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Distributions of Instantaneous Unit Hydrograph Peak Characteristics for Small River Networks

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Abstract

Diameter-conditioned distribution functions of peak ratio ("peak"/diameter) and position ratio ("time to peak"/diameter) of topological width functions of small networks were obtained analytically. Here diameter is also the main stream length of the network. This is done by using translation routing, which is the simplest case of linear routing schemes. A new concept, "junction configuration", which is a characteristic property for the IUH (instantaneous unit hydrograph) of the network, is introduced. Another scheme combining the concepts of diffusion routing and width function was applied, and diameter-conditioned distribution functions of dimensionless peaks and dimensionless positions of the IUH were obtained analytically. The results of translation routing and diffusion routing approaches were compared and the results of this study were compared with those of existing literature.

Key Words: Instantaneous unit hydrograph, river network, width function, translation routing, diffusion routing.

Küçük Akarsu Ağları için Enstantane Birim Hidrograf Pik Karakteristiklerinin Dağılımları

Özet

Küçük ağların topolojik genişlik fonksiyonlarının pik oranı ("pik"/çap) ve konum oranlarının ("pike olan zaman"/çap) çap koşullu dağılım fonksiyonları analitik olarak elde edilmiştir. Burada çap ağın ana kol uzunluğu olmaktadır. Bu, lineer öteleme modellerinin en basiti olan kinematik öteleme kullanılarak yapılmıştır. "Düğüm noktası konfigürasyonu" adı ile anılan ve EBH (enstantane birim hidrograf)'ın karakteristik bir özelliği olan yeni bir kavram ortaya konulmaktadır. Bir başka yaklaşım, difüzyon ötelemesi ile genişlik kavramının birleşimi kullanılarak EBH'ın boyutsuz pikleri ve boyutsuz konumlarının çap koşullu dağılım fonksiyonları analitik olarak elde edilmiştir. Kinematik öteleme yaklaşımı sonuçları ile difüzyon ötelemesi yaklaşımı sonuçları karşılaştırılmış, ayrıca bu çalışmanın sonuçları mevcut literatür sonuçları ile karşılaştırılmıştır.

Anahtar Sözcükler: Enstantane birim hidrograf, akarsu ağı, genişlik fonksiyonu, kinematik öteleme, difüzyon ötelemesi.

Introduction

In the planning, design, construction and operation phases of hydraulic structures the main variable to be known is the amount of river flow. There are several approaches for determining this variable. There are stochastic methods as well as methods which make use of the geomorphology of the river basin and the topology of the river network. The IUH (instantaneous unit hydrograph) of a river basin is a classical tool commonly used as a component in rainfall-runoff modeling. One of the approaches found in literature to determine the IUH in relation to geomorphological basin characteristics is the use of width function. The width function concept will be explained in a further paragraph in detail. In this approach IUH is similar to the width function in case of pure translation routing. Translation routing is the simplest method which can be used. Diffusion routing, a more sophisticated method, is also of subject which is expected to produce more realistic results.

In this study, firstly, for a given main stream length, the cumulative p.d.f.'s of the "peak" and "time to peak"s (position of the peak) of the IUH for small networks are obtained analytically, producing the IUH's with the width function approach. Secondly, the same thing is done by producing the IUH's with a method developed in a previous work (Oğuz, B., 1994), this new method is a combination of the concepts of diffusion routing and width function. A comparison is made of the results obtained by the two methods.

General Information About River Networks

A channel network has points farthest upstream known as *sources*, and a point farthest downstream known as the *outlet*. The point at which two channels combine to form one channel is called a *junction*. A link is a segment of channel network between two successive junctions or between the outlet and the first junction upstream or between a source and the first junction downstream. The *main stream length* of the network is denoted by λ. This magnitude is also called the *diameter* of the network.

