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Adaptive output tracking of distributed parameter systems

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Abstract: In this paper, we consider the unknown trajectory tracking problem for stable distributed parameter systems. The main assumptions are that trajectory signals are generated by an unknown finite-dimensional exosystem that is a marginally stable system and the tracking error is available for measurement. In order to achieve perfect error regulation, a frequency estimator scheme is proposed to estimate unknown exosystem parameters, and the control law that is designed based on geometric output regulation theory is revisited. The success of the proposed method is demonstrated on a parabolic heat equation and a first-order hyperbolic partial differential equation.

Key words: Output regulation, distributed parameter systems, frequency estimation, adaptive control

1. Introduction

In engineering applications, many physical phenomena are governed by partial differential equations (PDE). Since the dynamics of these phenomena vary with one or more spatial variables in addition to the time, they are called distributed parameter systems (DPS). Diffusion and heat processes, flexible structures, fluid flow systems, delay systems, chemical processes are some of the examples that belong to this class of systems. Control of distributed parameter systems can be handled by two main approaches: early lumping and late lumping. The first approach requires to approximate an infinite-dimensional model to a finite-dimensional model with a numerical approximation scheme i.e finite element method or modal approximation. Then, the finite-dimensional control theory is exploited to design a controller. Although this is the most common approach, the early lumping approach has many drawbacks. Large order finite-dimensional models after discretization, unmodeled dynamics due to neglecting spatial dynamics, the spillover effect are the drawbacks of early lumping approach [1, 2]. In contrast to the early lumping approach, a numerical approximation is not needed to design a controller in the late lumping approach. In other words, the spatial dynamics of a system are incorporated into the control design. This approach includes many difficulties because controller synthesis requires the solution of operator equations (e.g., the Riccati equation for the linear quadratic regulator problem, the operator regulator equations for output regulation). However, it may not be possible to find an analytic solution or a feasible numerical solution [3].

The error regulation problem, the theory of which is stated by Francis and Wonham, is one of the central problems in control theory. After theoretical fundamentals have been presented in [4], the output regulation

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theory has been extended to different classes of systems in the following years, e.g., nonlinear finite-dimensional systems [5]. The first attempts to develop the output regulation theory for the distributed parameter systems date back to the 1980s. The setpoint control problem for DPSs, PI control approach is developed in [6, 7]. A more generic approach has been presented after Wonham and Francis' geometric output regulation theory has been extended to DPSs that have bounded input and output operators in [8]. As a result of this development, the output regulation of DPSs has attracted great attention. In [9], the state feedback regulator design for regular systems is investigated. The dual observer-based controller for output regulation of Riesz-spectral systems is developed by Deutscher in [10]. In addition to this, the backstepping design is also used as a powerful tool for the output regulation problem for parabolic and hyperbolic PDEs in [11, 12]. The theoretical analysis for the infinite-dimensional exosystem case is handled in [13, 14]. State and error feedback problems for output regulation of regular hyperbolic systems are considered in [15]. A novel state and output regulator design for DPSs based on the separation principle is proposed in [16]. A reduced-order finite-dimensional controller design with the aid of the Galerkin approximation is constructed for the output regulation problem in [17]. The common point of the listed studies, frequencies of harmonic reference signals are assumed to be known. Although there are some recent studies that consider unknown external signals, they deal with just the unknown amplitude case [18, 19]. On the other hand, estimating unknown frequencies provides a more comprehensive solution to the output regulation problem for completely unknown exosystems. This problem has been addressed in [20] but the proposed solution requires canonical transformations. In [21], this limitation is annihilated but the proposed estimator requires the measurement of a reference signal and its derivatives e.g., noisy processing. In line with these results, the novelty of this paper can be summarized as follows. The proposed approach does not impose any structural limitations to design a controller and solves the adaptive output regulation problem without any lumping approach. Thus, truncation errors, e.g., modeling errors, are avoided.

The following sections are as follows. Notation and preliminaries are given in Section 2. Then, the output regulation theory is reviewed in Section 3. Section 4 covers the control design. The remaining sections are devoted to the simulation results and the conclusion.

2. Notation and preliminaries

The notation used throught the paper is as follow. $z \in [0,1]$ is the spatial variable and $t \geq 0$ denotes time. A is the system operator. $\sigma(A)$ and $\rho(A)$ are the spectrum and the resolvent set of A. $\mathcal{R}(\lambda, A) = (\lambda I - A)^{-1}$ is the resolvent operator where $\lambda \in \rho(A)$. $\mathcal{H}$ is the seperable Hilbert space and denotes the state space. U and Y are the input and output spaces. $\mathbb{C}$ is the field of complex numbers. $\mathcal{L}(\cdot,\cdot)$ denotes the set of linear bounded operators. $D(A)$ represents the domain of operator A and $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is an unbounded operator.

