Turkish Journal of Electrical Engineering & Computer Sciences

http://journals.tubitak.gov.tr/elektrik/

Research Article

Turk J Elec Eng & Comp Sci
(2020) 28: 1905 – 1915

doi:10.3906/elk-1909-102

Bases of polymatroids and problems on graphs

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Received: 19.09.2019 • **Accepted/Published Online:** 04.05.2020 • **Final Version:** 29.07.2020

Abstract: In the paper, we present new theorems to show that a Hamiltonian path and circuit on an undirected graph can be formulated in terms of bases of polymatroids or extended polymatroids associated with submodular functions defined on subsets of the node-set of a given graph. In this way, we give a new formulation of the well-known traveling salesman problem including constraints in these terms. The main result in the paper states that using a special base of the polymatroid, a Hamiltonian path on an undirected graph can be solved effectively. Since the determination of a Hamiltonian circuit can be reduced to finding a Hamiltonian path between some node and its adjacent nodes, an efficient Hamiltonian path algorithm will lead to solving the Hamiltonian circuit problem. Finding some special base is the main problem in solving these NP-hard problems.

Key words: Submodular function, bases of polymatroid, hamiltonian path and circuit, traveling salesman problem

1. Introduction

The theory of polymatroids has many applications in finding an optimal solution to many combinatorial optimization problems. This theory can also be used in designing approximation algorithms for some real-life problems. A deep understanding of the theory enables us to solve optimization problems over polymatroids structures. In this paper, we show that a Hamiltonian path and a circuit on an undirected graph can be formulated in terms of bases of polymatroids. *The Hamiltonian path problem* is a particular case of the following problem (as described below) as well as many other graph theoretical problems.

Let $G = (V, E)$ be a given undirected simple graph with the node set V and the edge set E, where $|V| = n$, $|E| = m$. Let d_v denote the degree of a node $v \in V$ and define $d := (d_v : v \in V)$. It is required to define a spanning subgraph $G_0 = (V, E_0)$ of $G = (V, E)$ under the condition that $b_v \leq d_v - k$, for all $v \in V$ where $k \geq 0$ is an integer and b_v is the degree of node v in G_0. Note that b_v can be zero in the subgraph $G_0 = (V, E_0)$ for some nodes v, that is, G_0 may contain some isolated nodes. As a matter of fact, in many combinatorial problems on graphs we have $b_v = 0$. These combinatorial problems can be formulated in terms of bases of polymatroids (or extended polymatriods) associated with certain submodular functions [1, 2] defined on subsets of node sets of given graphs.

The well-known maximum weight spanning tree problem is reduced to finding a maximum weight spanning subgraph G_0 for which the sum of node degrees is $2(m-n+1)$ and $b_v \geq 1$ for all $v \in V$ (see [3]). In [4], it was shown that a bipartite graph has a perfect matching if and only if the node degree vector $b = d - 1$ is a base of an extended polymatroid. Note that, a similar result has been shown in [5] for nonbipartite graphs. Based

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on this result an $O(n^3)$ algorithm was proposed for finding a perfectly matchable subgraph in a given bipartite graph.

This paper aims to show that a Hamiltonian path and circuit on an undirected graph can be formulated in terms of bases of polymatroids or extended polymatroids. This will allow us to reformulate the well-known "traveling salesman problem". It is known that related problems are NP-hard. Therefore, determining whether a graph has a Hamiltonian path or a circuit has a high time complexity. The Hamiltonian circuit problem has many applications such as time scheduling, the choice of travel routes and network topology [6, 7]. Thus, this paper will address a very important problem in graph theory and computer science.

Due to their similarities, problems related to Hamiltonian cycles are usually compared with Euler's problem. However, the techniques of solutions for these problems are very different. There is a very elegant (necessary and sufficient) condition for a graph to have Euler cycles. In the literature, there are many solutions that generate efficient algorithms for finding Euler Cycles. Some theorems provide sufficient conditions for the existence of a Hamiltonian circuit (see [8, 9]). However, the heavy conditions in those theorems make the results not applicable and make them unrealistic for practical use.

The next section is devoted to definitions and notations that will be used throughout the paper.