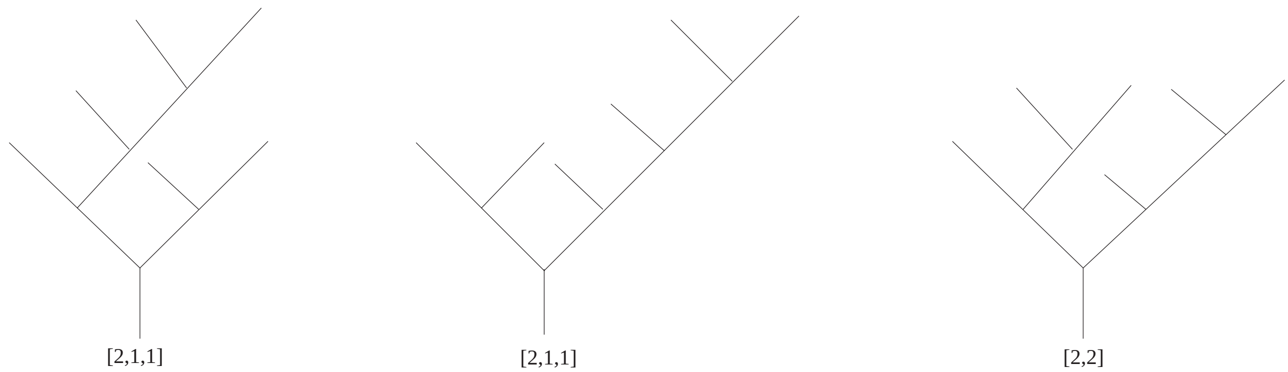


Figure 1. Three 6-source channel networks which are topologically different.

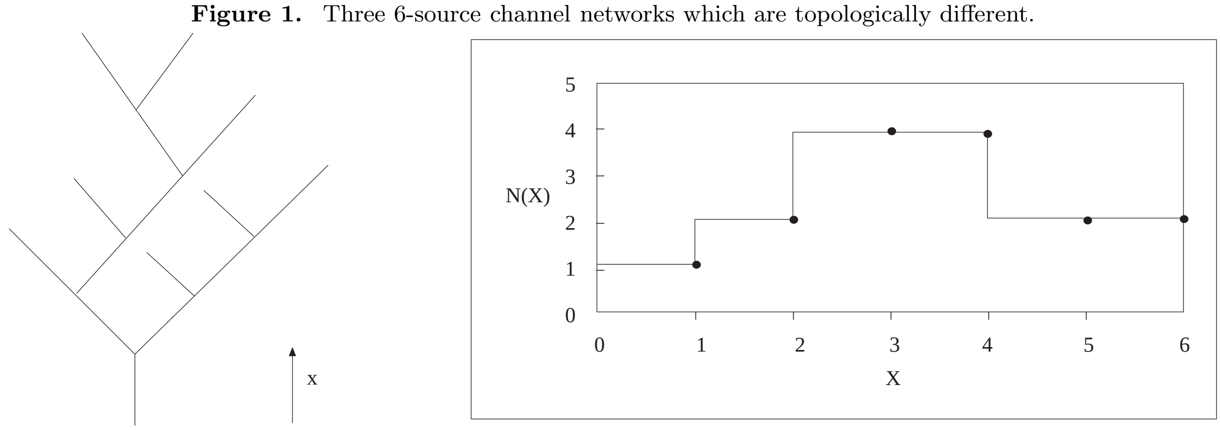


Figure 2. Hypothetical drainage basin and width function, N(x)

The important concept *topology* was first introduced into geomorphology by Shreve in 1966 (Smart, 1972). The number of sources, links and junctions of a channel network and the branching system of the network are topological characteristics of a network.

In Fig. 1, three channel networks with 6 sources are seen, of which the topologies are different. We know from previous literature that the number of topologically distinct channel networks (TDCN) would be 2, 5, 14 and 42 for networks with 3, 4, 5 and 6 sources, respectively (Smart, 1972). For example, a network with 5 sources may have 14 different network configurations and no more.

The *width function* is the "width", in a sense, of the network drawn against a distance x from the outlet. The width of the channel network is symbolized by the number of links (N(x)) that exist at a certain distance x from the outlet (Fig. 2). This concept is similar to what is known as the time-area diagram. Once the channel network is known, it is quite easy to determine its width function. The property of the network which will effect the hydrograph at the outlet is its width function. Given the number of sources, the number of all possible width functions remains much lower than the number of TDCN's.

A new concept, junction configuration, is defined as the numbers of junctions having i junctions between themselves and the outlet as $[a_1, a_2, \ldots, a_i, \ldots, a_n]$ where n=k-1(k:termination level of the network). In Fig. 1, 3 different TDCN's are seen for a network with 6 sources. The junction configuration of (a) and (b) are the same [2,1,1], whereas the junction configuration of (c) is different [2,2]. As an example [2,1,1], notation shows the number of junctions having only 1 junction between themselves and the outlet is 2; the number of junctions having 2 junctions between themselves and the outlet is 1; the number of junctions having 3 junctions between themselves and the outlet is 1. Each junction configuration has one width function corresponding to it.

