

Initial Strain Displacement Gradient Elastoplastic Boundary Element Formulation

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Abstract

This paper presents a robust formulation of the boundary element (BE) method for more complex elastoplastic applications such as elastoplastic frictional contact problems. The elastoplastic BE formulation is applied to standard test problems. The BE solutions are compared with the corresponding finite element (FE) solutions provided by a commercially available FE package, ABAQUS, and analytical or experimental results.

Key Words: Plasticity, efficiency, boundary element method

Başlangıç Şekil Değiştirme ve Deplasman Gradyenti Esasına Daylı Elastik Plastik Sınır Elemanları Formülasyonu

Özet

Bu makale elastoplastik sürtünmeli temas problemleri gibi daha karmaşık elastoplastik uygulamalar için sınır elemanlar(SE) yönteminin sağlıklı bir formülasyonunu sunmaktadır. Elastoplastik SE formülasyonu standart test problemlerine uygulanmıştır. SE sonuçları analitik veya deneysel sonuçların yanında sonlu elemanlar (FE) paket programı ABAQUS ile elde edilen FE sonuçlar ile karşılaştırılmıştır.

Anahtar Sözcükler: Plastisite, verimlilik, sınır elemanları yöntemi

Introduction

The boundary integral equation (BIE) method is well established as an accurate numerical tool particularly suitable for linear elastic problems. Due to its high resolution of stresses on the surface, the BIE approach has been shown to be well suited to problems involving stress concentration, fracture mechanics and contact analysis. However, its extension to non-linear problems is not widespread and is under developed when compared to the Finite Element (FE) method. In many non-linear BIE formulations

the interior of the solution domain has to be discretised, thus losing the main BIE advantage of surfaceonly modelling. Another difficulty encountered in the non-linear BIE formulations is the accurate evaluation of strongly singular integral functions.

The elastoplastic boundary element formulation has been covered in a number of publications; however, a degree of ambiguity was present in some implementations, which may explain why the BE method is lagging behind the FE method in non-linear applications. Because of the nature of boundary integral identities, the elasto-plastic BE needs

special attention in incremental-iterative processes. In the literature it can be observed that the most of the presented algorithms tend to cause a stiff response. Therefore the initial strain displacement gradient elastoplastic BE is presented with emphasis on a robust incremental-iterative process and efficiency to provide a foundation for extension to more complex problems such as contact mechanics and more complex plasticity problems.

Governing Equations

By considering a two-dimensional isotropic homogeneous body, which has the boundary S and the domain A during time-independent plastic deformation, the Navier rate equation for plane strain and plane stress can be written as follows:

$$\frac{\partial^2 \dot{u}_I}{\partial x_j \partial x_j} + \frac{1}{(1-2\nu)} \frac{\partial^2 \dot{u}_j}{\partial x_j \partial x_i} - \left(k_1 \frac{\partial \dot{\epsilon}_{jj}^p}{\partial x_i} + 2 \frac{\partial \dot{\epsilon}_{ij}^p}{\partial x_j} \right) = \frac{-\dot{f}_i}{\mu} \quad (1)$$

where \dot{u}_i is the displacement rate, \dot{f}_i is the body-force rate, $\dot{\epsilon}_{ij}^p$ is the plastic strain rate and the parameter k_1 is either 0 (for plane strain) or $1/(1-2\nu)$ (for plane stress). For a material obeying the von Mises yield criterion, the plastic strain rates are given by the following incremental elasto-plastic flow rules:

$$\dot{\epsilon}_{ij}^p = \frac{9}{4} \left(\frac{\dot{S}_{kl} \dot{\sigma}_{kl}}{H} \right) \frac{\dot{S}_{ij}}{(\dot{\sigma}_{eq})^2} \quad (2a)$$

$$\dot{\varphi}_{ij}^p = \frac{3}{2} \left(\frac{\dot{S}_{kl} \dot{\varphi}_{kl}}{1 + H/3\mu} \right) \frac{\dot{S}_{ij}}{(\dot{\sigma}_{eq})^2} \quad (2b)$$

where \dot{S}_{ij} and \dot{S}_{eq} denote the current deviatoric stress tensor and the equivalent stress respectively, $\dot{\epsilon}_{ij}$ is the total strain rate and H represents the plastic modulus. Axelsson and Samuelson [1979] proposed that the plastic strain rate is decomposed into its isotropic and kinematic parts as follows:

