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On the Global Stabilization of Nonlinear Systems via Switching Manifolds

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Abstract

The global stabilization of nonlinear systems is investigated by using switching surfaces. The nonlinear system is forced to a lower order switching manifold, which is designed to be stable by construction. Thus, the stability of the reduced-order system is guaranteed and parameter selection for the switching surface is avoided. The method is extended to a class of uncertain nonlinear systems and exemplified with some fictitious dynamic models.

1. Introduction

The Variable Structure System (VSS) Theory has been an active area of research for many years. Variable Structure Control (VSC) with a sliding mode was first described by former Soviet researchers and a survey paper with numerous references was written by Utkin [20]. The subject has attracted great interest and been investigated thoroughly by many authors [2, 4, 8, 9, 16, 22, 23]. One of the main issues in the design of switching controls is to construct the switching surface so that the system response slides along the surface to the origin. Switching surface design may be performed in an easier way when the system is given in some special form, such as controllable canonical form or regular form [14, 17, 19, 21, 22]. However, we are unlikely to be given such nonlinear systems. Thus, nonlinear systems, in general, need to be transformed to one of those forms via nonlinear transformations. The existence of these transformations, however, is not guaranteed for all nonlinear systems [12, 13, 14, 22] and the design of switching surfaces for this kind of nonlinear systems becomes more complicated.

In our previous work [4], we approached the stabilization problem of nonlinear systems in a different way. We utilized the idea of stabilizing dissipative systems in Hilbert space [1, 3]. Here, a generalized Lyapunov-like theory is used to develop switching surfaces directly, which are globally attracting by construction. If these surfaces can be designed around the stable manifold of the unforced system, global stabilization is guaranteed. If the unforced system does not have a stable manifold of proper dimension, then part of the control may be used to create one and then the remaining part of the control can be used to drive the system to this manifold.

In this paper, for the sake of completeness, we shall review the idea presented for nominal systems in [4]. Then, we shall extend the theory for a class of uncertain nonlinear systems.

In the next section, we shall study local systems defined on $\mathbf{R}^n$ by a linear analytic structure, i.e.,

$$\dot{x} = f(x) + G(x)u \tag{1.1}$$

where $f(x) \in \mathbf{R}^n$, $G(x) \in \mathbf{R}^{n\times m}$ and $u \in \mathbf{R}^m$ and we choose a function $\sigma(x)$ such that $\{x : \sigma(x) = 0\}$ is a smooth manifold through $x = 0$. Then we choose the control u so that $\sigma(x) \to 0$ as $t \to \infty$. Thus $\sigma(x)$ is a generalized Lyapunov function, although we shall now require

$$\dot{\sigma}(x) < 0 \text{ if } \sigma(x) > 0$$

$$\dot{\sigma}(x) > 0 \text{ if } \sigma(x) < 0$$

so that $\sigma(x) \to 0$.

In section 3, an extension of the theory to a class of uncertain nonlinear systems is presented. Finally, in section 4, the global construction of a stable manifold for (SI) systems on analytic manifolds is given. Thus, the systems are defined by vector fields V and W on a manifold X which are locally of the form $f(x)$ and $g(x)$. The theory of manifolds we require can be found in [10] and in Morse theory in [11].

2. Local Systems

In this section, we shall consider a nonlinear system whose local representation is given by (1.1). The control input is a *variable structure control* (VSC) which steers the system to a $(n-m)$ dimensional switching manifold and then maintains it on this hypersurface. The VSC design process can be divided into two phases; the switching manifold (or sliding surface) design and the construction of feedback gains necessary to drive the system's state to the switching manifold. Although it is possible to design a nonlinear surface, the switching manifold is, in general, designed as a linear surface, i.e., $\sigma(x) = Sx$ and the surface parameters (S) are chosen so that the system exhibits the desired behaviour. For instance, if the aim is to stabilize the nonlinear system, then S has to be determined such that the constrained system - $(n-m)$ order-reduced system - is stable. The reduced order system behaviour can be analyzed by means of an *equivalent control method* as proposed by [20,21]. The so-called equivalent control is the control necessary to satisfy the sliding mode regime, i.e.,

$$\dot{\sigma}(x) = S\dot{x} = S\big[f(x) + G(x)u_{eq}\big] = 0 \tag{2.1}$$

or,

$$u_{eq} = -\big[SG(x)\big]^{-1}\big[Sf(x)\big] \tag{2.2}$$

in which we assume that $SG(x)$ is nonsingular $\forall x$. Then, the dynamics of the system on the switching surface, i.e., the reduced order system equation is

$$\dot{x} = \big[I - G(x)[SG(x)]^{-1}S\big]f(x) \tag{2.3}$$

Thus, S is chosen such that (2.3) is stable. However, unless given in special forms such as canonical form or regular form, it is not easy to find surface parameters for general nonlinear systems. Moreover, there may not be such parameters which yield stabilization or other desired motions. On the other hand, in order to represent the nonlinear system in canonical form or regular form, we need nonlinear transformations which transform the system to one of the following coordinates:

$$\begin{aligned} \dot{y}_i &= y_{i+1} \text{ for } i = 1, \ldots, n-1 \\ \dot{y}_n &= \bar{f}(y) + \bar{g}(y)u \end{aligned} \tag{2.4}$$

or,

$$\begin{aligned} \dot{z}_1 &= \bar{f}_1(z_1, z_2) \\ \dot{z}_2 &= \bar{f}_2(z_1, z_2) + \bar{g}_2(z_1, z_2)u \end{aligned} \tag{2.5}$$

where $z_1 \in \mathbf{R}^{n-1}$ and $z_2 \in \mathbf{R}$. The first representation (2.4) is the canonical representation and the second is the regular form for a single input nonlinear system. It has been reported [12, 13] that the system (1.1) with a single input can only be transformed into (2.4) if and only if there exists a region Ω such that the following conditions hold:

(1) the vector fields $\{g(x), ad_f g(x), \ldots, ad_f^{n-1} g(x)\}$ are linearly independent in Ω,

(2) the set $\{g(x), ad_f g(x), \ldots, ad_f^{n-2} g(x)\}$ is involutive in Ω.

For a regular form, we need to solve some partial differential equations provided that a solution exists [14, 21].