For networks of a given diameter λ, the minimum and the maximum source numbers that can be of subject are given by the following equalities:

$$\mu_{min} = \lambda \quad \mu_{max} = 2^{\lambda-1} \tag{1}$$

(Agnese et al., 1998)

Translation Routing and the Width Function

Translation routing is one of the cases of linear routing for which the one-dimensional general flow equation can be solved. The simplest case is pure translation with constant velocity V. In the case of translation routing, it would be expected that the IUH can be described entirely by basin geometry. Agnese and D'Asaro (1990) argue that the distribution of link lengths do not play an important role in the prediction of peak characteristics. Thus one may assume constant link lengths (L).

If rainfall particles are injected instantaneously and uniformly at all the junctions and sources of the network in Fig. 2, these particles will travel a distance of a link length L during a time unit of L/V. The particle injected at the junction at level one will reach the outlet in one time unit. The particles injected at the junctions at level two will arrive at the outlet in two time units. In general, particles injected at junctions or sources at level i will reach the outlet in i time units. In other words, if N_i stands for the width function value at level i, then $N_i/\sum N_i$, percent of the total particles will arrive at the outlet in i time units. The total response of the network will be the same as the width function with the abscissa in units of time (L/V) instead of distance x. To find the unit response of the network at the outlet the width function values at all the levels, (N_i) must be divided by 2μ-1, which is the total number of sources and junctions. The unit response or the IUH can also be considered as the p.d.f. of the arrival times (T_a) of the particles to the outlet.

$$Pr(T_a = i\ \ time\ \ units) = N_i/(2\mu - 1) \tag{2}$$

λ-Conditioned CDF of Peaks and "Time to Peaks" of Width Function

Agnese et al., (1998) have obtained the λ-conditioned empirical probability densities of peak ratio (peak/λ) and position ratio ("time to peak"/λ) of the topological width function for λ ranging from 32 to 512. The networks they consider are large networks and their results are empirically obtained by simulation.

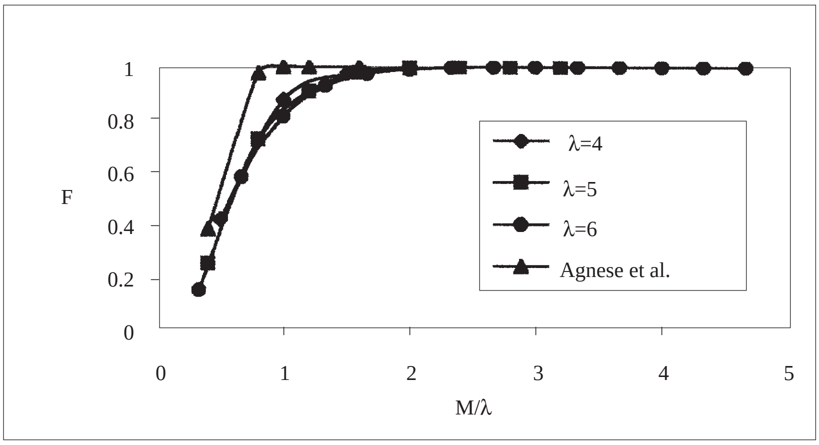
In this study, an analytical approach is used for the solution of the same problem. However, as will be explained further on, the use of this approach limited the work to small main stream lengths, that is, λ's. Agnese et al. (1998) have obtained probability densities of magnitudes which have certain dimensions. The aim of this study is to obtain the same distributions for dimensionless magnitudes.

For a given main stream length (λ), all the junction configurations must be determined, each of which will produce its own width function. For a given λ, the minimum and maximum numbers of sources that will be of subject are given by Eq. (1). Then by mathematical induction of the problem, it is seen that for a given λ and μ(magnitude=number of sources), the junction configuration $[a_1, a_2, \ldots, a_i, \ldots, a_{\lambda-2}]$ must be made up of (λ-2) positive integers, the sum of which is equal to (μ-2):

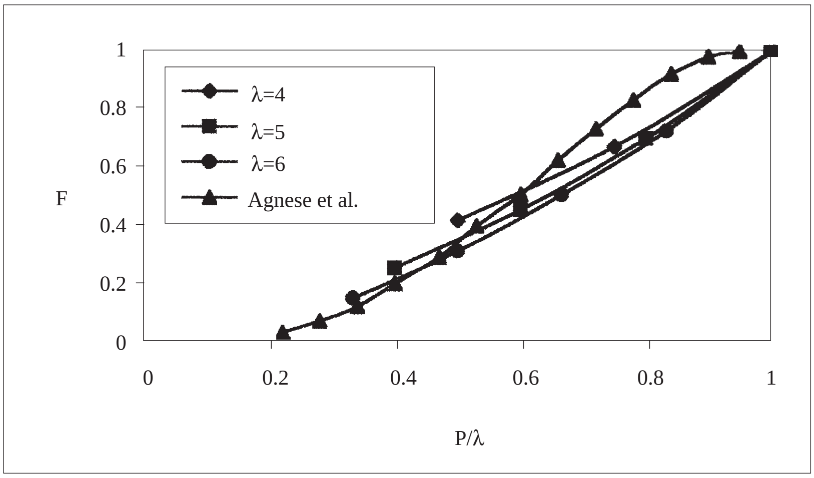
$$a_1 + a_2 + \ldots + a_{\lambda-2} = \mu - 2 \tag{3a}$$