Definition 1 *[22] An operator-valued function $\mathcal{T}(t)$ that satisfies the following properties*

$$\mathcal{T}(t+s) = \mathcal{T}(t)\mathcal{T}(s) \quad t, s \geq 0 \tag{1}$$

$$\mathcal{T}(0) = I \tag{2}$$

$$||\mathcal{T}(t)x_0 - x_0|| \to 0 \quad t \to 0^+ \quad \forall x_0 \in D(A) \tag{3}$$

is called as strongly continuous semigroup and denoted as C_0-semigroup.

The question of under which conditions an operator generates a strongly continuous semigroup has been investigated enormously in the literature. As a result of this, The Hille-Yoside and the Lümer-Phillips theorems have become the most famous theorems that are addressed by researchers. Due to the page restrictions, readers are referred to [23] for the details and the proofs of these theorems. In this paper, it is also assumed that operators are exponentially stable, and the exponential stability of a C_0-semigroup is characterized in the following theorem.

Theorem 1 *[23] Let the operator A is assumed to be generator of C_0-semigroup $\mathcal{T}(t)$ and $\lambda = \sigma + i\omega$. $\forall\lambda \in \rho(A)$, $\mathcal{T}(t)$ is exponentially stable if and only if*

$$||\mathcal{R}(\lambda, A)|| \leq M \tag{4}$$

is satisfied for some constant $M > 0$ and $Re\lambda \geq 0$.

3. Output regulation theory

Consider a distributed parameter system in the abstract form

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{5}$$

$$e(t) = Cx(t) - y_{ref}(t) \tag{6}$$

where $x \in \mathcal{H}$ expresses the state of the system and the operator A is assumed to be the infinitesimal generator of the exponential stable continuous C_0-semigroup, $B \in \mathcal{L}(U, \mathcal{H})$ is a bounded input operator, $C \in \mathcal{L}(\mathcal{H}, Y)$ is a bounded output operator and $u \in U$ is the input. The second term in equation (6) stands for the reference signal and it is assumed to be the output of the homogenous finite-dimensional system

$$\dot{w}(t) = Sw(t) \tag{7}$$

$$y_{ref}(t) = Qw(t) \tag{8}$$

which is called the exosystem. $S : \mathbb{C}^n \to \mathbb{C}^n$ is a finite-dimensional matrix and Q is also a finite-dimensional matrix in appropriate dimensions.

For the stable system (5)-(6) and the exosystem (7)-(8), the control purpose is to find the control law so that the output regulation is accomplished for any initial conditions $x_0 \in D(A)$ and $w_0 \in \mathbb{C}^n$, the tracking error decays to zero asymptotically.

Under the following assumptions,

1. The operator pair (A, B) is exponentially stabilizable.
2. The operator pair (A, C) is exponentially detectable.
3. Eigenvalues of S have zero real part and lie on the imaginary axis conjugately and simply.
4. The operators A and S have completely disjoint spectra $\sigma(A) \cap \sigma(S) = \emptyset$.

The solvability of the output regulation problem is presented by the following theorem.

Theorem 2 *[24] Assuming the above assumptions are satisfied and for a stable infinitesimal generator of continuous semigroup A and a marginally stable exosystem, there exists a control law $u(t) = \Gamma w(t)$ that* *achieves the error regulation if and only if the bounded operators* $\Pi \in \mathcal{L}(\mathbb{C}, \mathcal{H})$, $Ran(\Pi) \subset D(A)$, $\Gamma \in \mathcal{L}(\mathbb{C}, U)$ *exist and satisfy the equations*

$$\Pi S = A\Pi + B\Gamma \tag{9}$$

$$C\Pi = Q. \tag{10}$$

Proof Consider equations (9)-(10) have unique bounded solutions and assume that A is the generator of the exponential stable C_0-semigroup. For the initial conditions x_0, w_0 and the input $u(t) = \Gamma w(t)$, the forced solution of (5)

$$\begin{aligned} x(t) &= \mathcal{T}(t)x_0 + \int_0^t \mathcal{T}(t-\tau)B\Gamma w(\tau)d\tau \\ &= \mathcal{T}(t)x_0 + \int_0^t \mathcal{T}(t-\tau)(\Pi S - A\Pi)w(\tau)d\tau \\ &= \mathcal{T}(t)x_0 + \int_0^t \frac{d}{d\tau}(\mathcal{T}(t-\tau)\Pi w(\tau))d\tau \\ &= \mathcal{T}(t)x_0 + (\mathcal{T}(t-\tau)\Pi w(\tau))\Big|_0^t \\ &= \mathcal{T}(t)x_0 + \mathcal{T}(0)\Pi w(t) - \mathcal{T}(t)\Pi w_0 \end{aligned} \tag{11}$$