2. Basic notions and preliminary results

Let $G = (V, E)$ be a graph. For a vector $a \in R^V$ (or $a \in R^E$) and a subset $S \subseteq V$ (or $S \subseteq E$), define $a(S) = \sum_{v \in S} a_v$. Denote by $\overline{S} = V \setminus S$, the complement of S in V. Let $\gamma(S)$ and $\kappa(S)$ denote the subsets of edges having at least one of endpoints in $S \subseteq V$ and both endpoints in $S \subseteq V$, respectively. Consider the functions $f(S) = |(\gamma(S)|$ and $g(S) = |(\kappa(S)|$, defined on subsets of V. Obviously, both $f(S)$ and $g(S)$ are monotone functions. Moreover, it is well known that f is submodular and g is supermodular [1]. Hence, the function $f(S) - g(S)$ is submodular as well. The cut determined by a subset $S \subset V$ is denoted by $\delta(S)$. An edge with endpoints v and u is denoted by (v, u) and uv denotes the arc with the u and the tail v. We call the vector $d = (d_v : \; v \in V)$ as *the degree vector* of the graph G. We write $v \prec_L u$ if v precedes u in the linear ordering L of the nodes. From the definitions of the sets $\gamma(S)$ and $\kappa(S)$, it follows that

$$f(S) + g(S) = d(S),$$

and

$$f(S) - g(S) = |\delta(S)|$$

for the cut $\delta(S)$ determined by any $S \subset V$. The following sets in R^V associated with the functions f and g are called polymatroid and superpolymatroid [1, 2], respectively.

$$P(f) = \{x \in R^V : \; x \geq 0, \; x(S) \leq f(S), \; S \subseteq V\},$$

$$Q(g) = \{y \in R^V : \; y \geq 0, \; y(S) \geq g(S), \; S \subseteq V\}.$$

The following polytope associated with the functions $f - g$ is called extended polymatroid

$$EP(f-g) = \{w \in R^V ; \; w(S) \leq f(S) - g(S), \; S \subseteq V\}.$$

The vectors $x \in P(f)$ and $y \in Q(g)$ are called bases of the polymatroid and the superpolymatroid if $x(V) = f(V)$ and $y(V) = g(V)$, respectively. For any bases $x \in P(f)$ and $y \in Q(g)$, since $\gamma(V) = E = \kappa(V)$, (by the definition of the sets $\gamma(S)$ and $\kappa(S)$), we have

$$x(V) - y(V) = f(V) - g(V) = 0.$$

A vector $w \in EP(f-g)$ is a base of $EP(f-g)$ if $w(V) = 0$.

Let $x^L \in P(f)$ and $y^L \in Q(g)$ be the bases computed by the greedy algorithm with respect to any linear ordering L of the nodes. Our first observation is that, the difference $w^L = x^L - y^L$ of the bases x^L and y^L is a base of $EP(f-g)$ which can be also found by the greedy algorithm with respect to the linear ordering L of the nodes. From now on, we will write x, y and w for x^L, y^L and w^L, respectively when there is no ambiguity. Note that the following zero-sum equality

$$\sum_{w_v > 0} w_v = -\sum_{w_u \leq 0} w_u$$

holds for the base $w = x - y$.

According to the linear ordering of L nodes, one can orient the edges of the graph $G = (V, E)$ in such a way that the resulting digraph is an acyclic oriented graph. This requires each edge (v, u) to be replaced by an arc vu if $v \prec_L u$ or by an arc uv if $u \prec_L v$. The converse is also true: Each acyclic orientation of the edges of the graph $G = (V, E)$ defines a linear ordering L of its nodes. In an acyclic oriented graph $G = (V, E)$ with weights c_{vw} on arcs, let $\delta_+(v)$ be the set of arcs entering to node v and let $\delta_-(v)$ be the set of arcs leaving from node v.

We observe that the bases $x \in P(f)$ and $y \in Q(g)$ satisfy the equalities

$$|\delta_+(v)| = x_v, \ v \in V, \tag{1}$$

$$|\delta_-(v)| = y_v, \ v \in V. \tag{2}$$

In other words, x_v is the number of arcs leaving the node v and y_v is the number arcs entering the node v. In [1], it was shown that the equalities above are satisfied by any bases of $x \in P(f)$ and $y \in Q(g)$, where x and y are computed with respect to any linear ordering of the nodes in any graph. In [1], it was also proved that:

