_{N \smallsetminus c}) \in (S_c, S_{N \smallsetminus c})$. $\widetilde{s}_c$ is called an *improving deviation* for coalition c at strategy profile s if for each $i \in c$, $U_i(\widetilde{s}_c, s_{N \smallsetminus c}) \geq U_i(s)$, and for some $i \in c$, $U_i(\widetilde{s}_c, s_{N \smallsetminus c}) > U_i(s)$.

A strategy profile s is called c-*stable* if coalition c has no improving deviation at strategy profile s. A strategy profile s is called C-*stable* if it is c-*stable* for each coalition $c \in C$. The notions of super strong equilibrium and Nash equilibrium can be defined using this terminology as follows: A strategy profile is a *super strong equilibrium* if it is $\mathcal{P}_{\geq 1}(N)$-stable. A strategy profile is a *Nash equilibrium* if it is $\mathcal{P}_{=1}(N)$-stable, where $\mathcal{P}_{=1}(N) = \{c \subset N \mid |c| = 1\}$.

Next, we define the notions of a laminar equilibrium and a contiguous equilibrium.

- A coalition structure C is *laminar* if for any pair of coalitions $c_1, c_2 \in C$ such that $c_1 \cap c_2 \neq \emptyset$, either $c_1 \subseteq c_2$ or $c_2 \subseteq c_1$. Given a laminar coalition structure C, a strategy profile is a *laminar equilibrium* if it is C-stable.
- A coalition structure C is *contiguous* if there exists a permutation of agents $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ such that for each coalition $c \in C$, the agents in c are subsequently ordered under the permutation π. Given a contiguous coalition structure C, a strategy profile is a *contiguous equilibrium* if it is C-*stable*.

Definition 1 *A* resource selection game *(RSG)* $\mathcal{G}$ *is a triplet* $\langle N, R, f \rangle$ *where* $N = \{1, 2, \ldots, n\}$ *is a finite set of* agents*,* $R = \{1, 2, \ldots, m\}$ *is a finite set of* resources*, and* $f : (f_j)_{j=1}^m$ *is a profile of strictly monotonically increasing* cost functions *of the resources. We assume that* $f_j(0) = 0$ *for all* $j \in \{1, 2, \ldots, m\}$*. Each agent is to select exactly one resource to use. When* q *agents use resource* j*, each incurs a cost equal to* $f_j(q)$*. Each agent tries to minimize the cost that it incurs. In the rest of the paper we fix the game* $\mathcal{G} = \langle N, R, f \rangle$*.*

We say that agent i is allocated to resource j under strategy profile s if $s_i = j$. We define an *allocation* of agents to resources as a sequence $a : (a_j)_{j=1}^m$, where a_j denotes the set of agents allocated to resource j. Let $\mathcal{A}$ be the domain of allocations. Notice that an RSG is a game in strategic-form, where $S_i = R$ for each agent i, and the utility of agent i allocated to resource j is $-f_j(|a_j|)$. Since there is a one-to-one correspondence between the strategy profiles and allocations, we will speak of them interchangeably.

For an allocation a, we define the *makespan* of a as the maximum cost an agent incurs at allocation a, i.e., the makespan of a is equal to $max_{j \in R} f_j(|a_j|)$. We define the *minimum makespan* of $\mathcal{G}$ as the makespan of the allocation a for which the makespan is smallest, i.e., the minimum makespan of $\mathcal{G}$ is equal to $\min_{a \in \mathcal{A}} \{max_{j \in R} f_j(|a_j|)\}$. We use α to denote the minimum makespan. For each resource j, we define the *quota* q_j of resource j as the maximum number of agents that can be allocated to resource j without making the cost incurred at resource j greater than α, i.e., $q_j = max_z \ f_j(z) \leq \alpha$.

The resources are classified into two groups as follows: A resource j is called a *'type 1'* resource if it can attain a cost of α, i.e, if there exists a positive integer z such that $f_j(z) = \alpha$; and a resource is called a *'type 2'* resource if it cannot attain a cost of α. We also use T_1 and T_2 to denote the sets of type 1 and type 2 resources, respectively. Note that we cannot have $T_1 = \emptyset$ by definition of α. Note also that for each type 1 resource j, we have $q_j \geq 1$, whereas the quota of a type 2 resource may be 0. For each type 1 resource j, we use β_j to denote the cost of resource j when the number of agents allocated to resource j is one less than its quota, i.e., $\beta_j = f_j(q_j - 1)$; and we refer to β_j as resource j's *beta-value*.