2.1. Single Input (SI) Case

In this subsection, we shall consider a single input (SI) local system of the form

$$\dot{x} = f(x) + g(x)u \tag{2.6}$$

where $f(x)$ and $g(x) \in \mathbf{R}^n$ and $u \in \mathbf{R}$. Instead of transforming the nonlinear system to a special form and then choosing surface parameters, we shall approach the problem in a different way. Let $\sigma(x)$ be a smooth function such that all *level surfaces*, i.e., $\sigma(x) = const$, are $(n-1)$-dimensional smooth manifolds in and consider the control

$$u = \frac{c - \langle grad\sigma(x), f \rangle}{\langle grad\sigma(x), g \rangle} = \frac{c - L_f \sigma(x)}{L_g \sigma(x)} \tag{2.7}$$

where $L_{(\bullet)}$ is the Lie derivative with respect to (.) and $c < 0$ if $\sigma(x) > 0$ and $c > 0$ if $\sigma(x) < 0$. Then, the following theorem can be stated,

Theorem 1 *Suppose that a smooth function $\sigma(x)$ is given such that $L_g \sigma(x) \neq 0$ for all x. Then the surface $\sigma(x)$ is globally attracting with the control (2.7).*

Proof We have

$$\dot{\sigma}(x) = \frac{\partial \sigma}{\partial x} \dot{x} = \langle grad\sigma(x), f(x) \rangle + \langle grad\sigma(x), g(x) \rangle u = c$$

if u is given by (2.7). Hence, if $\sigma(x_0 > 0)$ then $c < 0$ and $\sigma(x) \to 0$ as $t \to -\sigma(x_0)/c$. Similarly, if $\sigma(x_0) < 0$ then $c > 0$ and $\sigma(x) \to 0$ again as $t \to -\sigma(x_0)/c$. □

The most obvious function, $\sigma(x)$, to choose is

$$\sigma(x) = ||x||^2 - r^2 \tag{2.8}$$

for some $r > 0$. Then we have

Corollary 1 *Suppose that the system (2.6) is locally controllable in the open set U near 0 and $B_r = \{x : ||x|| \leq r\} \subseteq U$. If $\langle x, g(x)\rangle \neq 0$ for all $x \in \mathbf{R}^n \setminus B_r$, then the system (2.6) is globally stabilizable.*

Proof Define $\sigma(x)$ as in (2.8); then $grad\sigma(x) = 2x$ and

$$u = \frac{c - \langle grad\sigma(x), f(x)\rangle}{\langle grad\sigma(x), g(x)\rangle} = \frac{c - 2\langle x, f(x)\rangle}{2\langle x, g(x)\rangle}$$

and since $\langle x, g(x)\rangle \neq 0$, the control is well defined. This control will drive any point from $x \in \mathbf{R}^n \setminus B_r$ to B_r and then the local controllability can be used. □

However, we are unlikely to have a control function, $g(x)$, such that $\langle grad\sigma(x), g(x)\rangle \neq 0$ and so a different control must be chosen near the set where $\langle grad\sigma(x), g(x)\rangle = 0$. Let $\Omega = \{x : \langle grad\sigma(x), g(x)\rangle = 0\}$ and let $\Omega_\epsilon = \{x : dist(x, \Omega) \leq \epsilon\}$ be an 'ϵ-neighbourhood' of Ω.

Theorem 2 *Suppose there exists a function $\sigma(x)$ such that the set Ω is an m-dimensional manifold for some $m < n$ and $\partial\Omega_\epsilon$ is an $(n-1)$- dimensional manifold for each $\epsilon > 0$. Moreover, suppose that, for some $\bar{\epsilon} > 0$, the set $\Omega_{\bar{\epsilon}}$ is invariant for some feedback control $u = u(x)$, with ϵ-limit set $\{0\}$ (i.e. the system is stabilizable in $\Omega_{\bar{\epsilon}}$.) Then the system (2.6) is globally stabilizable.*

Proof Parameterize $\sigma(x)$ so that $\Omega_{\bar{\epsilon}}$ is the set where $\sigma(x) = 0$ and that $\sigma(x) > 0$ in $\mathbf{R}^n \setminus \Omega_\epsilon$. Then the control (2.7) will drive the system to $\partial\Omega_{\bar{\epsilon}}$. We can then choose a stabilizing control in Ω_ϵ to drive the system to 0. □

Rather than choosing the surface, $\sigma(x) = 0$, arbitrarily, we may construct it to have some relation to the dynamics of the system with no control. Consider the nonlinear unforced system

$$\dot{x} = f(x), x \in \mathbf{R}^n \tag{2.9}$$

and its linearization about an equilibrium point,

$$\dot{x} = Ax \tag{2.10}$$

We shall refer to the Stable Manifold Theorem.

Theorem 3 ***(Stable Manifold Theorem)*** *Let E be an open subset of $\mathbf{R}$ containing the origin, let $f(x) \in C^1(E)$ and let ϕ_t be the flow of nonlinear unforced system (2.9). Suppose that $f(x) = 0$ and that the Jacobian $\frac{\partial f(0)}{\partial x}$ has k eigenvalues with a negative real part and $n-k$ eigenvalues with a positive real part. Then there exists a k-dimensional differentiable manifold S tangent to the stable subspace E^S of the linear system (2.10) at 0 such that for all $t \geq 0$, $\phi_t(S) \subset S$ and for all $x_0 \in S$,*

$$\lim_{t\to\infty} \phi_t(x_0) = 0,$$

and there exists an $(n-k)$*- dimensional differentiable manifold* U *tangent to the unstable subspace* E^U *of (2.6) at 0 such that for all* $t \le 0, \phi_t(U) \subset U$ *and for all* $x_0 \in U$*,*

$$\lim_{t \to -\infty} \phi_t(x_0) = 0.$$

Proof For the proof of the above theorem, refer to [15]. □

In general, we can always choose coordinates in E such that the nonlinear unforced system (2.9) is represented in the form

$$\begin{aligned} \dot{x}_1 &= J_1 x_1 + \tilde{f}_1(x_1, x_2) \\ \dot{x}_2 &= J_2 x_2 + \tilde{f}_2(x_1, x_2) \end{aligned} \tag{2.11}$$

where J_1 is an $(n-m) \times (n-m)$ matrix having all eigenvalues with a negative real part, J_2 is an $m \times m$ matrix having all eigenvalues with a positive real part and the functions $\tilde{f}_1(x_1 x_2)$ and $\tilde{f}_2(x_1, x_2)$ are $C^1(E)$ functions vanishing at $(x_1, x_2) = (0,0)$ together with all their first order derivatives. Clearly, x_1 and $\tilde{f}_1(x_1, x_2) \in \mathbf{R}^{n-m}, x_2$ and $\tilde{f}_2(x_1, x_2) \in \mathbf{R}^m$. The stable manifold for nonlinear unforced system (2.9) passes through (0,0) and is tangent to the subset of points whose x_2 coordinate is equal to 0. Thus, if the equation of the stable manifold is given by

$$x_2 = \varphi(x_1) \tag{2.12}$$

then the mapping $\varphi(x_1)$ satisfies

$$\varphi(0) = 0, \quad \frac{\partial \varphi}{\partial x_1}(0) = 0 \tag{2.13}$$

Moreover, this manifold is locally invariant for (2.11), which imposes on the mapping $\varphi(x_1)$ the constraint

$$\frac{\partial \varphi}{\partial x_1}\left[J_1 x_1 + \tilde{f}_1(x_1, \varphi(x_1))\right] = J_2 \varphi(x_1) + \tilde{f}_2(x_1, \varphi x_1)) \tag{2.14}$$

as easily deduced by differentiating (2.12) with respect to time, any solution curve $(x_1(t), x_2(t))$ of (2.11) which belongs to the stable manifold, i.e., satisfies $x_2(t) = \varphi(x_1(t))$ (See [13], for more details about definitions).