Claim 1 *If $x \in P(f)$ and $y \in Q(g)$ are any bases computed by the greedy algorithm with respect to any linear ordering L of the nodes, then*

$$x + y = d$$

and the difference $x - y = w$ is a base of $E(f-g)$, for which

$$\sum_{w_v > 0} w_v = -\sum_{w_v \leq 0} w_v.$$

Claim 2 . *For a given linear ordering $L = \{v_1, ...v_n\}$ of the nodes in V, the bases $x(L) \in P(f)$, $y(L) \in Q(g)$ and $w(L) \in EP(f-g)$ can be found in $O(m)$ time, where $n = |V|$ and $m = |E|$.*

Let $L = \{v_1, v_2, ..., v_n\}$ and $I = \{v_n, ..., v_2, v_1\}$ be two linear orderings of nodes.

Claim 3 *If the greedy algorithm defines the bases $x^1 \in P(f)$ and $x^2 \in P(f)$ with respect to the L and I, respectively, then $x^2 = y^1 = d - x^1 \in Q(g)$ and $x^1 = y^2 = d - x^2 \in Q(g)$.*

We will use the preceding claims in the theorems we will prove in the next section.

3. Hamiltonian path and bases of polymatroid

Let us consider an undirected graph $G = (V, E)$ with node set V, edge set E and unit weights on edges. Without loss of generality, we can assume that $d_v > 2$ for any node $v \in V$. Now consider the problem of defining a subgraph $G_0 = (V, E_0)$ of G such that $b_v = d_v - 2$ for all nodes in V, where the vector $b = (b_v; v \in V)$ denotes the degree of nodes in G_0. It is easy to see that after deleting edges in any set of all edge-disjoint cycles in the graph $G = (V, E)$, the resulting subgraph G_0 has the nodes with degree b_v. Hence, obtaining of G_0 is equivalent to finding a set of edge-disjoint cycles (EDCS) in the graph G. A solution to the latter problem can be found by solving the system of the linear equations similar to assignment constraints on the bipartite graph which is usually used in the formulation of the well known traveling salesman problem (see [10]).

Now, consider EDCS with the additional requirement that G_0 which is obtained by deleting edges of any Hamiltonian circuit in G is a spanning subgraph of G. Since $f(S) - g(S) = |\delta(S)|$ by Claim 2, this problem can be reduced to finding a 0 or 1 solution to the following system of linear equations and inequalities:

$$z(\delta(w)) = b_v, v \in V, \tag{3}$$

$$z(\delta(S)) \leq f(S) - g(S) - 2 \text{ for all } \emptyset \neq S \subset V. \tag{4}$$

where $z = (z_e : e \in E)$.

In fact, for $u = 1 - z$, the conditions (3) and (4) are transformed into the constraints of many classic formulations of the Hamiltonian circuit problems used in various models of the traveling salesman problem. Restrictions $z_e = 0$ or 1 for each $e \in E$ allow us to use different algorithms based on the branch and bound methods in the solution of the latter problem. Since the separation problem for (4) and vector $z \in R^E$ are reduced to finding at most n minimum cut problems [11], testing the validity of (4) can be checked in polynomial-time in all iterations of the branch and cut type methods. Below, we show that the equations (3) and (4) can be formulated using the variables for each node of the graph G.

Let $G_0 = (V, E_0)$ be a spanning subgraph obtained after removing the edges of a Hamiltonian circuit in some graph G. One can formally define the functions $f_0(S)$ and $g_0(S)$ with respect to the subgraph G_0 for any subset S of V. Let us define $P_0 = P(f_0)$, $Q_0 = Q(g_0)$ and $EP_0 = EP(f_0 - g_0)$. Using the claims in section 2, the degree b_v of nodes v in G_0, the equality

$$b_v = x_v^0 + y_v^0$$

must hold for all $x^0 \in P_0$ and $y^0 = b - x^0 \in Q_0$, where x^0 is a base defined by the greedy algorithm with respect to any linear ordering L of the nodes. In addition to the bases x^0 and y^0, consider the bases $x \in P(f)$ and $y \in Q(g)$ computed by the greedy algorithm with respect to L. Since f and g are monotone functions, $x_v \geq 0$ and $y_v \geq 0$ for any node $v \in V$. Now using the bases x^0 and y^0, we can define the spanning subgraph $G_0 = (V, E_0)$, where the edges in $\delta_+(v)$ are arcs leaving from node v and edges in $\delta_-(v)$ are arcs entering to node v in V. The condition $b_v = d_v - 2$, and the equalities (1) and (2) imply that one of the following equalities

$$x_v - x_v^0 = 2,\ y_v - y_v^0 = 0, \tag{5}$$

$$x_v - x_v^0 = 0,\ y_v - y_v^0 = 2, \tag{6}$$

$$x_v - x_v^0 = 1,\ y_v - y_v^0 = 1 \tag{7}$$

holds for each node $v \in V$.