and from the semigroup properties $\mathcal{T}(0) = I$, and

$$x(t) = \mathcal{T}(t)(x_0 - \Pi w_0) + \Pi w(t). \tag{12}$$

Then, the error equation (6)

$$e(t) = C\mathcal{T}(t)(x_0 - \Pi w_0) + C\Pi w(t) - Qw(t). \tag{13}$$

Since we have unique bounded solutions for (9)-(10), the equation (10) is satisfied. In addition to this, since $\mathcal{T}(t)$ is exponentially stable, the output tracking error $e(t)$ goes to zero [15]. Hereby, the sufficiency part is proved.

For the necessity, we can define the ideal steady state response of the control law and states as $x(t) = \Pi w(t)$ and $u(t) = \Gamma w(t)$. Then, if the transformation $\tilde{x} = x(t) - \Pi w(t)$ is substituted into equation (5)-(6)

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) - A\Pi w(t) + \Pi S w(t) + B\Gamma w(t) \tag{14}$$

$$e(t) = C\tilde{x} + C\Pi w(t) - Qw(t). \tag{15}$$

are obtained. Since A is the generator of the exponential stable C_0-semigroup, to ensure the asymptotic regulation of $e(t)$, the solutions of (9)-(10) have to result unique bounded Π and Γ. □

4. Controller design

4.1. Solution of regulator equations

To find the control law that is given in the Theorem 2, the solution procedure in [24] is followed. Consider the exosystem (7)-(8) models the reference signal $y_r(t) = Msin(\alpha t)$. Hence, the exosystem is expressed

$$S = \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix},\ Q = \begin{bmatrix} 1 & 0 \end{bmatrix},\ w(0) = \begin{bmatrix} 0 \\ M \end{bmatrix}.$$

Hence, if Π and Γ maps in the regulator equations are defined as $\Pi = \begin{bmatrix} \Pi_1(z) & \Pi_2(z) \end{bmatrix}$ and $\Gamma = \begin{bmatrix} \Gamma_1 & \Gamma_2 \end{bmatrix}$, the regulator equations (9)-(10) can be rewritten as

$$\begin{bmatrix} \Pi_1(z) & \Pi_2(z) \end{bmatrix} \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix} = A \begin{bmatrix} \Pi_1(z) & \Pi_2(z) \end{bmatrix} + B \begin{bmatrix} \Gamma_1 & \Gamma_2 \end{bmatrix} \tag{16}$$

and

$$C \begin{bmatrix} \Pi_1(z) & \Pi_2(z) \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{17}$$

This formulation yields two operator equations

$$-\alpha \Pi_2(z) - A\Pi_1(z) = B\Gamma_1 \tag{18}$$

$$\alpha \Pi_1(z) - A\Pi_2(z) = B\Gamma_2 \tag{19}$$

and if the equation (19) is multiplied by the $i = \sqrt{-1}$ and the yielding equation is

$$(i\alpha I - A)\Pi_1(z) + i(i\alpha I - A)\Pi_2(z) = Bi\Gamma_2 + B\Gamma_1. \tag{20}$$

Since the disjoint spectra (nonresonance) condition is satisfied, it is possible to write equation (20) as

$$\Pi_1(z) + i\Pi_2(z) = (i\alpha I - A)^{-1}Bi\Gamma_1 + (i\alpha I - A)^{-1}B\Gamma_2 \tag{21}$$

and from (17), the following identities

$$C\Pi_1(z) = 1,\ C\Pi_2(z) = 0 \tag{22}$$

are obtained. At this point, from [8] and [24], if equation (21) is multiplied by C and using the equalities (22)

$$1 = C(i\alpha I - A)^{-1}Bi\Gamma_1 + C(i\alpha I - A)^{-1}B\Gamma_2 \tag{23}$$

and Γ_1 and Γ_2 can be obtained as

$$\Gamma_1 = \frac{Re(\mathcal{P}(i\alpha))}{|\mathcal{P}(i\alpha)|^2}, \quad \Gamma_2 = -\frac{Im(\mathcal{P}(i\alpha))}{|\mathcal{P}(i\alpha)|^2} \tag{24}$$

where $\mathcal{P}(i\alpha) = C(i\alpha I - A)^{-1}B$.