Assume that (2.9) has a stable manifold $\mathbf{M} \subseteq \mathbf{R}^n$ of dimension $(n-1)$, and assume that $g(x)$ is transversal to $\mathbf{M}$ (except, possibly, at the origin). If $f(x)$ (and $g(x)$) are analytic then there exists a neighbourhood U of $\mathbf{M}$ in $\mathbf{R}^n$ and a function $\sigma(x)$ such that $\mathbf{M} = \{x : \sigma(x) = 0\}$ and $g(x)$ is transversal to the level curves $\mathbf{M}_\epsilon = \{x : \sigma(x) = \epsilon\} \cap U, \epsilon > 0$. The function $\sigma(x)$ is a Morse function [11] and its existence can be proved by elementary Morse theory (simply follow the dynamics determined by $g(x)$). The following theorem can be stated,

Theorem 4 *Let* V *denote the maximal neighbourhood of* $\mathbf{M}$ *on which* $\sigma(x)$ *can be chosen so that* $g(x)$ *is transversal to the level curves* $\mathbf{M}_\epsilon$*. Then the system (2.6) is globally stable on* V*.*

Proof As before, define the feedback control u by

$$\frac{-\langle grad(x), f(x)\rangle + c}{\langle grad\sigma(x), g(x)\rangle},$$

(with $c<0$ if $\sigma(x)>0$ and $c>0$ if $\sigma(x)<0$). Since $g(x)$ is transversal to $\mathbf{M}_\epsilon$ we have $\langle grad\sigma, g(x)\rangle \neq 0$ on V and so the control drives the system to $\mathbf{M}$. Now, since $\mathbf{M}$ is the stable manifold of (2.9), we can turn off the control when we reach $\mathbf{M}$ and follow the unforced system (2.9). □

Lemma 1 *Consider the nonlinear SI system (2.6). Let the linearized system of (2.6) about the equilibrium point be*

$$\dot{x} = Ax + bu \tag{2.15}$$

If the (A,b) pair is stabilizable, then a $(n-1)$-dimensional stable manifold can be created for the nonlinear system (2.6) locally by some state feedback.

Proof Consider again (2.6) and partition the control as $u = u_1 + u_2$, then

$$\dot{x} = f(x) + g(x)u_1 + g(x)u_2 = \bar{f}(x) + g(x)u_2$$

where $\bar{f}(x) = \dot{\bar{x}} = f(x) + g(x)u_1$ and the linearized system

$$\dot{\bar{x}} = A\bar{x} + bu_1$$

Since (A,b) is stabilizable, then there exists a linear state feedback such that $(n-1)$ eigenvalues of (2.9) are located at the left-hand side of the complex plane. It follows from theorem 3 that $\bar{f}(x)$ has an $(n-1)$ dimensional stable manifold. □

Remark 1 *It is clear that for different linear state feedback, $u_1 = u_1(x)$, a new $\bar{f}(x)$ will be obtained which leads to a different stable manifold. In other words, the shape and the dimension of the stable manifold will depend on the state feedback. Obviously, since the state feedback gains are free to choose - provided that lemma 1 is satisfied - then we can obtain infinitely many stable manifolds, i.e., a stable manifold is not unique.*

Remark 2 *Instead of linear state feedback, some nonlinear state feedback could also be added to u_1 in order to cancel some nonlinear terms in $f(x)$ and simplify $\bar{f}(x)$. This will also change the shape of the stable manifold.*

Example 1 *Consider the system $\dot{x} = f(x) + g(x)u$, where*

$$f(x) = \begin{bmatrix} -x_1 + x_2^2 \\ -2x_2 \\ 4x_3 + x_1^2 + x_2^3 \end{bmatrix} \quad \textit{and } g(x) = \begin{bmatrix} 1 - 4x_2^2x_3^2 \\ 1 + x_1^2 \\ 2 + x_1^2x_2^2x_3^2 \end{bmatrix}$$

which has an equilibrium point at (0,0,0). The linearized system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} u_1$$

has its eigenvalues at $\lambda_1 = -1, \lambda_2 = -2$, and $\lambda_3 = 4$, which guarantees a 2-dimensional local stable manifold according to Theorem 3. The stable manifold for $f(x)$ is

$$\sigma(x) = x_3 + \frac{1}{6}x_1^2 + \frac{1}{27}x_1x_2^2 + \frac{1}{10}x_2^3 + \frac{1}{324}x_2^4$$

The stable manifold for this particular example is plotted in figure 1. Note that the manifold, unlike the standard linear sliding surface, is a nonlinear surface. Then,

$$\begin{aligned} L_g\sigma \;=\; & \frac{1}{3}x_1 - \frac{4}{3}x_1x_2^2x_3^2 + \frac{71}{270}x_2^2 - \frac{4}{27}x_2^4x_3^2 + \frac{2}{27}x_1x_2(1+x_1^2) \\ & + \frac{4}{324}x_2^3(1+x_1^2) + x_1^2x_2^2(x_3^2+0.3) + 2 \end{aligned}$$

which is zero for some x*. Thus, we need to choose another stable manifold such that* $L_g\sigma \neq 0 \;\forall x$*. The manifold* $\bar{\sigma}(x) = 4x_3 + \frac{1}{3}x_1^3 + \frac{1}{3}x_2^3$ *satisfies*

$$L_g\bar{\sigma} = 8 + x_1^2 + x_2^2 + x_1^2x_2^2 \neq 0 \;\forall x.$$

Figure 2 gives $\bar{\sigma}(x)$ *which is again a nonlinear manifold. Then, applying the control*

$$u_1 = \frac{\frac{7}{3}x_1^3 + \frac{10}{3}x_2^3 - x_1^2x_2^2 - 4x_1^2 - 4x_2^3}{x_1^2 + x_2^2 + x_1^2x_2^2 + 8}$$

to the system, we have $\dot{x} = \bar{f}(x) + g(x)u_2$ *where*

$$\bar{f}(x) = \begin{bmatrix} -x_1 + x_2^2 + (1 - 4x_2^2x_3^2)u_1 \\ -2x_2 + (1+x_1^2)u_1 \\ 4x_3 + x_1^2 + x_2^3 + (2 + x_1^2x_2^2x_3^2)u_1 \end{bmatrix}$$

whose stable manifold is exactly $\bar{\sigma}(x) = 4x_3 + \frac{1}{3}x_1^2 + \frac{1}{3}x_2^3$*. Then the other (VSC) part of the control,* u_2*, is from (2.7)*

$$u_2 = \frac{c - L_{\bar{f}}\bar{\sigma}}{L_g\bar{\sigma}}$$

and $L_g\bar{\sigma} = 8 + x_1^2 + x_2^2 + x_1^2x_2^2 \neq 0 \;\forall x$*. Clearly, control* u_1 *is to create an* $(n-1)$*-dimensional local stable manifold and control* u_2 *is to force the new dynamics to the stable manifold. Figures 3 and 4 give the simulation results of example 1. For this simulation,* $c = -0.5sgn(\sigma(x))$ *and the initial conditions are* $\mathbf{x} = [1 \;\; \text{-}15 \;\; 0.4]^T$.