3. Computational intractability of finding a C-*stable* allocation

This section is devoted to establishing the computational complexity of deciding the existence of a C-*stable* allocation in an RSG. Theorem 1 below proves that the problem is NP-HARD even in the very restricted setting where (i) the size of each viable coalition is exactly two, (ii) and there are only two identical resources.

Theorem 1 *Given an RSG* $\mathcal{G} = \langle N, R, f \rangle$ *and a coalition structure* $C \subseteq \mathcal{P}_{\geq 1}(N)$, *it is* NP-HARD *to decide the existence of a* C*-stable allocation if* $|c| = 2$ *for each* $c \in C$, $|R| = 2$, *and* $f_1(.) = f_2(.)$.

Proof We prove Theorem 1 via a polynomial-time mapping reduction from the HALF-VERTEX-COVER problem, which is known to be NP-HARD.

In the HALF-VERTEX-COVER problem, we are given an undirected simple graph $G = \langle V, E \rangle$. Without loss of generality we can assume that $|V|$ is odd since the HALF-VERTEX-COVER problem is NP-HARD even then. Thus, let $|V| = 2k + 1$ for some positive integer k. We are asked to decide the existence of a subset $V' \subseteq V$ of vertices such that $|V'| \leq k$, and $u \in V'$ or $v \in V'$ for each edge $(u, v) \in E$. For a given HALF-VERTEX-COVER instance $\mathcal{H} = \langle V, E \rangle$, we construct the corresponding RSG instance $\mathcal{G} = \langle N, R, f \rangle$ and the coalition structure C as follows:

- There is an agent in $\mathcal{G}$ corresponding to each vertex of $\mathcal{H}$, i.e., $N = \{1, 2, \ldots, |V|\}$.
- There are two identical resources in the RSG instance $\mathcal{G}$, i.e., $|R| = 2$, and $f_1(.) = f_2(.)$.
- For every edge $(i, j) \in E$ of $\mathcal{H}$, there is a corresponding coalition in $c \in C$ such that the coalition members of c are the agents corresponding to the vertices i and j, i.e., $c = \{i, j\} \in C$.

Note that $\mathcal{G}$ and C are as described in the theorem statement, i.e., $|c| = 2$ for all $c \in C$, $|R| = 2$, and $f_1(.) = f_2(.)$. Since $f_1(.) = f_2(.)$, we will refer to them as $f(.)$ in the rest of the proof.

To complete the proof all we need is to show the following: There exists a subset $V' \subseteq V$ of vertices of $\mathcal{H}$ such that $|V'| \leq k$, and $u \in V'$ or $v \in V'$ for each edge $(u,v) \in E$ if and only if there exists a C*-stable* allocation a in $\mathcal{G}$.

(Only If) Suppose that there exists a subset $V' \subseteq V$ of vertices of $\mathcal{H}$ such that $|V'| \leq k$, and $u \in V'$ or $v \in V'$ for each edge $(u,v) \in E$. We now show that the allocation $a = (V', N \setminus V')$ is a C*-stable* allocation in $\mathcal{G}$ by showing that a is c*-stable* for any $c \in C$. Note that $|a_1| \leq k$ and thus $|a_2| \geq k+1$. Hence, the cost incurred by the agents allocated to resource 1 $f(V')$ is strictly less than that of the agents allocated to resource 2. Let $c = \{i,j\}$ be an arbitrary coalition in C. Since for every $(u,v) \in E$, we have at least one of u and v in V', we have $|a_1 \cap c| \in \{1,2\}$. It is easy to see that none of the deviations of c is an improving deviation.