Following inequalities must be satisfied since the number of junctions at level 1 or i cannot exceed twice the number of junctions at level i-1.

$$a_1 \leq 2, a_2 \leq 2a_1, \ldots, a_{i+1} \leq 2a_i, \ldots , a_{\lambda-2} \leq 2a_{\lambda-3} \tag{3b}$$

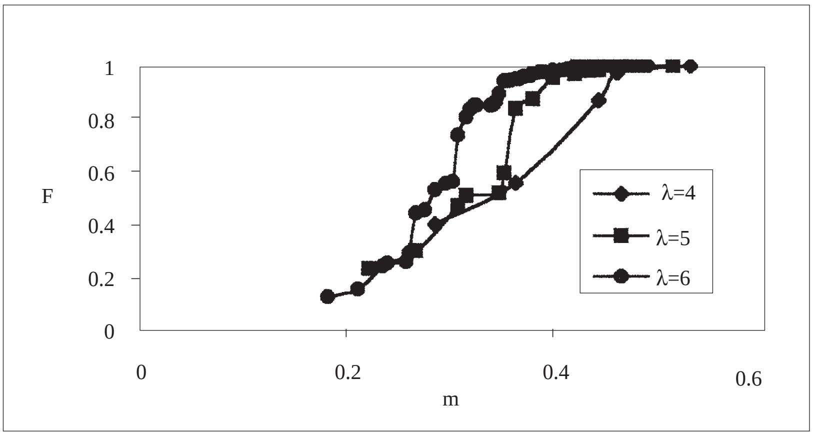


(a)

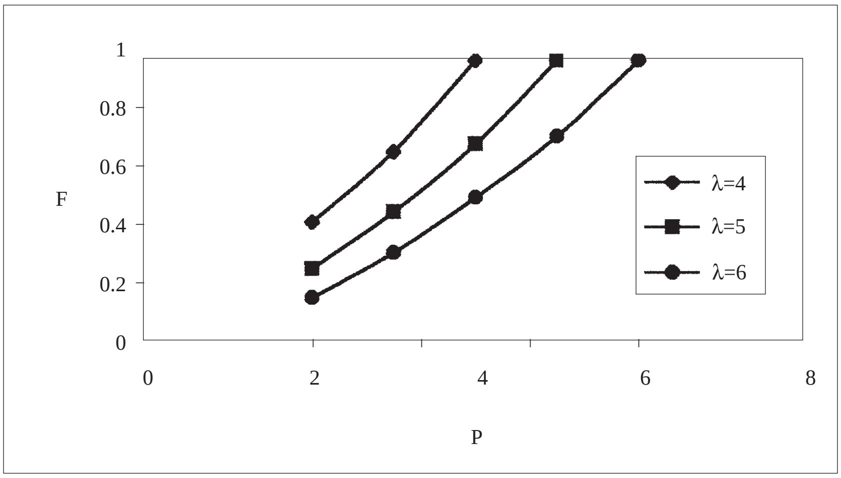


(b)

Figure 3. λ-conditioned distribution functions of the peak ratios (a) and position ratios (b) of the topological width functions.



(a)



(b)

Figure 4. λ-conditioned distribution functions of the normalized prak (a) and position (b) of the topological width functions.

Then, the problem of finding all junction configurations $[a_1, a_2, \ldots, a_i, \ldots, a_{\lambda-2}]$ for given λ within an interval of μ_{min} and μ_{max} for that λ turns into a problem of partitioning of mathematics, satisfying the conditions given in eq. 3b. This is done with the help of *Mathematica 3.02*, and all possible junction configurations for λ=4,5,6 are determined. The width functions (the "peaks" and "time to peaks") corresponding to the junction configurations can be obtained easily.