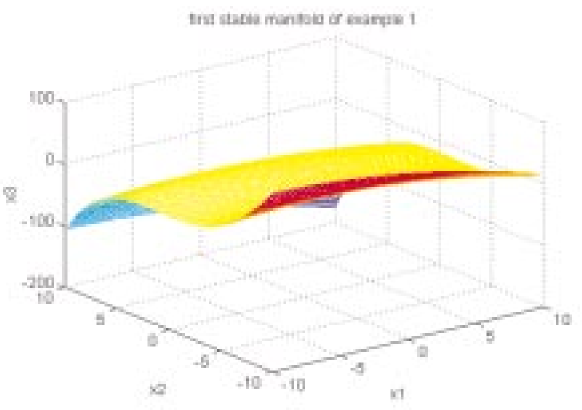

Figure 1. Stable manifold, $\sigma(x)$.

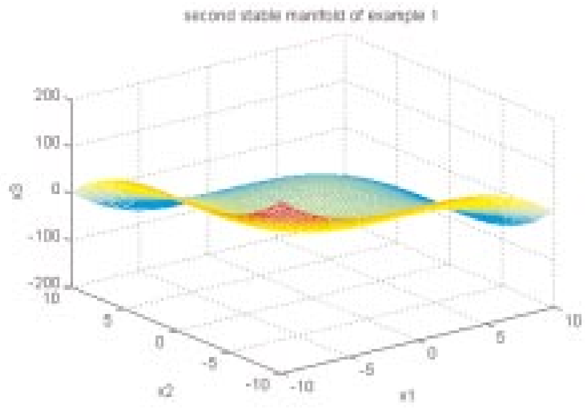

Figure 2. Stable manifold, $\overline{\sigma}(x)$.

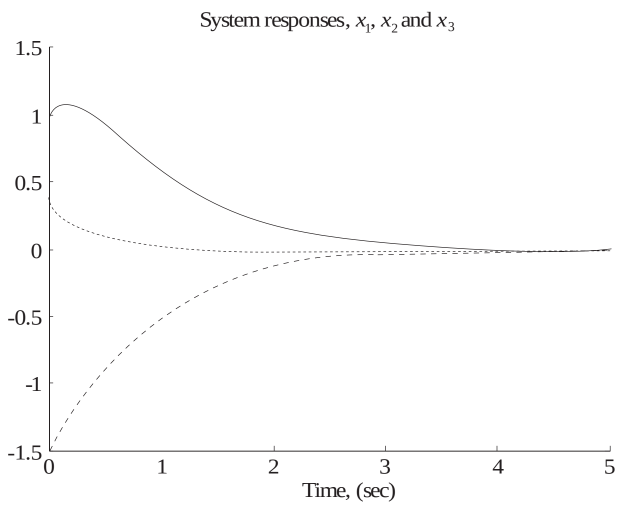

Figure 3. System response.

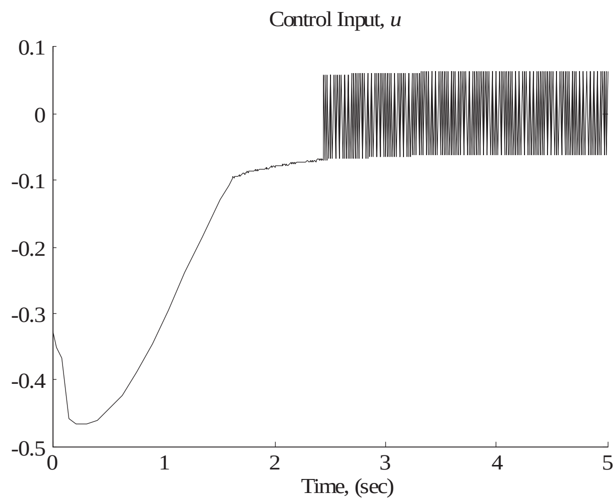

Figure 4. Control input.

Remark 3 *In order to keep the system on the manifold, the control input u in (2.7) changes sign, which brings high frequency control chattering. This can be alleviated by considering instead of (2.7), the control*

$$u_{fn} = \frac{-c\left[\frac{\sigma(x)}{|\sigma(x)|+\delta}\right] - L_f\sigma(x)}{L_g\sigma(x)} \tag{2.16}$$

where $c > 0$ and δ is a positive small number. In this case, if $\sigma(x) > 0$, for instance, $\dot{\sigma} = -c\left(\frac{\sigma(x)}{|\sigma(x)|+\delta}\right)$ and so when $\sigma(x)$ is small, this is approximated $\dot{\sigma}(x) = -c\frac{\sigma(x)}{\delta}$ i.e., $\sigma(t) = \sigma(x_0)e^{-ct/\delta}$ so that the switching surface is never reached. This control is simply keeping the system close to the switching surface, which is chosen so as to be stable.

Another possibility for smoothing the control signal is to use a saturation function, i.e.,

$$u_{sat} = \frac{-csat(\sigma(x)) - L_f\sigma(x)}{L_g\sigma(x)} \tag{2.17}$$

where $c > 0, \delta$ is a positive small number and

$$sat(\sigma(x)) = \begin{cases} \sigma(x) \ \ if \ |\sigma(x)| \leq \delta \\ sgn(\sigma(x)) \ \ otherwise \end{cases}$$

By using this control, the switching function becomes

$$\dot{\sigma}(x) = -csat(\sigma(x)) = \begin{cases} -c\sigma(x) \ \ if \ |\sigma(x)| \leq \delta \\ -csgn(\sigma(x)) \ \ otherwise \end{cases}$$

which means that the control steers the system into a strip and inside the strip, the system tends to reach the stable manifold.

2.2. Multi Input (MI) Case

In this subsection, the methodology presented in subsection 2.1 is extended to a multi-input (MI) case. Let the nonlinear system be defined by

$$\dot{x} = f(x) + G(x)u = f(x) + \sum_{i=1}^{m} g_i(x)u_i \tag{2.18}$$

where $f(x)$ and $g_2 \cdot (x) \in \mathbf{R}^n$ and $u \in \mathbf{R}^m$. The control will force the sysytem to an $(n-m)$-dimensional switching manifold and then keep the system on the hypersurface after hitting it. Thus, m equations are necessary to define the $(n-m)$-dimensional switching manifold. Let the i^{th} element of the switching manifold vector be $\sigma_i(x)$. Then, in the vicinity of the switching manifold,

$$\begin{array}{c} \dot{\sigma}_i(x) < 0 \text{ if } \sigma_i(x) > 0 \\ \dot{\sigma}_i(x) > 0 \text{ if } \sigma(_i(x) < 0 \end{array} \quad \text{for } i = 1, 2, \ldots, m \tag{2.19}$$

so that $\sigma_i(x) \to 0$. We now have

Theorem 5 *Let the m dimensional vector $\mathbf{L}_f$ and $(m \times m)$ matrix $\mathbf{L}_G$ be*

$$\mathbf{L}_f = \begin{bmatrix} L_f\sigma_1(x) \\ \vdots \\ L_f\sigma_m(x) \end{bmatrix} \qquad \mathbf{L}_G = \begin{bmatrix} L_{g_1}\sigma_1(x) & \cdots & L_{g_m}\sigma_1(x) \\ \vdots & \ddots & \vdots \\ L_{g_1}\sigma_m(x) & \cdots & L_{g_m}\sigma_m(x) \end{bmatrix} \tag{2.20}$$