The presented approach shows that parameters of the exosystem corresponding to frequencies of the reference trajectory and the Γ depends on it. If the matrix S is solved,

$$\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = \begin{bmatrix} cos(\alpha t) & sin(\alpha t) \\ -sin(\alpha t) & cos(\alpha t) \end{bmatrix} \begin{bmatrix} w_{10} \\ w_{20} \end{bmatrix} \tag{25}$$

$$w_0 = T_S(-t)w(t) \tag{26}$$

where $T_S(-t)$ is the inverse of the state transition matrix $T_S(t)$. The initial conditions that correspond to the amplitude of the reference signal are also unknown. To cope with this problem, an adaptive observer that estimates parameters and states of the exosystem has to be used. Consequently, enhanced estimation either initial conditions or frequency become equivalent and amplitude of unknown reference signal is also determined in addition.

4.2. Frequency estimator scheme

In order to construct the control law $u(t) = \Gamma(\hat{\alpha})\hat{w}(t)$, the frequency estimator which is presented in [25], is proposed. Consider the exosystem (7) in the following companion canonical form

$$\hat{S} = \begin{bmatrix} 0 & 1 \\ -\alpha^2 & 0 \end{bmatrix} \tag{27}$$

and

$$\dot{v}_1 = v_2 \tag{28}$$

$$\dot{v}_2 = -\alpha^2 v_1 \tag{29}$$

$$y_r = k_1 v_1 + k_2 v_2 \tag{30}$$

where $k_1, k_2, \gamma > 0$ and there exists a nonzero unknown parameter (α) dependent a similarity transformation matrix such that $\hat{w} = P(\hat{\alpha})\hat{v}$ [26]. The proposed estimator for the exosystem in the new canonical form (28)-(30) is in the form of

$$\dot{\hat{v}}_1 = \hat{v}_2 + \frac{1}{k_2}(\hat{e} - e) \tag{31}$$

$$\dot{\hat{v}}_2 = -\hat{v}_1\hat{v}_3 + (\hat{e} - e) \tag{32}$$

$$\dot{\hat{v}}_3 = -\gamma\hat{v}_1(\hat{e} - e) \tag{33}$$

$$\hat{y}_r = k_1\hat{v}_1 + k_2\hat{v}_2 \tag{34}$$

where $\hat{e}(t) = Cx(t) - \hat{y}_r$. It is obvious that $\hat{v}_3$ is square of unknown frequency $\hat{\alpha}$. Therefore, it requires an adaptive processing to reach unknown $\hat{\alpha}$.

Theorem 3 *For the stable system (5)-(6), the adaptive frequency estimator (31)-(34) and the control law* $u(t) = \Gamma(\hat{\alpha})\hat{w}(t)$ *solve the adaptive tracking problem such that* $\lim_{t\to\infty} e(t) = 0$ *and* $\sqrt{\hat{v}_3} \to \alpha$ *as* $t \to \infty$.

Proof In order to prove the Theorem 3, the stability of the adaptive frequency estimator has to be proved. Therefore, first the error dynamics for the frequency estimator are defined as $e_1 = v_1 - \hat{v}_1$, $e_2 = v_2 - \hat{v}_2$ and $e_3 = \hat{v}_3 - \alpha^2$. Following, the error dynamics are

$$\dot{e}_1 = -\frac{k_1}{k_2}e_1 \tag{35}$$

$$\dot{e}_2 = -(\alpha^2 + k_1)e_1 - k_2 e_2 + \hat{v}_1 e_3 \tag{36}$$

$$\dot{e}_3 = -\gamma\hat{v}_1(k_1 e_1 + k_2 e_2). \tag{37}$$

Then, the Lyapunov function candidate is

$$V(e) = e^T P_1 e = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} \frac{c_1}{2} & \frac{k_1}{2} & 0 \\ \frac{k_1}{2} & \frac{k_2}{2} & 0 \\ 0 & 0 & \frac{1}{2\gamma} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \tag{38}$$

and

$$P_1 = \begin{bmatrix} \frac{c_1}{2} & \frac{k_1}{2} & 0 \\ \frac{k_1}{2} & \frac{k_2}{2} & 0 \\ 0 & 0 & \frac{1}{2\gamma} \end{bmatrix} \tag{39}$$

where P_1 is a Hermitian matrix and $k_1, k_2, \gamma > 0$. Some conditions on k_1, k_2 and c_1 have to be imposed to show that the Lyapunov function candidate is radially unbounded. These conditions can be derived by using the principle minor test for the positive definiteness of the matrix P_1. An easy calculation shows that the conditions $c_1 > 0$ and $c_1 k_2 > k_1^2$ have to be satisfied. Then, if the Lyapunov function candidate (42) is rewritten, the following equation

$$V(e) = \frac{c_1}{2}e_1^2 + \frac{k_2}{2}e_2^2 + \frac{1}{2\gamma}e_3^2 + k_1 e_1 e_2. \tag{40}$$

is obtained. By taking the derivative of equation (40) and substituting the error dynamics into it,