Since G_0 itself is a graph, the zero sum equality holds for the base $z^0 = x^0 - y^0$ in EP_0. Thus, the base $x^0 \in P_0$ and $y^0 \in Q_0$ can be defined by using (5)-(7) to construct the required spanning subgraph G_0. In order to construct G_0, we need to consider acyclic or-graph $G(L)$ obtained by orienting the edges in E in accordance with L. Then, it is necessary to use x_v and y_v (the components of the bases x and y) in such a way that the conditions $x_v^0 = |\delta_+(v)|$ and $y_v^0 = |\delta_-(v)|$ hold in $G_0 = (V, E_0)$ (see (1) and (2)). Since the determination of a Hamiltonian circuit can be reduced to finding a Hamiltonian path between nodes v_1 and w for all adjacent nodes w with v_1, we first show that some bases of $P(f)$ can be used for this purpose. Let the degree d_{v_1} of a node v_1 be minimum. Clearly, a Hamiltonian path between nodes v_1 and w and the edge (v_1, w) together is a Hamiltonian circuit. Suppose that G contains the edge (v_1, v_n) and we are seeking a Hamiltonian path Γ between the nodes v_1 and v_n. If a Hamiltonian path Γ visits nodes according to the order in $L = \{v_1, v_2, ..., v_n\}$, then using the greedy formulas for computing the bases $x \in P(f)$ and $y \in Q(g)$ with respect to L, we have $x_1 = d_1$, $y_1 = 0$ and $x_n = 0$, $y_n = d_n$. Obviously, if G_0 is a spanning subgraph obtained after deleting all edges of Γ in G, then $x_v - x_v^0 = 1$, $y_v - y_v^0 = 0$ when $v = v_1$ and $x_v - x_v^0 = 0$, $y_v - y_v^0 = 1$ when $v = v_n$, for the bases $x^0 \in P_0$ and $y^0 \in Q_0$ computed with respect to L. Let b be a vector, where $b_v = d_v - 1$ for $v = v_1, v_n$ and $b_v = d_v - 2$ for each $v \in V \setminus \{v_1, v_n\}$.

Theorem 1 *Let $L = \{v_1, v_2, ..., v_n\}$ be a linear ordering of nodes and let x be a base of P(f) and $y = d - x$. The graph $G = (V, E)$ has a Hamiltonian path between nodes v_1 and v_n if and only if the vectors x^0 and y^0 defined by using the equalities (5)-(7) satisfy the following conditions:*

- $x_v^0 = d_v - 1$, $y_v^0 = 0$ *when* $v = v_1$,
- $x_v^0 = 0$, $y_v^0 = d_v - 1$ *when* $v = v_n$,
- $b_v = x_v^0 + y_v^0$ *for each* $v \in V \setminus \{v_1, v_n\}$.

Proof ($\Rightarrow$) Let us assume that the graph $G = (V, E)$ has a Hamiltonian path between nodes v_1 and v_n that visits nodes according to an order $L = \{v_1, v_2, ..., v_n\}$. Let $G_{1n} = (V, E_{1n})$ denote the graph obtained deleting the edge (v_1, v_n) in G. Since the graph $G_{1n} = (V, E_{1n})$ has a Hamiltonian path, it implies that there are arcs wv and uv in the $G_{1n}(L)$ for any $v \neq v_1, v_n$. Let x be a base defined by the greedy algorithm with respect to L. Since G has a Hamiltonian path and $f(S)$ is a monotone function, $0 < x_v < d_v$ for any $v \neq v_1, v_n$ in V. Defining x_v^0 and y_v^0 using the equality (7) requires that the edges in the Hamiltonian path are to be deleted in G_{1n}. Hence, we can set $x_v^0 = d_1 - 1$, $y_v = y_v^0$ for $v = v_1$ and $x_v = x_v^0$, $y_v^0 = d_n - 1$ for $v = v_n$ and $b_v = x_v^0 + y_v^0$ for $v \in V \setminus \{v_1, v_n\}$.