If $\mathbf{L}_G$ is nonsingular $\forall x$, then the control

$$u = \mathbf{L}_G^{-1}(\mathbf{c} - \mathbf{L}_f) \tag{2.21}$$

will drive the system (2.18) to the switching surface $\sigma(x)$ provided that

$$c = \begin{bmatrix} -c_1 sgn(\sigma_1(x)) \\ \vdots \\ -c_m sgn(\sigma_m(x)) \end{bmatrix} \quad c_i > 0 \quad \text{for } i = 1, 2, \ldots, m$$

Proof Each component of the switching manifold should satisfy (2.19). Thus,

$$\begin{aligned}\dot{\sigma}(x) &= \frac{\partial \sigma_i(x)}{\partial x}x = \langle grad\sigma_i(x), f(x)\rangle + \langle grad\sigma_i(x), g_1(x)\rangle u_1 \\ &+ \cdots + \langle grad\sigma_i(x), g_m(x)\rangle u_m\end{aligned}$$

for $i = 1, 2, \ldots, m$ or in compact form

$$\dot{\sigma}(x) = \mathbf{L}_f + \mathbf{L}_G u \tag{2.22}$$

where $\mathbf{L}_f$ and $\mathbf{L}_G$ are given in (2.20). Since $\mathbf{L}_G$ is nonsingular $\forall x$, then the control (2.21) is well defined and (2.22) becomes $\dot{\sigma}(x) = \mathbf{c}$ and decouples. Choosing $\mathbf{c}$ as stated, (2.19) is satisfied. □

As defined for the single input case in subsection 2.1, an $(n-m)$-dimensional stable switching manifold may be constructed by first splitting up the control into two parts and then applying one of them to get a $f(x)$ whose linearization about the equilibrium point has $(n-m)$ negative eigenvalues.

Example 2 *Consider the nonlinear system* $\dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2$ *where*

$$f(x) = \begin{bmatrix} -x_1 + x_2^2 \\ 2x_2 \\ 3x_3 + x_1^2 + x_2^3 \end{bmatrix}, g_1(x) = \begin{bmatrix} -2 \\ 1 + x_3^2 \\ 1 + \frac{4}{5}x_1 \end{bmatrix} \text{ and } g_g(x) = \begin{bmatrix} 5x_1 \\ 1 - x_2^2 \\ 3 \end{bmatrix}$$

which has an equilibrium point at (0,0,0). The eigenvalues of the linearized system are $\lambda_1 = -1, \lambda_2 = 2,$ *and* $\lambda_3 = 3$*. Thus, the system has a 1-dimensional local stable manifold (Theorem 3). The stable manifold for* $f(x)$ *is*

$$\sigma_1(x) = x_2, \qquad \sigma_2(x) = x_3 + \frac{1}{5}x_1^2$$

From (2.20)

$$\mathbf{L}_G = \begin{bmatrix} 1 + x_3^2 & 1 - x_2^2 \\ 1 & 3 + 2x_1^2 \end{bmatrix}$$

and $det\mathbf{L}_G = 2 + x_2^2 + 2x_1^2 + 3x_3^2 + 2x_1^2x_2^2 > 0, \forall x$. *Then* u *is well defined. Simulation results are given in Figures 5, 6 and 7 for* $c_1 = -0.25sgn(\sigma_1(x)), c_2 = -0.25sgn(\sigma_2(x))$, *and the initial conditions* $\mathbf{x} = [-1 \;\; 0.8 \;\; 1]^T$.

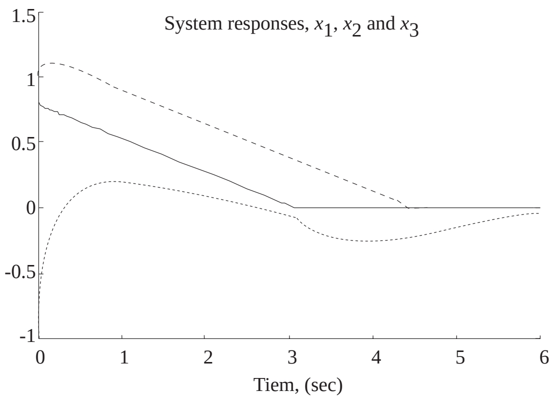

Figure 5. System response.

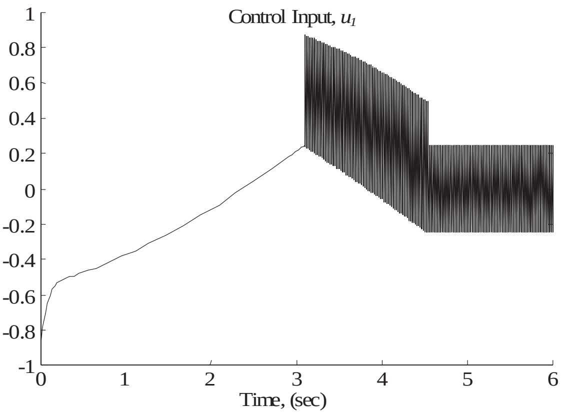

Figure 6. Control input u_1.

One could also get an $(n-1)$-dimensional stable switching manifold by locating the $(n-1)$ eigenvalues of the linearized system to the left hand-side of the complex plane if the linearized system is at least stabilizable (***Lemma 1***). Let $\sigma(x)$ be a smooth function such that all the *level curves* $\sigma(x) = const$ are $(n-1)$-dimensional smooth manifolds in $\mathbf{R}^n$, and consider the control vector whose i^{th} element is defined by

$$u_i = \frac{\langle grad\sigma(x), g_i(x)\rangle(-\langle grad\sigma(x), f(x)\rangle + c)}{\sum_{k=1}^{m}\langle grad\sigma(x), g_k(x)\rangle^2} = \frac{L_{g_i}\sigma(x)(-L_f\sigma(x)+c)}{\sum_{k=1}^{m}(L_{g_k}\sigma(x))^2} \quad \text{for } i = 1,2,\ldots,m \tag{2.23}$$

where $L_{(\bullet)}$ is the Lie derivative with respect to (.) and $c < 0$ if $\sigma(x) > 0$ and $c > 0$ if $\sigma(x) < 0$.