$$\dot{V}(e) = -(c_1\frac{k_1}{k_2} + k_1\alpha^2 + k_1^2)e_1^2 - k_2^2 e_2^2 - (k_2\alpha^2 + 2k_1k_2 + \frac{k_1^2}{k_2})e_1 e_2 \tag{41}$$

is obtained and in the matrix form

$$\dot{V}(e) = e^T P_2 e = \begin{bmatrix} e_1 & e_2 \end{bmatrix} \begin{bmatrix} -(c_1\frac{k_1}{k_2} + k_1\alpha^2 + k_1^2) & -\frac{1}{2}(k_2\alpha^2 + 2k_1k_2 + \frac{k_1^2}{k_2}) \\ -\frac{1}{2}(k_2\alpha^2 + 2k_1k_2 + \frac{k_1^2}{k_2}) & -k_2^2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}. \tag{42}$$

and

$$P_2 = \begin{bmatrix} -(c_1\frac{k_1}{k_2} + k_1\alpha^2 + k_1^2) & -\frac{1}{2}(k_2\alpha^2 + 2k_1k_2 + \frac{k_1^2}{k_2}) \\ -\frac{1}{2}(k_2\alpha^2 + 2k_1k_2 + \frac{k_1^2}{k_2}) & -k_2^2 \end{bmatrix}. \tag{43}$$

In this case, the condition on c_1 is given by

$$c_1 > \frac{\alpha^4 k_2^4 + 2\alpha^2 k_1^2 k_2^2 + k_1^4 + 4k_1^3 k_2^2}{4k_1 k_2^3} \tag{44}$$

to ensure negative definiteness. Thus, $c_1 k_2 > k_1^2$ and (44) have to be assured. To ensure this, the following condition is imposed

$$c_1 > max\left(\frac{\alpha^4 k_2^4 + 2\alpha^2 k_1^2 k_2^2 + k_1^4 + 4k_1^3 k_2^2}{4k_1 k_2^3}, \frac{k_1^2}{k_2}\right). \tag{45}$$

In this way, it is shown that $\dot{V}(e) \leq 0$, if the condition (45) is satisfied. However, $\dot{V}(e) \leq 0$ is not sufficient to indicate the error dynamics decay asymptotically to zero, so the LaSalle Invariance Principle is utilized.

According to the LaSalle Invariance Principle, $e_1 = 0$ and $e_2 = 0$ imply that

$$\dot{\hat{v}}_1 = \hat{v}_2 \tag{46}$$

$$\dot{\hat{v}}_2 = -\hat{v}_1\hat{v}_3 \tag{47}$$

$$\dot{\hat{v}}_3 = 0 \tag{48}$$

$$\dot{e}_1 = 0 \tag{49}$$

$$\dot{e}_2 = 0 = \hat{v}_3 e_3 \tag{50}$$

$$\dot{e}_3 = 0 \tag{51}$$

and the above equations show that $e_3 = 0$. Thus, the stability of the error dynamics is shown [25]. It results that the control law $u(t) = \Gamma(\hat{\alpha})\hat{w}(t)$ converges to the ideal control law due to the fact that $\sqrt{\hat{v}_3} \to \alpha$ as $t \to \infty$. □

5. Simulation results

In this section, the proposed method is illustrated on the first-order hyperbolic PDE and the heat equation. The algorithm that is used, is given in Table 1.

Algorithm 1: Finite difference simulation algorithm

Input: Spatial sampling rate dz, time sampling rate dt, simulation time T_f, initial conditions, length of domain L

```
for t = 1 ··· T_f/dt do
    Run exosystem dynamics
    Run the frequency estimator to estimate unkown frequency
    Apply the transformation matrix P(α̂)
    Compute the control law
    for z = 1 ··· L/dz do
        Simulate the plant
    end
end
```

Example 1 *The first example for the proposed method considers the error regulation of the first-order hyperbolic PDE that is given in [15]. The following first order hyperbolic PDE is given as*

$$\frac{\partial \xi}{\partial t}(z,t) = \nu \frac{\partial \xi}{\partial z}(z,t) + \mathcal{F}(z)\xi(z,t) + u(t), \tag{52}$$

$$\xi(0,t) = 0 \tag{53}$$

$$\xi(z,0) = x_0 \tag{54}$$

$$y(t) = \xi(1,t) \tag{55}$$

where $\xi(z,t) \in \mathcal{H}$ is the state of the system. $\mathcal{F}(z)$ is a bounded operator that depends on a spatial variable. In order to define the system governed by (52)-(55) in the abstract form, let $A = v\frac{d}{dz} + \mathcal{F}(z).I$ with the domain