A second problem related to a junction configuration that must be solved is the probability of a junction configuration for a given λ. It is assumed that the probabilities of bifurcation and non-bifurcation are equal to 0.5 and that the branching of level i+1 independent of that of level i. As an example, looking at one of the [2,1,1] networks in Fig. 1, it is seen that, at the second level, the two links both succeed in bifurcating. At the third level, only one of the four links succeeds in bifurcating and the other three fail to bifurcate. At the fourth level, only one of the two links succeeds and the other fails. Therefore, the probability of junction configuration [2,1,1] can be written following the Bernoulli distribution as

$$Pr[2,1,1] = \underbrace{\binom{2}{2} 0.5^2 0.5^0}_{\text{Prob. of 2 successes at 2 trials}} \times \underbrace{\binom{4}{1} 0.5^1 0.5^3}_{\text{Prob. of 1 success at 4 trials}} \times \underbrace{\binom{2}{1} 0.5^1 0.5^1}_{\text{Prob. of 1 success at 2 trials}}$$

If this procedure is generalized for junction configuration $[a_1, a_2, \ldots, a_i, \ldots, a_n]$, the following expression is obtained,

$$Pr[a_1, a_2, \ldots, a_i, \ldots a_n] = \prod_{i=1}^{n} \binom{2a_{i-1}}{a_i} 0.5^{a_i} 0.5^{(2a_{i-1}-a_i)} \qquad a_0 = 1 \tag{4}$$

which can be simplified as

$$Pr[a_1, a_2, \ldots, a_i, \ldots a_n] = \prod_{i=1}^{n} \binom{2a_{i-1}}{a_i} 0.5^{2a_{i-1}} \quad a_0 = 1 \tag{5}$$

However, the probabilities of junction configurations as computed above for a certain λ do not sum up to one. Therefore, after calculating the probabilities of all the junction configurations which form a population for a certain λ, these probabilities are normalized by dividing them by their sum. Thus, for each λ, all possible "peak" and "time to peak" values of the width function with their corresponding normalized probabilities are found.

Then the λ-conditioned c.d.f.'s of the peak ratios (M/λ) and the position ratios (P/λ) of the topological width function are obtained for λ=4,5,6 (Fig. 3a,b). M and P stand for the peak and the position of the peak of the width function. λ main stream length is in units of [kL]. The units of M and P are [V/L] and [L/V] respectively. Consequently, peak ratio and position ratio are in units of [V/kL^2] and [1/Vk], respectively. In these figures, the results of Agnese et al. (1998) obtained for large networks by simulation are shown for comparison. The c.d.f.'s of Agnese et al. are not very accurate since these curves are transferred into Figs. 3a, b by reading values from their figures. The agreement of results obtained analytically in this study and the results of Agnese et al. (1998) obtained by simulation is quite good. Agnese et al. found that E[M/λ]≅0.5 independent of λ. This is also true for Fig. 3a. Agnese et al.(1998) give a value of E[M/λ]=0.60 for λ=32, and one can conclude from their results that a greater E[M/λ] must be expected for smaller λ's. The P/λ value corresponding to F(0.50) in Fig 3b is 0.7, in agreement with the above cited findings. The λ-conditioned c.d.f.'s of Fig.3a for λ=4, 5, 6 and of Fig. 3b for λ=5, 6 (the c.d.f. for λ=4 has only 3 points) are quite close, showing the scale invariance property of the peak and distance to peak of the topological width functions.

A second way of expressing the topological width function would by *normalizing* it so that it can correspond to the IUH of the network. The normalized peak is equal to

$$m = M/(2\mu - 1) \tag{6}$$

(Gupta and Waymire, 1983; Agnese et al., 1998). The c.d.f. of the *normalized* peak and the c.d.f. of the position corresponding to this case are given in Figs. 4a and b. The simulation data results of Agnese et al. (1998) show that E[m] for λ=6 is 0.28, which is exactly the value read from Fig. 4a for λ=6 corresponding to F(0.50). At this point, it should be noted that the normalized peak (m) in Fig 4a is the response of the network to unit input and its dimension is [1/T] and the dimension of P in Fig 4b is [T].

Diffusion Routing with Width Function

There are several methods used for flow routing. Diffusion wave routing is one type of the distributed (hydraulic) models used for this aim. Diffusion routing can be considered as a more realistic model than translation routing. In this model, the impulse response function is given as follows:

$$h(x,t;\beta) = x(4\pi\beta_2 t^3)^{-1/2} exp[-(4\beta_2 t)^{-1}(\beta_1 t - x)^2] \tag{7}$$

$$\beta_1 = 1.5V \tag{8}$$

and

$$\beta_2 = (2SB)^{-1} q(1 - F^2) \tag{9}$$

In eq.'s 6 and 7,
V is velocity, B is width,
q is discharge, F is the Froude number,
S_o is slope.