Lemma 2 *Assume that a stable manifold $\sigma(x)$ is given such that*

$$\sum_{k=1}^{m}(L_{g_k}\sigma(x))^2 \neq \qquad \forall x \tag{2.24}$$

Then the surface $\sigma(x)$ is globally attracting with the control (2.23)

Proof Simply take the derivative of $\sigma(x)$,

$$\dot{\sigma} = \frac{\partial\sigma(x)}{\partial x}\dot{x} = \langle grad\sigma(x), f(x)\rangle + \sum_{i=1}^{m}u_i\langle grad\sigma(x), g_i(x)\rangle$$

and substitute u_i into the equation. Then, $\dot{\sigma} = c$. Hence, if $\sigma(x_0) > 0$ then $c < 0$ and $\sigma \to 0$ as $t \to -\sigma(x_0)/c$. Similarly, if $\sigma(x_0) < 0$ then $c > 0$ and $\sigma \to 0$ again as $t \to -\sigma(x_0)/c$. $\square$

Example 3 *Consider the nonlinear system $\dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2$ where*

$$f(x) = \begin{bmatrix} -x_1 + x_2^2 \\ -2x_2 \\ 3x_3 + x_2^3 \end{bmatrix}, g_1(x) = \begin{bmatrix} -2 \\ 1 + x_3^2 \\ 1 + 4x_1^2 \end{bmatrix} \text{ and } g_2(x) = \begin{bmatrix} 5x_1 \\ 1 - x_2^2 \\ 3 \end{bmatrix}$$

which has an equilibrium point at (0,0,0). The eigenvalues of the linearized system are $\lambda_1 = -1, \lambda_2 = -2$, and $\lambda_3 = 3$. Thus, the system has a 2-dimensional local stable manifold (Theorem 3). The stable manifold for $f(x)$ is $\sigma(x) = x_3 + \frac{1}{9}x_2^3$ and

$$\sum_{k=1}^{2}(L_{g_k}\sigma)^2 = \left(1 + 4x_1^2 + \frac{1}{3}x_2^2 + \frac{1}{3}x_2^2x_3^2\right)^2 + \left(3 + \frac{1}{3}x_2^2 - \frac{1}{3}x_2^4\right)^2 \neq 0 \quad \forall x$$

Then u_1 and u_2 are well defined. Simulation results for example 3 are given in Figures 8, 9 and 10 for $c = -0.25sgn(\sigma(x))$, and the initial conditions $\mathbf{x} = [-1\ \ 0.8\ \ 1]^T$.

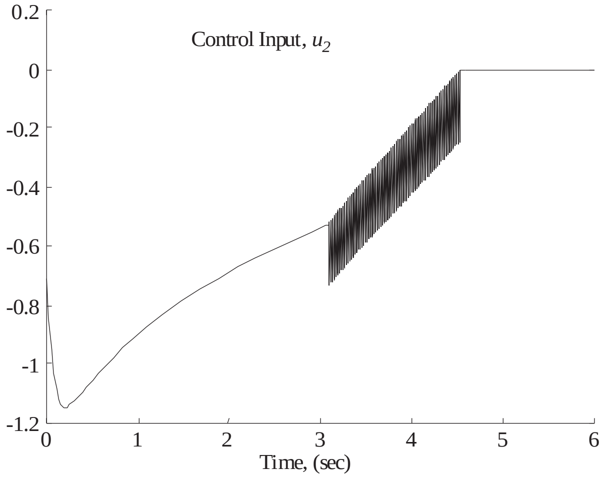

Figure 7. Control input u_2.

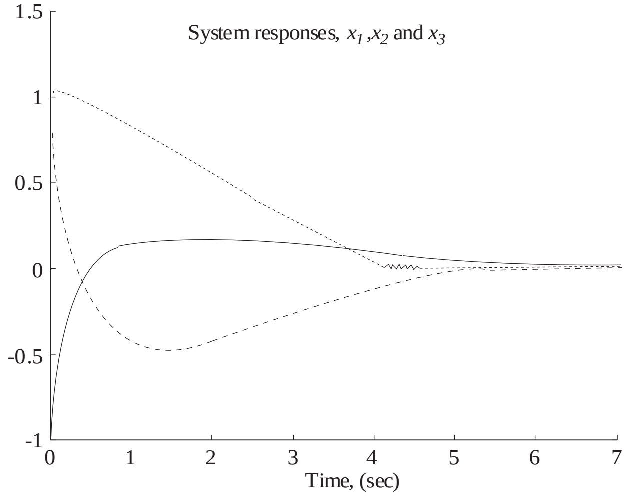

Figure 8. System response.

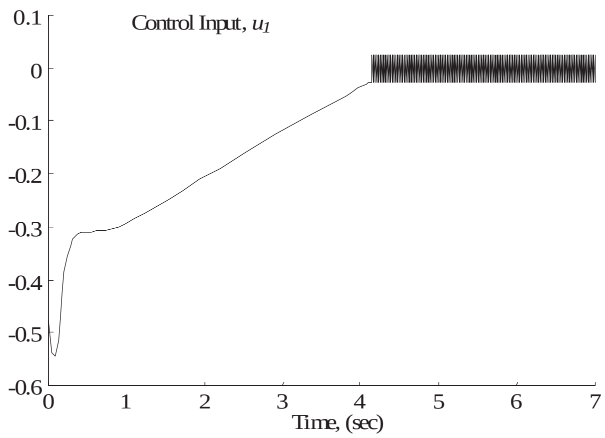

Figure 9. Control input u_1.

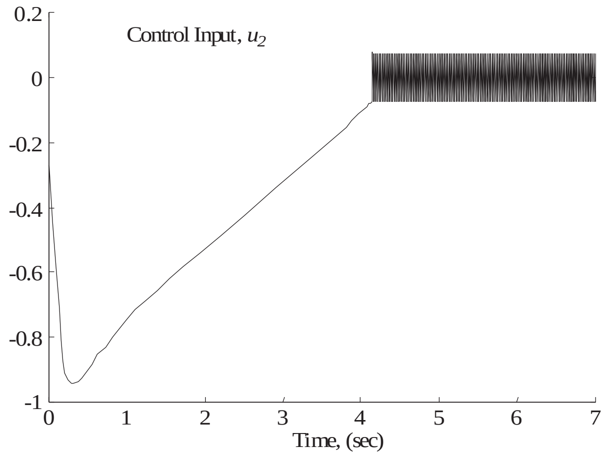

Figure 10. Control input u_2.

Remark 4 *Note that in examples 2 and 3, only the VSC part of the controls is given since the unforced systems already have stable manifolds of proper dimensions. Thus, there are no state feedback control parts in either examples for obtaining stable switching manifolds of appropriate dimensions.*

Remark 5 *One could obtain an $(n-k)$-dimensional stable manifold (where $k < m$) and attempt to force the system to that manifold. In this case, it would be possible to turn off the $(m-k)$ VSC part of the controls, which again yields a system whose stable manifold dimension and control input numbers are appropriate.*

3. Extension to Uncertain Systems

So far, the theory has been established for nominal systems, i.e., we have assumed that system models are exact and there is no external disturbance. However, exact models may not be easily determined for all systems and external disturbances cannot be estimated all the time. Hence, real systems may not be exactly represented by their models and consequently the control-derived model becomes an irrelevant input into the system as the unmodelled part deteriorates the system response.

On the other hand, uncertainties and/or disturbances can be estimated within some bounds by doing experiments or observing the system response. Thus, these factors should be added to the model dynamics and the controller is to be designed so that the effects of uncertainties and/or disturbances are eliminated.