$$D(A) = \left\{ \vartheta \in H \mid \vartheta(z) \text{ is absolutely continuous, } \frac{d\vartheta}{dz} \in H \text{ and } \vartheta(0) = 0 \right\} \tag{56}$$

the output operator is $Cx(z) = x(1)$. *It is shown in [27] that* A *generates an exponentially stable* C_0*-semigroup if* $v < 0$. *The admissibility of the unbounded operator* C *is given in [15]. In order to compute the transfer function of equation (52), the numerical scheme that is proposed in [28], is addressed. To simulate equation (52)-(55), the upwind differencing scheme is chosen for the numerical stability purpose and the sampling rates are* $dt = 0.0005$ *for the time variable and* $dz = 0.001$ *for the spatial variable. The initial condition, the bounded space varying functions and parameters are set to* $\xi_0(z) = 10z^2(3/2 - z)$, $\mathcal{F}(z) = tan(z)$, $v = -1$ *and the reference signal to be tracked is* $y_{ref}(t) = 3sin(4t)$. *The parameters of the frequency estimator are chosen as* $k_1 = 4$, $k_2 = 1$ *and* $\gamma = 45$. *The tracking performance of the system (52)-(55) is shown in Figure 1-2.*

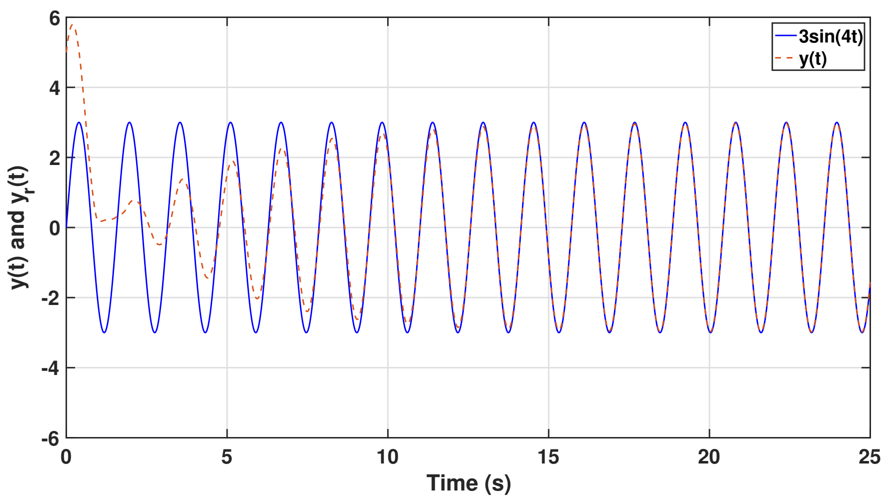

Figure 1. The output of equation (52) for the control $u(t) = \Gamma(\hat{\alpha})\hat{w}(t)$.

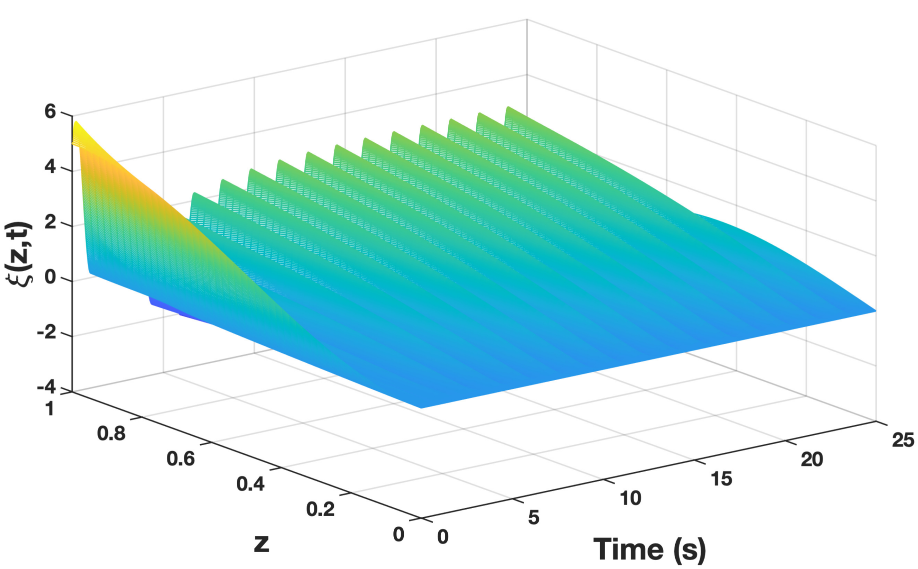

Figure 2. The dynamic behavior of the state $\xi(z,t)$ of first order hyperbolic PDE for the control $u(t) = \Gamma(\hat{\alpha})\hat{w}(t)$.