(If) Suppose that there exists a C*-stable* allocation a in $\mathcal{G}$. Since $N = 2k+1$, there exists a resource that is allocated to at most k agents (and thus the other resource is allocated to at least $k+1$ agents) by the pigeonhole principle. Without loss of generality, let resource 1 be this resource, i.e., $|a_1| \leq k$ and $|a_2| \geq k+1$. Since a is C-stable, a is c*-stable* for any $c \in C$. Let $c = \{i,j\}$ be an arbitrary coalition of the coalition structure C. Notice that $|a_1 \cap c| \in \{1,2\}$, since otherwise ($c \subseteq a_2$) the deviation where agent i moves to resource 1 is an improving deviation. Since $|a_1 \cap c| \in \{1,2\}$ for every $c \in C$, the subset of V that corresponds to the agents in a_1 covers all the edges in E. □

4. On efficient computation of equilibria in two-resource RSGs

This section is devoted to our computational tractability results in two-resource RSGs. More specifically, we show that given a two-resource RSG $\mathcal{G}$ and a coalition structure $C \subseteq \mathcal{P}_{\geq 1}(N)$, a C*-stable* allocation a can be computed in polynomial time if (i) C is laminar, or if (ii) C is contiguous and for each $c \in C$, $|c| \neq 2$.

Before proceeding with our results, we present first two characterization results from the existing literature. These results become useful in showing our results later on. The first result is due to Anshelevich et al. [1], who characterized the set of Nash equilibrium allocations in an RSG.

Characterization 1 *(Anshelevich et al. [1]) In an RSG $\mathcal{G} = \langle N, R, f \rangle$, an allocation a is a Nash equilibrium if and only if: (i) each resource $j \in T_1$ is allocated either q_j or $q_j - 1$ agents; (ii) a resource $j \in T_1$ is allocated exactly q_j agents; and (iii) each resource $j \in T_2$ is allocated exactly q_j agents.*

Let a be a Nash equilibrium allocation. At allocation a, we refer to a resource $j \in T_1$ as a *'high'* resource if $|a_j| = q_j$, and we denote the set of high resources with H. Similarly, we refer to a resource $j \in T_1$ as a *'low'* resource if $|a_j| = q_j - 1$, and we denote the set of low resources with L. Note that $H \neq \emptyset$ (due to Characterization 1), $H \cap L = \emptyset$, and $H \cup L = T_1$.

Now, consider a two-resource RSG $\mathcal{G}$. Note that RSGs are a subclass of congestion games, and therefore, in $\mathcal{G}$ there always exists a Nash equilibrium allocation. (See [18] for the existence of Nash equilibrium in congestion games.) Also, by Characterization 1 it is easy to see that a Nash equilibrium allocation can be efficiently computed (for instance, via a simple greedy algorithm; see [1]). At some Nash equilibrium allocation a, note that if we have $L = \emptyset$, it means that we have $L = \emptyset$ at *any* Nash equilibrium allocation. Also, $L = \emptyset$ implies that a is c*-stable* for each $c \subseteq N$ and hence a is C*-stable* for every coalition structure C. (These observations are straightforward. Nonetheless, if necessary, the reader may refer to Anshelevich et al. [1]). Therefore, in the rest of the paper, without loss of generality we will assume that in $\mathcal{G}$: (i) both resources are of type 1, and (ii) at every Nash equilibrium allocation a, one of the resources is a high resource and the other is a low resource. The second characterization result that we present, due to Caskurlu et al. [5], gives necessary and sufficient conditions for an allocation to be c-*stable* in two-resource RSGs.

Characterization 2 *(Caskurlu et al. [5]) Let* $\mathcal{G} = \langle N, R, f \rangle$ *be a two-resource RSG such that both of the resources are of type 1. Let* a *be a Nash equilibrium allocation such that one of the resources (say* h*) is a high resource, and the other (say* l*) is a low resource. Let* $c \in N$ *be a coalition of agents. The allocation* a *is* c-stable *if and only if the conditions below are satisfied:*

($C1$) If $|a_l \cap c| = 0$ then $|a_h \cap c| \leq 1$.

($C2$) If $\beta_h = \beta_l$ and $|a_l \cap c| > 0$ then $|a_h \cap c| \leq |a_l \cap c| + 1$.