The impulse response function (Eq.(7)) is given for one-dimensional routing of flows in channels that are wide and rectangular, and in which the frictional effects are assumed to follow the Chezy law (Troutman and Karlinger, 1985).

Eq. (7) shows the response of a channel subject to an instantaneous upstream input which is at a distance of x (impulse response function). Eq. (7) can be interpreted as the p.d.f. corresponding to the travel time of a drop travelling a distance of x. This equation is not a dimensionless equation, the response h(x, t;β) is in units of [1/T].

In case the following definitions are made,

$$h^\star = \frac{h}{SV/y} \text{dimensionless response} \tag{10a}$$

$$t^\star = \frac{t}{y/SV} \text{dimensionless time} \tag{10b}$$

$$x^\star = \frac{x}{y/S} \text{dimensionless distance} \tag{10c}$$

the non-dimensionalized form of Eq. (7) can be obtained as follows:

$$h^\star(x^\star, t^\star) = x^\star[2\pi(1-F^2)t^{\star 3}]^{-1/2} \; exp\left[-\frac{(1.5t^\star - x^\star)^2}{2(1-F^2)t^\star}\right] \tag{11}$$

where y is the depth of water.

Since Eq. (7) and Eq. (11) can be interpreted as p.d.f.'s, the area under them must be equal to unity.

Introduction of diffusion routing into the width function can be done by the following formulation:

$$U(t) = \frac{\sum_x h(x,t), N(x)}{\sum N(x)} \tag{12}$$

U(t) will be the IUH satisfying the condition that the area underneath it is equal to unity. If we use the dimensionless response $h^\star(x^\star, t^\star)$, then we obtain the dimensionless IUH $U^\star$ versus dimensionless time $t^\star$. Here $U^\star$ is defined as follows: (Oğuz, 1994).

$$U^\star = \frac{U}{SV/y} \tag{13}$$

λ-Conditioned C.D.F.of Peaks and "Time to Peaks" of Diffusion Routing Hydrograph

Eq. (11) is applied to all the networks symbolized by their junction configurations for λ=4,5,6. In order to perform diffusion routing, values must be assigned to Froude number, depth of water and the slope. These values are selected as F=0.2, y=1m and S=0.0003 as common values in nature. Thus V=0.626 m/sec.

In order to find the IUH of a network, calculations at different x distances from the outlet must be made. These x distances are selected at equal Δx distances from the previous one. Also calculations at Δt intervals must be made. In this case, Δx and Δt are chosen as 50m and 1 min, respectively. The link lengths of the networks were assumed to be equal and a value of 1000 m was assigned. Thus the c.d.f.'s of peaks $m^\star$ and positions $P^\star$ of the dimensionless IUH of diffusion routing (width function combination) for the above case are given in Figs. 5a, b.

In order to compare the results of diffusion routing-width function scheme, which are dimensionless (Figs. 5a, b), with those of translation routing-normalized width function scheme, which have dimensions (Figs. 4a, b), Figs. 4a and b must be transformed into a dimensionless form. The peak of the normalized width function, m, in Fig. 4a has a dimension of [1/T]. One time interval belonging to the normalized width function must be equal to the link length divided by the velocity,

$$\frac{1000}{0.626}\frac{m}{m/sec} = 1597 sec$$

Thus the dimensionless time using Eq. (10b) is,

$$t^\star = \frac{t}{y/SV} = \frac{1597}{1/(0.0003 \times 0.626)} = 0.3$$

Since the peak of the normalized width function m in Fig. 4a has a dimension of [1/T], the c.d.f. of dimensionless width function peak $m^\star$ is obtained (Fig. 6a) by dividing m values by 0.3. Parallel to this, the P values in Fig. 4b have a dimension of [T]. The c.d.f. of dimensionless position $P^\star$ is obtained (Fig. 6b) by multiplying the P values by 0.3. Now dimensionless c.d.f.'s of IUH peak characteristics of Fig. 5a,b can be compared.

Looking at Figs. 5a and 6a, almost the same results are obtained with respect to the median. The corresponding dimensionless IUH peak values ($m^\star$) by diffusion routing corresponding to F(0.50) for λ=6,5,4 are 0.9, 1.1, 1.4, respectively. On the other hand, the values ($m^\star$) by translation routing corresponding to F(0.50) for λ=6,5,4 are 0.96, 1.05, 1.15, showing a great similarity. However, for other F values, the agreement between the diffusion routing results and those of translation routing are not very good due to the greater variance of the translation routing dimensionless IUH peak values than that of those of diffusion routing.