Example 2 *Consider the heat equation*

$$\frac{\partial \xi}{\partial t}(z,t) = \frac{\partial^2 \xi}{\partial z^2}(z,t) + b(z)u \tag{57}$$

$$\xi(0,t) = 0 \tag{58}$$

$$\frac{\partial \xi}{\partial z}(1,t) = 0 \tag{59}$$

$$\xi(z,0) = x_0 \tag{60}$$

for which is in the abstract form $A = \frac{d^2}{dx^2}$ *with the domain,*

$$D(A) = \{\vartheta \in \mathcal{H} : \ \vartheta(0) = 0, \ \vartheta'(1) = 0\}. \tag{61}$$

The input operator $B = b(z)$, $b(z) = \frac{1}{2v_0} 1_{[z_0 - v_0, z_0 + v_0]}$ *where*

$$1_{[z_0-v_0,z_0+v_0]} = \begin{cases} 1, \ z \in [z_0 - v_0, z_0 + v_0] \\ 0, \ otherwise \end{cases} \tag{62}$$

and the output operator $C\varphi = \int\limits_0^1 c(z)\xi(z,t)dz$ *and* $c(z) = \frac{1}{2v_1} 1_{[z_1-v_1,z_1+v_1]}$. *By taking the laplace transform of equation (57) and the boundary conditions* ($\hat{\xi}(0,s) = 0$, $\hat{\xi}_z(1,s) = 0$),

$$s\hat{\xi}(z,s) = \hat{\xi}_{zz}(z,s) + b(z)\hat{u}(s) \tag{63}$$

$$\begin{bmatrix} \hat{\xi}_z(z,s) \\ \hat{\xi}_{zz}(z,s) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ s & 0 \end{bmatrix} \begin{bmatrix} \hat{\xi}(z,s) \\ \hat{\xi}_z(z,s) \end{bmatrix} - \begin{bmatrix} 0 \\ b(z) \end{bmatrix} \hat{u}(s) \tag{64}$$

$$\begin{aligned} \begin{bmatrix} \hat{\xi}(z,s) \\ \hat{\xi}_z(z,s) \end{bmatrix} &= \begin{bmatrix} cosh(\sqrt{s}z) & \frac{1}{\sqrt{s}} sinh(\sqrt{s}z) \\ sinh(\sqrt{s}z)\sqrt{s} & cosh(\sqrt{s}z) \end{bmatrix} \begin{bmatrix} 0 \\ \hat{\xi}_z(0,s) \end{bmatrix} \\ &- \frac{1}{2v_0} \int_0^1 \begin{bmatrix} cosh(\sqrt{s}(z-\tau)) & \frac{1}{\sqrt{s}} sinh(\sqrt{s}(z-\tau)) \\ sinh(\sqrt{s}(z-\tau))\sqrt{s} & cosh(\sqrt{s}(z-\tau)) \end{bmatrix} \begin{bmatrix} 0 \\ b(z) \end{bmatrix} \hat{u}(s)d\tau \end{aligned} \tag{65}$$

and from equation (65), the following equations

$$\hat{\xi}(z,s) = \frac{1}{\sqrt{s}} sinh(\sqrt{s}z)\hat{\xi}_z(0,s) - \frac{1}{2v_0}\int_0^1 \frac{1}{\sqrt{s}} sinh(\sqrt{s}(z-\tau))b(z)\hat{u}(s)d\tau \tag{66}$$

$$\hat{\xi}_z(z,s) = cosh(\sqrt{s})\hat{\xi}_z(0,s) - \frac{1}{2v_0}\int_0^1 cosh(\sqrt{s}(z-\tau))b(z)\hat{u}(s)d\tau \tag{67}$$

are obtained. If the boundary condition for $z = 1$ ($\hat{\xi}_z(1,s) = 0$) *is used*

$$\hat{\xi}_z(0,s) = \frac{1}{2v_0}\frac{1}{cosh(\sqrt{s})}\int_0^1 cosh(\sqrt{s}(1-\tau))b(z)\hat{u}(s)d\tau, \tag{68}$$

then, from the output of the heat equation

$$\hat{y}(s) = \frac{1}{2v_0}\int_0^1 c(z)\hat{\xi}(z,s)dx \tag{69}$$

and substituting $\hat{\xi}(z,s)$ into the output equation, the explicit transfer function of the heat equation is obtained as

$$\mathcal{P}(s) = \frac{1}{\sqrt{s}cosh(\sqrt{s})} \frac{cosh(\frac{\sqrt{s}}{2}) - 1}{\frac{\sqrt{s}}{2}} \frac{sinh(\frac{\sqrt{s}}{2})}{\frac{\sqrt{s}}{2}}. \tag{70}$$