($C3$) If $\beta_h < \beta_l$ and $|a_l \cap c| > 0$ then $|a_h \cap c| \leq |a_l \cap c|$.

Let $\mathcal{G} = \langle N, R, f \rangle$ be a two-resource RSG as described above: i.e., $|R| = |T_1| = 2$, and $|N| = q_1 + q_2 - 1$ (so that at a Nash equilibrium allocation, one resource in T_1 is high and the other is low). Without loss of generality, we assume that $\beta_1 \leq \beta_2$. Recall that we are studying the existence of laminar and contiguous equilibria in $\mathcal{G}$. We will study the cases when $\beta_1 = \beta_2$ and $\beta_1 < \beta_2$. We begin with the easier case, when $\beta_1 = \beta_2$. Below, in Lemma 1, we show that if $\beta_1 = \beta_2$, for any given contiguous coalition structure C there *always* exists an allocation a such that a is both C-*stable* and a Nash equilibrium, and this allocation can be computed efficiently. Note that this result is directly applicable for laminar coalition structures, too, since Caskurlu et al. [5] showed that a coalition structures which is laminar is also contiguous. We will deal with the case when $\beta_1 < \beta_2$ later on in Subsection 4.1.

Lemma 1 *Let* $\mathcal{G} = \langle N, R, f \rangle$ *be a two-resource RSG where both resources are of type 1 and* $\beta_1 = \beta_2$. *Let* C *be a contiguous coalition structure. Then, there exists an allocation* a *such that* a *is both* C-stable *and a Nash equilibrium, and such an allocation can be computed in polynomial time.*

Proof Let $\mathcal{G} = \langle N, R, f \rangle$ be a two-resource RSG where both resources are of type 1 and $\beta_1 = \beta_2$. Let C be a contiguous coalition structure. Without loss of generality, assume that: (i) $q_1 \leq q_2$, and (ii) for each $c \in C$, the agents in c are subsequently ordered under the identity permutation $\pi_I = (1, 2, \ldots, n)$ of N. Algorithm 1 below, which runs in polynomial time, returns an allocation as desired: i.e., it returns an allocation a such that a is C-*stable* and a Nash equilibrium. We explain below why Algorithm 1 returns the desired allocation a.

Algorithm 1 Polynomial-time algorithm that computes a C-*stable* allocation for the case $\beta_1 = \beta_2$.

Initially $a_1 = a_2 = \emptyset$
for $i = 1, \ldots, 2 \cdot q_1 - 1$ **do**
  if $i \mod 2 = 1$ **then**
    $a_1 = a_1 \cup \{i\}$
  else
    $a_2 = a_2 \cup \{i\}$
for $i = 2 \cdot q_1, \ldots, n$ **do**
  $a_2 = a_2 \cup \{i\}$

In Algorithm 1, agents are allocated to resources in a round-robin fashion until resource 1 is allocated q_1 agents and becomes a high resource. And then all the remaining agents are allocated to resource 2: In sum, resource 2 is allocated $q_2 - 1$ agents and it becomes a low resource. Let a be the allocation returned by Algorithm 1. Note that by Characterization 1, the allocation a is a Nash equilibrium. To see that a is C-*stable*, consider a coalition $c \in C$. Since C is contiguous, the agents in c are subsequently ordered in π_I. Let $c = \{i, i+1, \ldots j\}$ where $i, j \in N$ and $i \leq j$. We will show that a is c-*stable* using Characterization 2: Note that the condition $(C3)$ is not applicable since $\beta_1 = \beta_2$. Also, note that if $a_2 \cap c = \emptyset$ then $|c| = 1$, and hence at a the condition $(C1)$ is satisfied. Now, consider an agent $k \in c$ (i.e., $i \leq k \leq j$). Note that if $k \in a_1$ and $k < j$, Algorithm 1 allocates agent $k+1 \in c$ to resource 2. This implies that $|a_1 \cap c| \leq |a_2 \cap c| + 1$. But then the condition $(C2)$ of Characterization 2 is also satisfied. Therefore, a is c-*stable*. Since our choice of c was arbitrary, this also shows that the allocation a is C-*stable*. This completes our proof. □