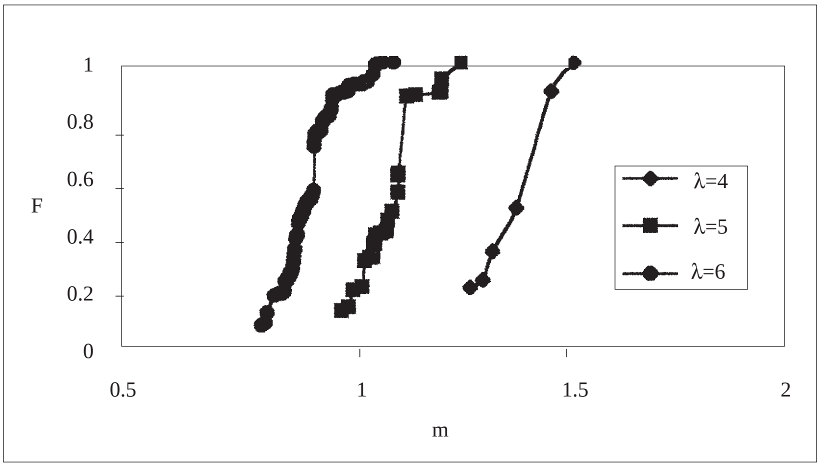
Comparing Figs. 5b and 6b, it is seen that the dimensionless position of IUH peak ($P^{\star}$) remains greater for the translation routing case than the diffusion routing case in general. The variance of $P^{\star}$ for the translation routing case is greater than that for the diffusion routing case.

Conclusions

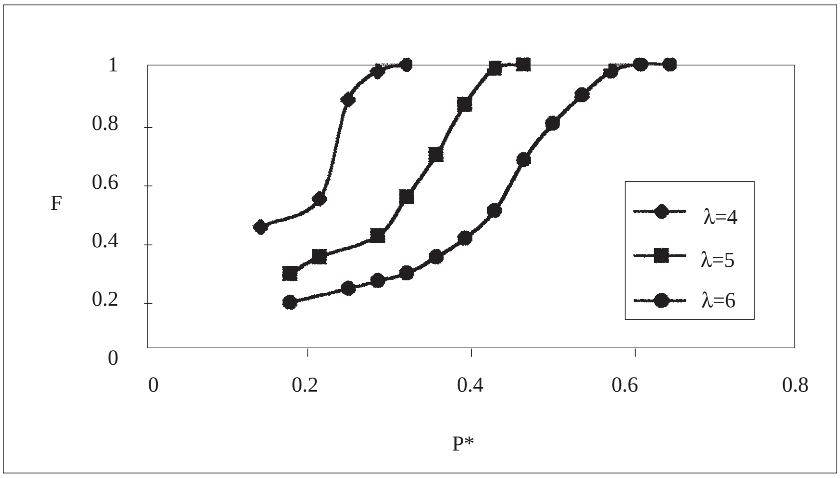
Distribution functions of IUH peak ratios and position ratios obtained for small networks by an analytical approach agree well with the results of Agnese et al. (1998) obtained by simulation for large networks. The analytical results show the scale invariance properties of the peak of the width function.

Distribution functions of dimensionless IUH peaks for diffusion routing and for translation routing coincide with respect to the median, with the variance of the latter remaining greater than that of the former.

The dimensionless position of the IUH peak remains smaller for the diffusion routing case than the translation routing case; i.e., the peak is seen earlier for a diffusion routing case under the same conditions as for a translation routing case. The variance of the dimensionless position is greater for the translation routing case than for the diffusion routing case.



(a)



(b)

Figure 5. λ-conditioned distribution functions of dimensionless peaks (a) and dimensionless positions (b) of the IUH obtained by diffusion routing.