Forward time central space algorithm is chosen for the simulation purpose and the sampling rates are picked as $dt = 0.00001$ *for the time variable and* $dz = 0.005$ *for the spatial variable to assure the numerical stability. For the initial condition* $x_0(z) = 4cos(\pi z)$, *the reference signal* $y_{ref}(t) = sin(2t)$ *and parameters* $k_1 = 2$, $k_2 = 1$, $\gamma = 35$, $x_0 = 0.75$, $v_0 = x_1 = v_1 = 0.25$, *simulation results are shown in Figure 3-4.*

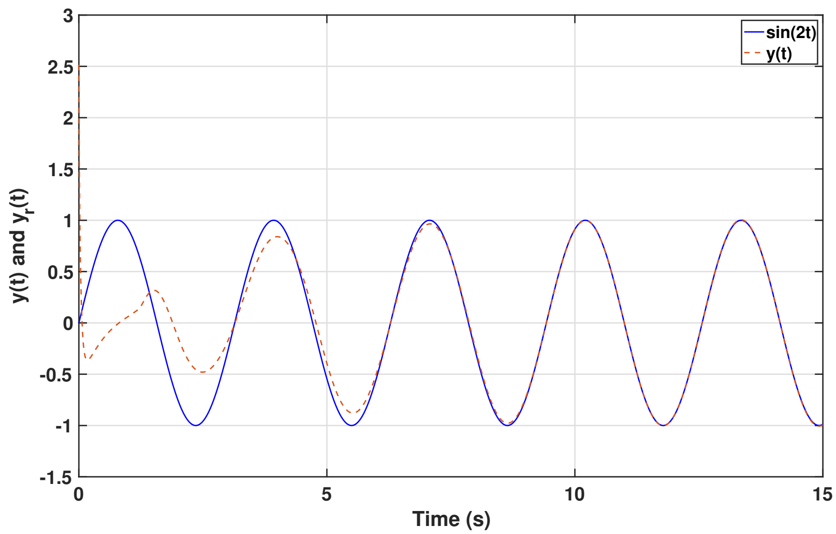

Figure 3. The output of equation (52) for the control $u(t) = \Gamma(\hat{\alpha})\hat{w}(t)$.

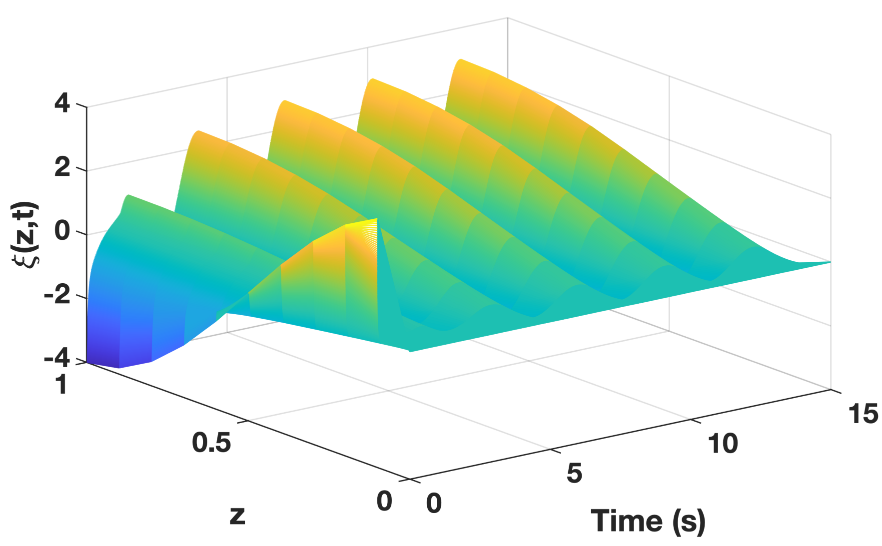

Figure 4. The dynamic behavior of the state $\xi(z,t)$ of heat equation for the control $u(t) = \Gamma(\hat{\alpha})\hat{w}(t)$.

6. Conclusion

In this research, the adaptive tracking problem is considered for stable distributed parameter systems. To do that, the output regulation theory is addressed to design the control law that depends on unknown reference frequencies. An adaptive observer is utilized to estimate the unknown frequency. The first-order hyperbolic PDE and the parabolic heat equation are chosen to show the success of the proposed method. The simulation results show that the proposed technique achieves perfect regulation even in the unknown exosystem case. Our main contribution can be summarized such that we are not using obligatory canonical forms or state transformation for the plant. In addition, we keep the infinite dimensionality of the system in the controller structure as opposed to truncation methods. As future works, there are two possible discussions. The first one is how to design a control law that depends on unknown parameters so that post processing is avoided. The second one is to consider the output feedback case.

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