4.1. Computing equilibrium using a fitting set

This subsection is devoted to our analysis when $\beta_1 < \beta_2$ in a two-resource RSG. Our findings are as follows: In Theorem 2, we show that for any laminar coalition structure C, we can compute in polynomial time an allocation a such that a is both C-*stable* and a Nash equilibrium. In Theorem 3, we show that for a contiguous coalition structure C, if no coalition in C consists of exactly two agents, then there exists an allocation a such that a is both C-*stable* and a Nash equilibrium, and this allocation can be computed in polynomial time. In our proofs, the main idea is as follows: For a given coalition structure C, we refer to a subset of agents $F \subset N$ satisfying some condition as a *fitting set*. When a fitting set exists for C, we present a polynomial-time algorithm that takes a fitting set as part of its input and then computes a C-*stable* allocation. We then conclude our proof(s) by showing that for a coalition structure C a fitting set always exists and it can be computed in polynomial-time if (i) C is laminar, or if (ii) C is a contiguous and for each $c \in C$, $|c| \neq 2$.

Our fitting set notion, defined below, is simple: Given a coalition structure C, a fitting set $F \subset N$ is a subset of agents such that for each $c \in C$, $|c| > 1$, the set F includes at least one agent in c but no more than half the agents in c. Note that it may be that for C a fitting set may not exist.

Definition 2 *A subset of agents $F \subset N$ is a fitting set for coalition structure C if for each $c \in C$ such that $|c| > 1$, F satisfies $1 \leq |F \cap c| \leq \lfloor \frac{|c|}{2} \rfloor$.*

Below in Lemma 2, using the notion of a fitting set we show the following: In a two-resource RSG, if a fitting set exists for coalition structure C, then we can compute in polynomial time an allocation a such that a is C-*stable* and a Nash equilibrium. Lemma 2 plays a key role in showing our Theorems 2 and 3.

Lemma 2 *Let $\mathcal{G} = \langle N, R, f \rangle$ be a two-resource RSG where both resources are of type 1 and $\beta_1 < \beta_2$. Let C be a coalition structure for which a fitting set F exists. Then, there exists an allocation a such that a is both* C-stable *and a Nash equilibrium, and given F such an allocation can be computed in polynomial time.*

Proof Let $\mathcal{G}$, C, and F be as described in the statement. Algorithm 2 below, which runs in polynomial time, returns an allocation as desired: i.e., it returns an allocation a such that a is C-*stable* and a Nash equilibrium. We explain below why Algorithm 2 returns the desired allocation a. Since Algorithm 2 conditions on the size of F, we will consider the two cases pertaining to the size of F separately.

Algorithm 2 Polynomial-time algorithm that computes a C-*stable* allocation for the case $\beta_1 < \beta_2$.

```
Initially a_1 = a_2 = ∅
if |F| ≤ q_1 − 1 then
    Let K ⊂ N be such that F ⊆ K and |K| = q_1 − 1
    a_1 = a_1 ∪ K
    a_2 = a_2 ∪ N \ K
else
    Let K ⊂ N be such that K ⊆ F and |K| = q_1
    a_1 = a_1 ∪ K
    a_2 = a_2 ∪ N \ K
```

(If) Suppose that $|F| \leq q_1 - 1$. Then, $a_1 = K$ where $F \subseteq K$ and $|K| = q_1 - 1$. Also, we have $|a_2| = q_2$ since $n = q_1 + q_2 - 1$. Hence, at a resource 1 is a low resource and resource 2 is a high resource. Note that by Characterization 1, the allocation a is a Nash equilibrium. To see that a is C-*stable*, consider a coalition $c \in C$. We will show that a is c-*stable* using Characterization 2: Note that the conditions $(C2)$ and $(C3)$ are not applicable since $\beta_1 < \beta_2$ and resource 1 is the low resource. Also, note that if $|c| = 1$, then a is c-*stable* since a is a Nash equilibrium. So, let $|c| > 1$. Note that Condition $(C1)$ of Characterization 2 holds at a since $|a_1 \cap c| \neq 0$. Therefore, a is c-*stable*. Since our choice of c was arbitrary, this also shows that the allocation a is C-*stable*.