($\Leftarrow$) Since we seek a Hamiltonian path between nodes v_1 and v_n, without loss of generality, we assume that $v_1 \prec_L v$ and $v \prec_L v_n$ for any node $v \in V \setminus \{v_1, v_n\}$ in any linear ordering L of nodes. Let x be a base of the polymatroid $P(f)$ defined with respect to some linear ordering L of the nodes in V and $y = d - x$. Then we can set $x_v - x_v^0 = 1$ for $v = v_1$ and $y_v - y_v^0 = 1$ for $v = v_n$ since the graph G is a connected graph. Let x_v^0 and y_v^0 be defined using the equalities (5)-(7) for nodes $v = v_2, ..., v_{n-1}$ such that the condition $b_v = x_v^0 + y_v^0$

holds for $v \in V$. Suppose that graph G has not a Hamiltonian path between nodes v_1 and v_n. Then, in G, there is no a spanning tree T having nodes with degree $<= 2$ (see [10]). That is, in T, at least 1 node v_k has degree > 2. To obtain a subgraph $G_0 = (V, E_{1n})$ to determine x_v^0 and y_v^0 with respect to any linear ordering $L = \{v_1, v_2, ..., v_n\}$ by using the equalities (5)-(7), one must delete at least 3 edges incident to the node v_k. Hence,the last condition does not hold for x_v^0 and y_v^0 defined by (5)-(7) when $v = v_k$, which contradicts with the theorem. □

We are going to use this theorem to give a new formulation for TSP as follows. First, note that a Hamiltonian circuit cannot be defined in the same way as a Hamiltonian path, since the subgraph G_0 with degree b_v can be obtained by deleting edges in any set of subcycles in G, even if

$$x_v^0 + y_v^0 = b_v = d_v - 2,\ v \in V$$

for x_v^0 and y_v^0 (defined by the equalities (5)-(7)) used for the construction of G_0. Consequently, in order to define a Hamiltonian circuit in the spirit of Theorem 1, some additional subcycle deletion constraints must be held for x_v^0 and y_v^0 defined by (5)-(7).

Theorem 1 states that: The vectors $x^0 = (x_v^0; v \in V)$ and $y^0 = (y_v^0; v \in V)$, defined by removing some incident arcs with each node v, are bases of P_0 and Q_0 defined by the greedy algorithm with respect to L as well. That is, $y^0 = b - x^0$. Therefore, in order to a spanning subgraph $G_0 = (V, E_0)$ obtained after deleting a Hamiltonian path between some pair of nodes v_1 and v_n, we must first define some linear ordering L of the nodes.

However, when $x_v \geq 2$ and $y_v \geq 2$ for some nodes $v \in V$, there are some uncertainties in the determination of the values x_v^0 and y_v^0 for node v, since the choice of equality in (5)-(7) is not clear for computing x_v^0 and y_v^0. The reason is that each equality in (5)-(7) is suitable for this case. But, we can choose the equality (5) when $x_v = 0$ and (6) when $y_v = 0$ to define x_v^0 and y_v^0 for v. In order to reduce ambiguity, it is necessary to compute a base $x \in P(f)$ such that either $x_v = 0$ or $y_v = 0$ for the maximum number of nodes v in V, where $y = d - x$. Such a base can be defined after finding a maximum independent set Π of nodes in G, since either $x_v = 0$ or $y_v = 0$ for $v \in \Pi$, by the formulas of the greedy algorithm when $x \in P(f)$ is defined with respect to the linear ordering $L = (\Pi, V \setminus \Pi)$ of nodes. Since the maximum independent set problem is NP-hard [12], it is impractical to use this way directly to define x^0 and y^0.

Now, we consider which base of $P(f)$ is more suitable for finding a Hamiltonian path Γ. Let Γ visit nodes according to its order in $L = \{v_1, v_2, ..., v_n\}$.