In this section, the theory discussed in Section II is extended to a class of uncertain nonlinear systems. Assume that uncertainty on $g(x)$ is negligible and that the SI system is given by

$$\dot{x} = f(x) + \Delta f(x) + g(x)u \tag{3.1}$$

where $\Delta f(x)$ represents the uncertainties on $f(x)$. Note that equation (3.1) is a summation of three vectors, namely $f(x), \Delta f(x)$, and $g(x)u$, which define the next position of x (see Figure 11). Consider now the stable manifold, $\sigma(x)$, of the nominal system. From Theorem 3, it is clear that the vector $f(x)$ is tangent to the stable manifold $\sigma(x)$ when the system is on the manifold (see Figure 12). If the vector $\Delta f(x)$ were zero, then it would be possible to turn off the control since the $f(x)$ dynamics would follow the stable manifold $\sigma(x)$.

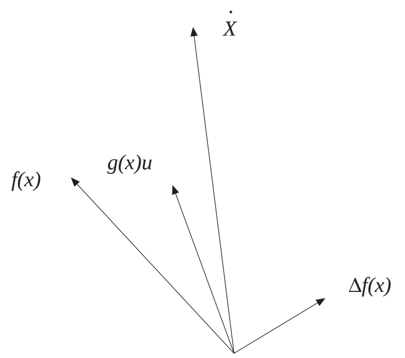

Figure 11. Explanation of system motion.

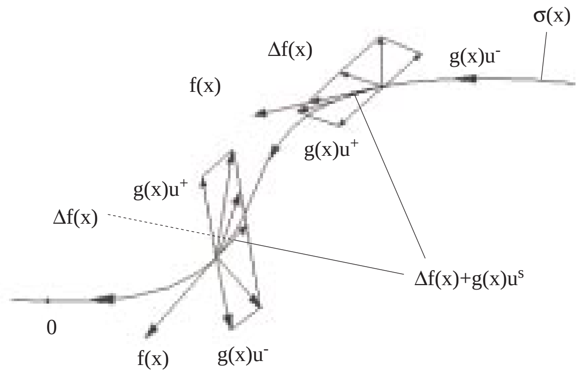

Figure 12. Planar motion of the system on the stable manifold.

When $\Delta f(x)$ is introduced to the system, the system will leave the stable manifold if the control is turned off. Nevertheless, there always exists a control u^s which keeps the system on $\sigma(x)$ provided that $L_g\sigma(x) \neq 0$. This control simply creates another vector tangent to the stable manifold. The new vector can be defined by

$$F^s(x) := \Delta f(x) + g(x)u^s$$

which yields $\dot{x} = f(x) + F^s(x)$.

The component of the two vectors $f(x)$ and $F^s(x)$ must be directed to the origin in order to have a stable motion. Since $f(x)$ is always directed to the origin, the magnitude of $F^s(x)$ should be less than the magnitude of $f(x)$. This implies that

$$||f(x)|| > ||F^s(x)|| \tag{3.2}$$

Notice that inequality (3.2) can be satisfied if

$$||\Delta f(x)|| < ||f(x)||, \forall x \tag{3.3}$$

The uncertainty condition given by inequality (3.3) simply implies that the magnitude of unmodelled dynamics should not be greater that the magnitude of modelled dynamics.

If the system is not on the stable manifold, $\sigma(x)$, then the controller should drive the system to $\sigma(x)$. This is achieved by defining the control, u, from

$$\dot{\sigma}(x) = L_f\sigma(x) + L_{\Delta f}\sigma(x) + L_g\sigma(x)u$$

which results in

$$u = \frac{-(\delta + |L_{\Delta f}\sigma(x)|_{\max})sgn(\sigma(x)) - L_f\sigma(x)}{L_g\sigma(x)} \tag{3.4}$$

so that $\dot{\sigma}(x) = -(\delta + |L_{\Delta f}\sigma(x)|_{\max})sgn(\sigma(x)) + L_{\Delta f}\sigma(x)$ where δ is a small positive number. If the bond of $\Delta f(x)$ is known, then $|L_{\Delta f}\sigma(x)|_{\max}$ can be computed, so that $\sigma(x) \to$ as $t \to \infty$.

Example 4 *Consider the system given in example 1. Now assume there are some uncertain parameters in the system.*

$$\dot{x} = f(x) + \Delta f(x) + g(x)u$$

where

$$\Delta f(x) = \begin{bmatrix} \alpha_1 x_3^2 \\ \alpha_2 x_1 + \alpha_3 x_2^2 \\ 0 \end{bmatrix},$$

$$\alpha_1 = -0.9\sin(t/3), \alpha_2 = -0.9\cos(2t) \text{ and } \alpha_3 = -0.9\cos(t/2).$$

The VSC part of the control

$$u_2 = \frac{c - L_{\bar{f}}\bar{\sigma}(x)}{L_g\bar{\sigma}(x)}$$

where $c = -(\delta + |L_{\Delta f}\bar{\sigma}(x)|_{\max})sgn(\bar{\sigma}(x))$ *and*

$$\left|L_{\Delta f}\bar{\sigma}\right|_{\max} = \left|x_1^2 x_3^2 \alpha_1\right| + \left|x_2^2(\alpha_2 x_1 + \alpha_3 x_2^2\right| = x_1^2 x_3^2 \left|\alpha_1\right|_{\max} + x_2^2\left|x_1\right|\left|\alpha_2\right|_{\max} + x_2^4\left|\alpha_3\right|_{\max}$$

Figures 13 and 14 show system responses and control input without $\left|L_{\Delta f}\sigma\right|_{\max}$. *The modified control with* $\left|L_{\Delta f}\sigma\right|_{\max}$ *is applied to the system and the results are given in Figures 15 and 16. For this simulation we have taken* $\left|\alpha_1\right|_{\max} = \left|\alpha_2\right|_{\max} = \left|\alpha_3\right|_{\max} = 0.95, \delta = 0.5$, *and the initial conditions are* $\mathbf{x} = [1 \ \ -1.5 \ \ 0.4]^T$.

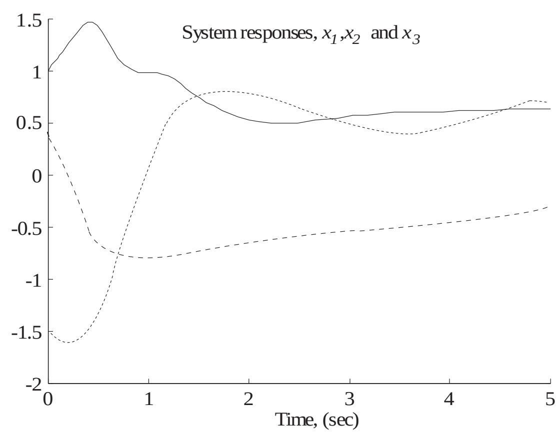

Figure 13. System response without $|L_{\Delta f}\sigma(x)|$ term.