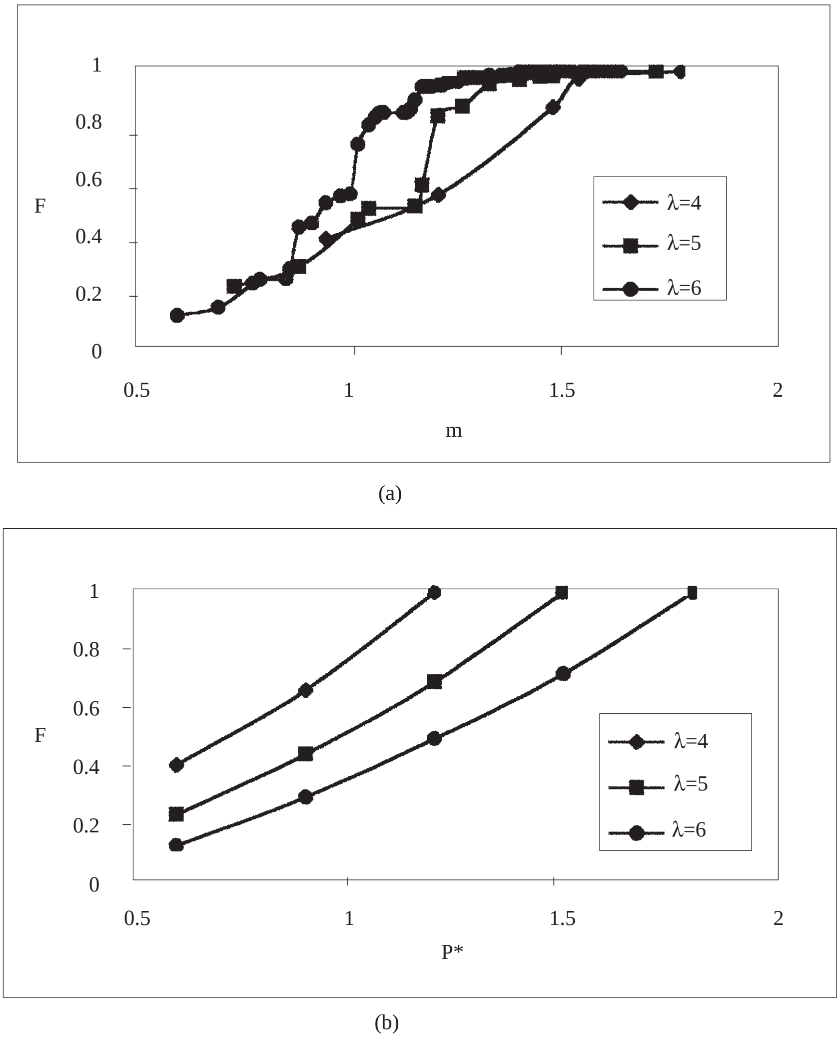


Figure 6. λ-conditioned distribution functions of dimensionless peaks (a) and dimensionless positions (b) of the width function.

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List of Symbols

B	width of channel network
E(.)	expected value (.)
F	Froude number
h	impulse response function
$h^\star$	dimensionless response
k	termination level of network
L	link length
m	normalized peak of width function
$m^\star$	dimensionless peak of IUH
M	peak of width function
N_i	number of links at level i
N(x)	width function
P	position of peak of width function
$P^\star$	dimensionless position of IUH
Pr	probability
q	discharge
S	slope of channel network
t	time
$t^\star$	dimensionless time

T_a	arrival time of the particles to the outlet	x_j	number of links at level j
U	IUH ordinate	y	depth
$U^\star$	dimensionless IUH ordinate	β_1	advective velocity (celerity)
V	velocity	β_2	diffusion coefficient
x	distance	λ	main stream length, diameter of network
$x^\star$	dimensionless distance	μ	source number, magnitude of network

References

Agnese, C., and D'Asaro, F., Comment on "Predictors of the Peak Width for Networks with Exponential Links" by B. M. Troutman and M. R. Karlinger, Stochastic Hydrol. Hydraul., 4, 83-87, 1990.

Agnese, A., Criminisi, A. and D'Asaro, F., "Scale Invariance Properties of the Peak of the Width Function in Topologically Random Networks", Water Resour. Res., 34 (6), 1571-1583, 1998.

Gupta, V. K., and Waymire, E., "On the Formulation of an Analytical Approach to Understand Hydrological Response at the Basin Scale", J. Hydrol., Vol. 65, No.1/3 95-124, 1983.

Oğuz, B., "An Approach to the Derivation of the Instantaneous Unit Hydrograph of a Topologically Random Channel Network", Procs. of NATO ASI Defence from Floods and Floodplain Management, (eds. Gardiner, J., Starosolszky, Ö. and Yevjevich, V.), 1994.

Smart, J. S., "Channel Networks", Advances in Hydroscience, (ed. Ven Te Chow), Vol. 8, Academic Press, 1972.

Troutman, B. M., and Karlinger, M. R., "Unit Hydrograph Approximations Assuming Linear Flow Through Topologically Random Channel Networks", Water Resour. Res., 21(5), 743-754, 1985.

Troutman, B. M., and Karlinger, M. R., "Averaging Properties of Channel Networks Using Methods in Stochastic Branching Theory", Scale Problems in Hydrology, (eds. Gupta, V. K., Rodriguez-Iturbe I. and Wood E. F.), D. Reidel Publishing Comp., 1986.