Definition 1 *A linear ordering $L = \{v_1, v_2, ..., v_n\}$ of nodes is called tracing if the inequality $0 < x_v < d_v$ holds for any node $v \neq v_1, v_n$ and the base $x = (x_v : v \in V) \in P(f)$ defined by the greedy algorithm with respect to L.*

For convenience, let us denote by $L(1, n)$ a tracing linear ordering L such that $v_1 \prec_L v$ and $v \prec_L v_n$ for any $v \neq v_1, v_n$. Let us define $t_{v_i} = t_i$ for $t = x, y, z, x^0, y^0, z^0$. It is easy to show that the visiting order of nodes in a Hamiltonian path is a tracing linear order of nodes, but the inverse is not true. Suppose that $L = \{v_1, v_2, ..., v_n\}$ is a tracing linear ordering. Let $G(L)$ be the acyclic graph obtained after orienting the edges in E according to the tracing linear ordering L of nodes. Now, using equality (7), we can remove 2 arcs

vw (leaving the node v) and uv (entering the node uv) in G. Set $x_v - x_v^0 = 1$, $y_v - y_v^0 = 1$ for any node $v \neq v_1, v_n$ and $x_1 - x_1^0 = 1$, $y_1 = y_1^0$, $x_n = x_n^0$, $y_n - y_n^0 = 1$. However, since vw is an arc entering w and uv is an arc leaving u, deleting these arcs may result in $b_t \neq x_t^0 + y_t^0$ for node $t = u$ or $t = w$.

In order to get a tracing linear ordering of nodes, we may assume that the graph G does not have a node v with $d_v \leq 2$, using a well known theorem in [3], which states that: A sequence of integers $b_1, b_2, ..., b_n$ are the degrees of nodes of a simple k-edge connected graph with n nodes if and only if $b_i \geq k$ for all $i = 1, ..., n$. Recall that the number of edge connectivity of a graph is the number of edges in a minimum cut in G. The proof of this theorem in [13] can be viewed as the algorithm required to construct the required graph.

Let k be the edge connectivity of a graph $G = (V, E)$. Thus, the edge connectivity of a subgraph $G_0 = (V, E_0)$ cannot be more than $k - 1$. Clearly, for the case $k = 2$, either $b_v = 1$ or $b_v = 0$. Therefore, a node v can be removed from G if $d_v \leq 2$. This is because the edges (u, v) and (v, w) can be replaced by the edge (u, w) to determine the values of x_v^0 and y_v^0. However, we need to keep the inequality $b_v \geq k - 1$ when edges incident to v are deleted for nodes $v = v_1, v_2, ..., v_n$ according to tracing linear ordering $L = \{v_1, v_2, ..., v_n\}$.

In order to determine a tracing linear ordering $L(1, n)$ of nodes, we can use the idea of the well-known depth-first search traversal of nodes as follows: First, we set $L(1, n) := \{v_1\}$ and mark all edges if one of the end nodes is v_1. In each step, we choose an adjacent node $v \neq v_n$ with one of the nodes in $L(1, n)$ so that there is at least 1 unmarked edge with one of the end nodes v in the graph $G = (V, E)$. Then, we mark all the unmarked edges with one of end nodes v in G and set $L(1, n) = \{L(1, n), v\}$.
This process is repeated until either all edges with one of the end nodes v_n are marked or some node $s \notin L(1, n)$ whose all incident edges are marked is chosen. In the first case, $L = (v_1, v_2, ..., v_n)$ is a tracing linear ordering of nodes, since there is at least one unmarked edge at each step that implies $0 < x_v < d_v$ for any $v \neq v_1, v_n$. In the second case, if a node $s \neq v_n$ with at least 1 unmarked edge cannot be chosen. Then all edges (s, w) are marked and $w \in L(1, n)$. Hence, all paths connecting the node s and every node $v \notin L(1, n)$ contains marked edges. Let $v \notin L(1, n)$ be a node so that there is at least 1 marked edge (w, v) with end node $w \in L(1, n)$. Consider a shortest path between s and v passing through nodes $s = s_1, s_2, ..., s_{p+1} = v$. Hence, $s_2, ..., s_p \in L(1, n)$ and $s_k \prec s$ for $k = 2, ..., p$. Then, in $L(1, n)$, the order of nodes $s = s_1, s_2, ..., s_p$ can be changed as follows:

$$s \prec s_2, ..., \prec s_p \prec v$$

, and we set $L(1, n) := (L(1, n), v)$. By repeating this process, all edges incident to the node v_n can be either marked or not marked. In the latter case, the graph does not have the required Hamiltonian path, in the former case $L(1, n)$ is a tracing linear ordering.