(Else) Suppose that $|F| > q_1 - 1$. Then, $a_1 = K$ where $K \subseteq F$ and $|K| = q_1$. Also, we have $|a_2| = q_2 - 1$ since $n = q_1 + q_2 - 1$. Hence, at a resource 1 is a high resource and resource 2 is a low resource. Note that by Characterization 1, the allocation a is a Nash equilibrium. To see that a is C-*stable*, consider a coalition $c \in C$. We will show that a is c-*stable* using Characterization 2: Note that Condition $(C2)$ is not applicable since $\beta_1 < \beta_2$. Also, note that if $|c| = 1$, then a is c-*stable* since a is a Nash equilibrium. So, let $|c| > 1$. Note that Condition $(C1)$ holds at a since $|a_2 \cap c| \geq \lceil \frac{|c|}{2} \rceil > 0$. Also, note that Condition $(C3)$ of Characterization 2 also holds at a since $|a_2 \cap c| = |c| - |a_1 \cap c| \geq \lceil \frac{|c|}{2} \rceil \geq \lfloor \frac{|c|}{2} \rfloor \geq |a_1 \cap c|$. Therefore, a is c-*stable*. Since our choice of c was arbitrary, this also shows that the allocation a is C-*stable*. □

We are now ready to present Theorem 2.

Theorem 2 *Let $\mathcal{G} = \langle N, R, f \rangle$ be a two-resource RSG and let C be a laminar coalition structure. Then, there exists an allocation a, and which can be computed in polynomial time, such that a is both C-*stable *and a Nash equilibrium.*

Proof Let $\mathcal{G} = \langle N, R, f \rangle$ be a two-resource RSG and let C be a laminar coalition structure. All we need is to show that we can find a fitting set for C in polynomial-time.

For each $c \in C$ let $\mathcal{L}_{>1}(c) = \{c' \in C \mid c' \subsetneq c \text{ and } |c'| > 1\}$, i.e., $\mathcal{L}_{>1}(c)$ is the set of nonsingleton coalitions in C that is a subset of coalition c. Notice that $\mathcal{L}_{>1}(c') \subsetneq \mathcal{L}_{>1}(c)$ and $|c'| < |c|$ for each coalition $c' \in \mathcal{L}_{>1}(c)$.

Let $\mathcal{R}_{>1} = \{c \in C \mid \mathcal{L}_{>1}(c) = \emptyset \text{ and } |c| > 1\}$. Notice that a nonsingleton coalition $c \in C$ is in $\mathcal{R}_{>1}$ if and only if no nonsingleton subset $c' \subsetneq c$ is in C.

Let $c, c' \in \mathcal{R}_{>1}$ such that $c \neq c'$. We now show that c and c' are disjoint, i.e., $c \cap c' = \emptyset$. Assume that $c \cap c' \neq \emptyset$. Since C is a laminar coalition structure and $c, c' \in C$ it should be that $c \subset c'$ or $c' \subset c$. Without loss of generality, assume $c' \subset c$. Note that $|c'| > 1$ since $c' \in \mathcal{R}_{>1}$. But then $c' \in \mathcal{L}_{>1}(c)$. This contradicts with $c \in \mathcal{R}_{>1}$.

We construct a set of agents $F \subset N$ by choosing exactly one agent arbitrarily from each coalition in $\mathcal{R}_{>1}$. Note that F can be clearly computed in polynomial-time. We need to show that F is a fitting set, i.e., for each $c \in C$ where $|c| > 1$, it is the case that $1 \leq |F \cap c| \leq \lfloor \frac{|c|}{2} \rfloor$.

Let $c \in C$ be such that $|c| > 1$.

Assume that $c \in \mathcal{R}_{>1}$. Then, $|F \cap c| = 1$, since F is constructed by choosing exactly one agent from each coalition in $\mathcal{R}_{>1}$, and thus $1 \leq |F \cap c| \leq \lfloor \frac{|c|}{2} \rfloor$.