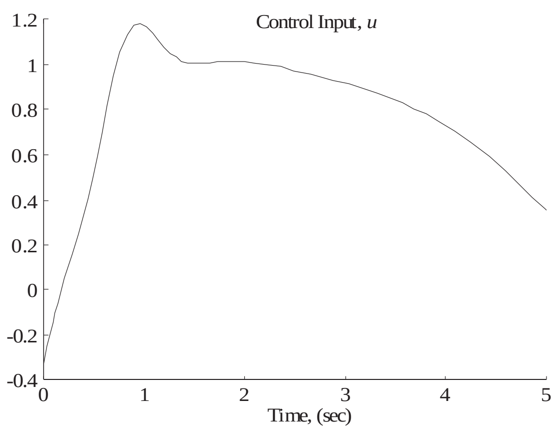

Figure 14. Control input without $|L_{\Delta f}\sigma(x)|$ term.

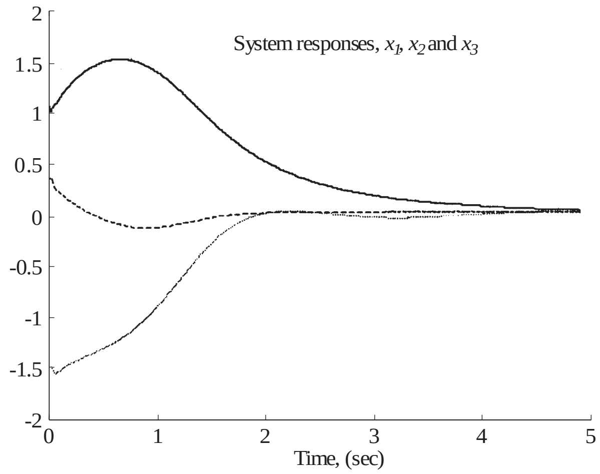

Figure 15. System response with $|L_{\Delta f}\sigma(x)|$ term.

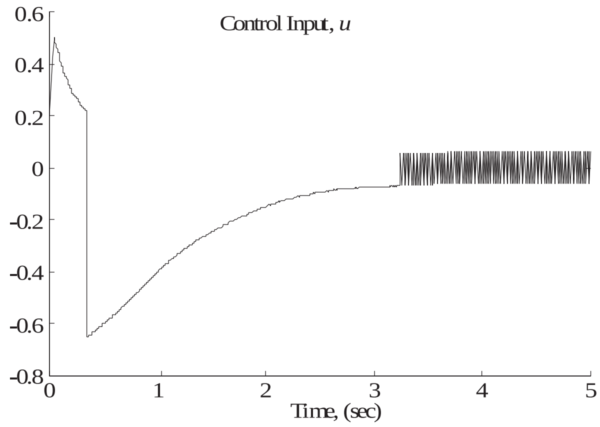

Figure 16. Control input with $|L_{\Delta f}\sigma(x)|$ term.

4. Global Theory

We shall present the global construction of a stable switching manifold for single input (SI) nonlinear systems in this section. The idea can easily be extended to multi input (MI) nonlinear systems. Let X be a compact differentiable manifold of dimension n and let V, W be vector fields on X. The controlled vector field $V + uW$ has the local representation

$$\dot{x} = f(x) + ug(x) \tag{4.1}$$

in the coordinates $x : N \to \mathbf{R}^n$ for some open set $N \subseteq X$. If $S \subseteq X$ is a smooth submanifold of X of dimension $n-1$ (i.e. a hypersurface) then S and the vector W_x are **transversal** if

$$TS \oplus \mathbf{R}W_x = TX \tag{4.2}$$

Suppose that $p \in X$ is an equilibrium point of V, i.e. $V_p = 0$. It is well known [11] that the total index of vector field V on X is given by the Euler characteristic of $X, \gamma(X)$. Then X has at least one equilibrium point if $\gamma(X) > 0$. Let (4.1) be a local representation of the system at p, where $x : U \to \mathbf{R}^n$ is a coordinate system in the neighbourhood U of p with $x(p) = 0$. We shall assume that $(f(x), g(x))$ is linearizable and the linearized system is stabilizable at p so that we may write

$$\dot{x} = Ax + f^{(2)}(x) + u(g(0) + g^{(1)}(x)) = Ax + bu + f^{(2)}(x) + ug^{(1)}(x)$$

where $A = \frac{\partial f}{\partial x}(0), b = g(0), f^{(2)}(x) = f(x) - Ax, g^{(1)}(x) = g(x) - g(0)$. Now write $u = u_1 + u_2$ and choose $u_1 = kx$ to stabilize (A, b). Then we have

$$\dot{x} = (A + bk)x + f^{(2)}(x) + (kx + u_2)g^{(1)}(x) + u_2 b$$

Now choose an $(n-1)$-dimensional stable submanifold $S \subseteq U$ of the system such that $b+g^{(1)}(x)$ is transversal to S. Then S can be defined by a function $\sigma(x)$ such that $S = \{x \in U : \sigma(x) = 0\}$. If $y : U' \to \mathbf{R}^n$ is another coordinate neighbourhood such that $U \cap U' \neq \emptyset$and $S \cap U' \neq \emptyset$. Then we can extend S as follows:

if $y = h(x)$, then u_1 is extended to $u_1 = kh^{-1}(y)$ in U' and S is extended into U' as the union of all trajectories of the system

$$\dot{y} = (A + bk)h^{-1}(y) + f^{(2)}(h^{-1}(y)) + (kh^{(1)}(y))g^{(1)}(h^{-1}(y)) + u_2(b + g^{(1)}(h^{-1}(y)))$$

in U passing through S in $U \cap U'$. In this way we obtain the maximal extension of S to X on which $b + g^{(1)}(h^{-1}(y))$ is transversal to the submanifold. Let S_m denote this maximal $(n-1)$-dimensional stable submanifold of the system (V, W). It is defined by a set of equations

$$S_m = \{m \in X : \sigma_i(x) = 0, i \in U_i\}$$

where $\{U_i\}_{1 \leq i \leq L}$ is a set of coordinate neighbourhoods. These functions $\sigma_i(i)$ piece together to form a section of the real line bundle over X. Finally we integrate the partial differential equation $grad\sigma = b + g^{(1)}(x)$ from S_m and define the region $\overline{\mathbf{M'}}$ just as in the local case. Then the local control

$$u_2 = \frac{-\langle grad\sigma, f(x)\rangle + c}{\langle grad\sigma, g(x)\rangle}$$

will drive all the points in $\overline{\mathbf{M'}}$ to S_m, which is then a stable manifold using $u_2 = 0$.

Conclusions

In this paper, we have presented a new method of design for switching manifolds for general nonlinear systems. It has been shown that the stabilization of many kinds of nonlinear systems can be achieved by first designing a stable manifold of lower dimension for the system and then using a switching control to steer the system to this submanifold. The method does not uses nonlinear transformations to bring the system to a special form and we do not need to choose switching surface parameters to stabilize the nonlinear system. The idea is also extended to a class of uncertain nonlinear systems. The method easily extends to global systems on differentiable manifolds, giving a truly global control method for nonlinear systems.

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