The definition of tracing $L(1, n)$ follows that it can be defined in polynomial-time, since for going out from node s, enumerating shortest paths can be defined by one of the well-known polynomial-time algorithms when all edges have unit weight. Therefore, using a tracing linear ordering can lead to new simple Hamiltonian path algorithms by carefully choosing deleting edges for conforming that $b_v = x_v^0 + y_v^0$ in $G_0 = (V, E_0)$.

The computational testing of such an algorithm is the topic of a future project. To clarify some difficulty arising in the work of this type algorithm, the reader can try to find $L(1, 6)$ with respect to the graph in Figure 1, if $L(1, 6) = \{v_1, v_4\}$ at the second step.

In the next section, we give a new formulation for the traveling salesman problem based on the above results.

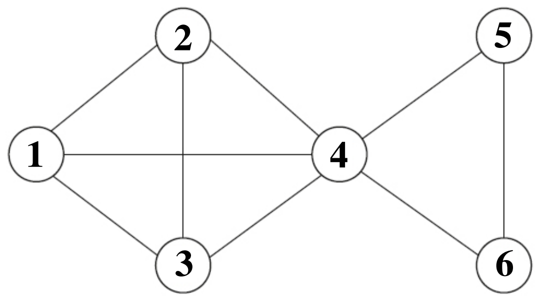

Figure 1. Non-Hamiltonian graph $G = (V, E)$.

4. Traveling salesman problem formulation

The purpose of this section is to obtain a new formulation for the traveling salesman problem in variables for nodes. In order to achieve this purpose, we first consider a system of linear inequalities that can be used in describing an unknown spanning subgraph $G_0 = (V, E_0)$ obtained by deleting the edges of a Hamiltonian circuit in the graph G. We denote the polymatroid and the superpolymatroid with respect to the spanning subgraph $G_0 = (V, E_0)$ by P_0 and Q_0, respectively. Let $\overline{x} \in P(f)$ be a base defined by the greedy algorithm with respect to some linear ordering L of nodes and $\overline{y} = d - \overline{x}$. Let $h \in P_0$ be an unknown base which is also defined by the greedy algorithm with respect to L. Then, the equalities (5)-(7) can be rewritten as

$$0 \leq \overline{x}_v - h_v \leq 2, \text{ for all } v \in V, \tag{8}$$

$$0 \leq \overline{y}_v - t_v \leq 2, \text{ for all } v \in V. \tag{9}$$

In addition to these constraints, the bases $h \in P_0$ and $t \in Q_0$ must satisfy condition

$$h_v + t_v = d_v - 2, \text{ for all } v \in V, \tag{10}$$

$$h(V) - t(V) = 0, \tag{11}$$

as $G_0 = (V, E_0)$ is itself a graph.

Definition 2 *We say that numbers h_v and t_v realize the acyclic copy $G(L)$ of a spanning subgraph $G_0 = (V, E_0)$ obtained by orienting its edges according to the linear ordering L (the greedy algorithm defines a base $\overline{x} \in P(f)$) if $h_v = |\delta_+(v)|$ and $t_v = |\delta_-(v)|$, where h_v is the number of the leaving edges from a node v and t_v is the number of entering edges to v in $G(L)$.*

It can be easily proved that if there are integers h_v and t_v satisfying (8)-(11) (with respect to given bases $\overline{x} \in P(f)$ and $y \in Q(g)$], then these numbers realize $G_0 = (V, E_0)$ obtained after deleting some set C of edge-disjoint cycles in the graph G. For example, after deleting the bold lines in the graph shown in Figure 2, the subgraph G_0 contains 3 isolated nodes for which

$$h_v = t_v = f(\{v\}) = g(\{v\}) = 0.$$

Thus, it needs to have some constraints to remove edge-disjoint cycles. Clearly, if $S \subset V$ is a node set of some cycle in C, then $h(S) - t(S) = f(S) - g(S)$. Therefore, the following condition is an edge-disjoint cycles deleting constraints.

$$h(S) - t(S) \leq f(S) - g(S) - 2, \text{ for all } \emptyset \neq S \subset V. \tag{12}$$

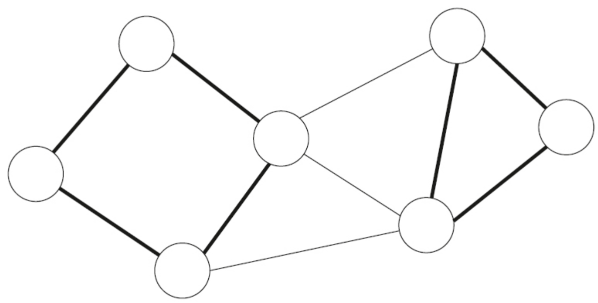

Figure 2. Edge-disjoint cycles in G.