Assume that $c \notin \mathcal{R}_{>1}$. Then $\mathcal{L}_{>1}(c) \neq \emptyset$. We now show that $\mathcal{L}_{>1}(c) \cap \mathcal{R}_{>1} \neq \emptyset$. Let $c' \in \mathcal{L}_{>1}(c)$ be such that $|c'| \leq |c^*|$ for each $c^* \in \mathcal{L}_{>1}(c)$. Assume that $c' \notin \mathcal{R}_{>1}$. But then $\mathcal{L}_{>1}(c') \neq \emptyset$ since $|c'| > 1$. This contradicts that $|c'| \leq |c^*|$ for each $c^* \in \mathcal{L}_{>1}(c)$. Thus, $c' \in \mathcal{R}_{>1}$, and $\mathcal{L}_{>1}(c) \cap \mathcal{R}_{>1} \neq \emptyset$. Let $k = |\mathcal{L}_{>1}(c) \cap \mathcal{R}_{>1}|$. Notice that $|F \cap c| = k$ since F is constructed by choosing exactly one agent from each coalition in $\mathcal{R}_{>1}$. We know that $k \geq 1$. Since the coalitions in $\mathcal{R}_{>1}$ are disjoint and no coalition in $\mathcal{R}_{>1}$ is a singleton coalition, we have $|c| \geq 2k$. Thus, $1 \leq |F \cap c| \leq \lfloor \frac{|c|}{2} \rfloor$.

This completes our proof. □

As shown in Theorem 2, for each laminar coalition structure C there exists a fitting set F. As it turns out, however, this finding cannot be generalized for contiguous coalition structures. We give an example below in Figure 2: Consider the coalition structure C such that the circles in Figure 2 correspond to the coalitions in C (i.e., a circle in the figure corresponds to the coalition consisting of agents that lie inside that circle). Obviously, the coalition structure C is contiguous. It is easy to verify that there exists no fitting set for C. This is indeed not surprising: In a two-resource RSG, by Lemmas 1 and 2, we know that a C-*stable* allocation exists if C has a fitting set. But Caskurlu et al. [5] showed via an example that in a two-resource RSG a C-*stable* allocation may not exist for a contiguous coalition structure. Indeed, in their example, the coalition structure considered is precisely the one depicted in Figure 2. Nonetheless, below in Theorem 3, we show that a positive existence result can be obtained if no coalition consists of exactly two agents: We show that in a two-resource RSG, for a contiguous coalition structure C such that for each $c \in C$, $|c| \neq 2$, there always exists a fitting set, and hence, an allocation always exists, and can be computed in polynomial time, which is C-*stable* and a Nash equilibrium.

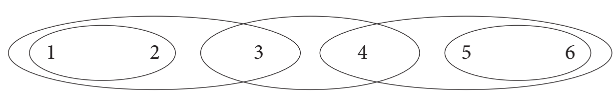


Figure 2. A contiguous coalition structure without a fitting set.

Theorem 3 *Let* $\mathcal{G} = \langle N, R, f \rangle$ *be a two-resource RSG. Let* C *be a contiguous coalition structure such that* $|c| \neq 2$ *for each* $c \in C$. *Then, there exists an allocation* a, *and which can be computed in polynomial time, such that* a *is both* C-stable *and a Nash equilibrium.*

Proof Let $\mathcal{G} = \langle N, R, f \rangle$ be a two-resource RSG and let C be a contiguous coalition structure such that $|c| \neq 2$ for each $c \in C$. Without loss of generality, suppose that for each $c \in C$ the agents in c are subsequently ordered under the identity permutation $\pi_I = (1, 2, \ldots, n)$ of N. By Lemmas 1 and 2, to show the desired result it is sufficient to show that we can find a fitting set for C in polynomial-time.

For agent $i \in N$, let $e(i) = \{c \in C \mid i \in c \text{ and } |c| > 1\}$; i.e., $e(i)$ is the set of nonsingleton coalitions in C that include agent i. We say that i *covers* the nonsingleton coalitions in $e(i)$. (The idea is that when we pick agent i, we would be picking at least one agent from each coalition in $e(i)$.) For a subset of agents F, let $E(F)$ denote the set of nonsingleton coalitions covered by agents in F (i.e., $E(F) = \bigcup_{i \in F} e(i)$). Algorithm 3 below constructs a set F which turns out to be a fitting set for C and hence proves the desired result. To ease of understanding, we first explain verbally what Algorithm 3 does. Initially the set F is empty. Then we iterate over agents in N in ascending order and update the set F as follows:

- If $F = \emptyset$, we add agent i to the set F and proceed (with the next agent in the ascending order).
- If $F \neq \emptyset$, let j be the last agent added to the set F. Then we check whether $E(F \cup \{i\})$ is equal to $E(F \cup \{i\} \setminus \{j\})$. If the answer is yes, we remove agent j from F, add agent i to F, and then proceed. If the answer is no, we check whether or not the set $E(F)$ expands if agent i is added to F. If the answer is yes, we add agent i to F and then proceed. If the answer is no, we keep F unchanged and proceed.

In Algorithm 3, the set of agents F is implemented as a *stack* data structure. Note that the *pop*() method removes and returns the least recently added element of the stack, while the *peek*() method only returns (does not remove) the least recently added element of the stack. If F is empty, then *pop*() and *peek*() methods return $\emptyset$.

Algorithm 3 Polynomial-time algorithm that computes a fitting set.

1: **Let** F **be an *empty set of agents*.**
2: **for *agent*** $i = 1, \ldots, n$ **do**
3:     **if** $E(F \cup \{i\}) = E(F \setminus \{F.peek()\} \cup \{i\})$ **then**
4:         $F.pop()$
5:         $F.push(i)$
6:     **else if** $e(i) \not\subseteq E(F)$ **then**
7:         $F.push(i)$
8: **return** F

To show that the set F returned by Algorithm 3 is a fitting set for C, we need to show that for each $c \in C$, $|c| > 1$, we have $1 \leq |F \cap c| \leq \lfloor \frac{|c|}{2} \rfloor$. Consider an arbitrary c such that $c \in C$ and $|c| > 1$.

We now show that $1 \leq |F \cap c|$. By way of contradiction, suppose that $F \cap c = \emptyset$, i.e., $c \notin E(F)$. Note that when Algorithm 3 runs if at some point a nonsingleton coalition $c' \in C$ is covered by F, then c' continues to be covered until the algorithm terminates. Since $F \cap c = \emptyset$, it means that no agent in c is ever pushed to the stack throughout the execution of the algorithm. Consider an agent $i \in N$. Then, in the algorithm, before the i^{th} iteration, we must have $e(i) \not\subseteq E(F)$. But then it is clear that Algorithm 3 pushes agent i to the stack at the i^{th} iteration (at line 7 of Algorithm 3), contradicting that $F \cap c = \emptyset$. Thus, we obtain that $1 \leq |F \cap c|$.

We now show that $|F \cap c| \leq \lfloor \frac{|c|}{2} \rfloor$. Let $i, j \in F$ be such that $j > i$. We first show that $j > i + 2$. By way of contradiction, suppose that $j \in \{i + 1, i + 2\}$.

Suppose that $j \neq n$. Then, $j + 1 \in N$. Consider the $(j + 1)^{\text{th}}$ iteration of the algorithm. Consider a coalition $c' \in C$ such that $c' \in e(j)$. (Note that by definitions of $e(j)$ and C, $|c'| \neq 1$ and $|c'| \neq 2$.) It is easy to observe that either $c' \in e(i)$ or $c' \in e(j + 1)$. But then, since $i \in F$, j is replaced by $j + 1$ at this iteration. But then $j \notin F$, a contradiction.

Now, suppose that $j = n$. Consider a coalition $c' \in C$ such that $c' \in e(j)$. But then $c' \in e(i)$ since $|c| \geq 3$. But then j would not be pushed to the stack in the j^{th} iteration of the algorithm without i being popped, contradicting that $i, j \in F$.

Thus, we conclude that if $i, j \in F$ and $j > i$ then $j > i+2$. In other words, we showed that if $i \in F$ then $i+1 \notin F$ and $i+2 \notin F$. Therefore, we obtain that $|F \cap c| \leq \lceil \frac{|c|}{3} \rceil$ and $\lceil \frac{|c|}{3} \rceil \leq \lfloor \frac{|c|}{2} \rfloor$ (since $|c| \geq 3$). This concludes our proof. □

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