Theorem 2 *For the base $\overline{x} \in P(f)$ defined by the greedy algorithm with respect to a linear ordering L of nodes, finding an integer solution $h = (h_v : v \in V)$ and $t = (t_v : v \in V)$ to the system of linear equations and inequalities (8)-(12) realizes $G_0(L)$ [the or-graph copy of a spanning subgraph $G_0 = (V, E_0)$] obtained after deleting all edges of a Hamiltonian circuit (if exists) in the graph G.*

Proof Suppose that the system of linear equations and inequalities (8)-(12) have integer solutions h_v and t_v for some fixed base $\overline{x} \in P(f)$ defined by the greedy algorithm with respect to some linear ordering L. Let $y = d - x$. It follows from the equalities (8)-(10) that one of the equalities (5)-(7) holds for some $h_v = \overline{x}_v$ and $t_v = \overline{y}_v$. This implies that one of the equalities (5)-(7) is used in the determination of h_v and t_v with respect to the base $\overline{x}$ and $\overline{y} = d - \overline{x}$. Thus, the condition $h_v + t_v = d_v - 2 = b_v$ holds for each node $v \in V$ and $t(V) = b(V) - h(V)$ and $2t(V) = 2h(V) = b(V)$ by (11). So, we can set $h_v = |\delta_+(v)|$ and $t_v = |\delta_-(v)|$ for each node $v \in V$. Now, we can claim that numbers h_v and t_v realize the acyclic copy of a subgraph G_0 with degree b_v of nodes v. Otherwise, one of the 2 conditions (10) or (11) is not satisfied. Using (12), we see that these numbers cannot realize some set of edge-disjoint cycles C in the graph G, because $h(S) - t(S) = f(S) - g(S)$ for some node set S of edge-disjoint subcycle. $\square$

Now, let us consider a graph $G = (V, E)$ and a nonnegative cost c_e for each edge in E. Let the base $\overline{x} \in P(f)$ be defined by the greedy algorithm with respect to some linear ordering L of nodes and let $y = d - \overline{x}$. By Theorem 2, the problem of spanning subgraph obtained after removing all the in a Hamiltonian circuit, can be formulated as follows on the or-graph $G(L)$:

$$\max \sum_{e \in E} c_e z_e$$

subject to (8)-(12) and

$$z(\delta_-(v)) = h_v, z(\delta_+(w)) = t_w, \text{ for } \ e = (v, w) \in E$$

$$z_e = 0 \text{ or } 1, \ e \in E.$$

Clearly, this can be regarded as a new model of the traveling salesman problem.

5. Concluding remarks

It is well known that the topological properties of graphs play an essential role in solving combinatorial problems over polymatroids. The properties that the greedy algorithm defines polymatroid bases with respect to a given linear ordering of nodes in a given graph and models of combinatorial problems in terms of polymatroid bases, together may be used for some investigations of these problems. The result in the paper is to show that TSP can be formulated in term of bases of polymatroids. Since we do not know the unambiguous connections between NP and P, it is difficult to come up with a polynomial-time algorithm for solving TSP by using only the above described specifics. As a future investigation, based on constraints (8)-(12), one can propose a fast approximation algorithm to solve TSP, since polymatroid approaches are an effective means of solving many combinatorial problems on graphs. Furthermore, designing a polynomial-time algorithm for solving the following question is also a future investigation: What tracing linear ordering L can be defined in polynomial-time so that $b_w = x_w^0 + y_w^0$ for base $x \in P(f)$ defined with respect to L?

Based on a positive answer to this question, a polynomial-time algorithm can be developed for finding an optimal solution to TSP and as a result, we could get that $NP = P$.

Acknowledgments

The author is grateful to the anonymous referees for their constructive comments and valuable suggestions which have helped us very much to improve the quality of the paper.

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