$$d\varphi_{ij}^{p(i)} = M d\varphi_{ij}^p \quad (3a)$$

$$d\varphi_{ij}^{p(k)} = (1 - M) d\varphi_{ij}^p \quad (3b)$$

in which M is defined as the mixed hardening parameter, which is equal to -1 for isotropic softening, 0 for kinematic hardening and 1 for isotropic hardening. The slope of the stress-plastic strain curve in a uniaxial tensile test, H, is given as follows:

$$H = \frac{d\sigma_{eq}}{d\epsilon_{eq}^{p(i)}} \quad (4)$$

Analytical Boundary Element Formulation for Elastoplasticity

One way of the dealing with the plastic terms in the Navier rate equations is to treat the derivatives of the plastic stress rates (see Henry and Banerjee(1988)) or the plastic strain rates (see, for example, Mendelson (1973)) as a kind of body force. Alternatively, by considering the plastic strain rates to be initial strain rates and then modifying Betti's reciprocal theorem to include plasticity, the pseudo-boundary integral equation for the initial strain approach can be written as follows (see, for example, Lee (1983)):

$$C_{ij}(P) \dot{u}_j(P) + \int_S T_{ij}(P, Q) \dot{u}_j(Q) dS(Q) = \int_S U_{ij}(P, Q) \dot{t}_j(Q) dS(Q) + \int_A U_{ij}(P, q) \dot{f}_j(q) dA(q) + \int_A W_{kij}(P, q) \dot{\epsilon}_{ij}^p(q) dA(q) \quad (5)$$

In this expression U_{ij} and T_{ij} are fundamental displacements and tractions at x in the jth direction at the field point Q or q due to a unit load at load point P acting in the ith direction. W_{kij} represents the third-order tensor for the stress at the field point due to a unit load at load point P. The tensors U_{ij} , T_{ij} , W_{kij} , derived from the fundamental solution to Kelvin's problems in two dimensions, are given as follows:

$$U_{ij}(P, Q) = -\frac{1}{8\pi(1-\nu)\mu} \left\{ \delta_{ij}(3-4\nu) \ln r - \frac{\partial r}{\partial x_i} \frac{\partial r}{\partial x_j} \right\} \quad (6)$$

$$T_{ij}(P, Q) = -\frac{(1-2\nu)}{4\pi(1-\nu)} \frac{1}{r} \left\{ (\delta_{ij} + \frac{2}{(1-2\nu)} \frac{\partial r}{\partial x_i} \frac{\partial r}{\partial x_j}) \frac{\partial r}{\partial n} - \frac{\partial r}{\partial x_i} n_j + \frac{\partial r}{\partial x_j} n_i \right\} \quad (7)$$

$$W_{kij}(P, q) = -\frac{1}{4\pi(1-\nu)} \frac{1}{r} \left\{ (1-2\nu) (\delta_{jk} \frac{\partial r}{\partial x_i} + \delta_{ik} \frac{\partial r}{\partial x_j} - k \delta_{ij} \frac{\partial r}{\partial x_k}) + 2 \frac{\partial r}{\partial x_i} \frac{\partial r}{\partial x_j} \frac{\partial r}{\partial x_k} \right\} \quad (8)$$

where the parameter k is either equal to 1 (plane stress) or $1/(1-2\nu)$ (plane strain). Finally, the free-term tensor, C_{ij} , is defined as follows:

$$C_{ij}(P) + \delta_{ij} + \lim_{\epsilon \rightarrow 0} \int_{S_\epsilon(P)} T_{ij}(P, Q) dS(Q) \quad (9)$$

BE Elasto-plastic Approaches

In order to include the effect of plasticity in the solution domain, additional domain integrals appear in the boundary integral equations and they display strongly and weakly singular behaviour, of the order of $1/r^2$ and $1/r$ respectively, in two-dimensional applications. The techniques for evaluation of the strong singular integrals can be classified into two main groups called the indirect and direct approaches.

From the literature it can be seen that indirect techniques need to consider a known reference solution such as an admissible stress field (see Henry and Banerjee [1988]) and a constant plastic strain field (see Brebbia et al, [1984]). In the application of this approach, a subregion formulation can be used in order to avoid the discretisation of the entire domain to be solved.

The direct approaches, which are commonly used, are based on the regularisation of the strongly singular integrals, except the one presented by Banerjee and Raveendra [1986] in which the strongly singular integrals are excluded from the solution domain using a small circle (or sphere for three-dimensional applications), where plastic strain is assumed to be constant, and calculated analytically. The regularisation procedure of the strongly singular integrals can be performed using polar coordinate transformation and Taylor series expansion (see Guiggiani et al. [1992]). The advantage of this approach is the handling of the hyper-singular integral equation (HSE), in which the singular integral is of higher order singularity than the strongly singular integral equation (SSE), with high accuracy for any arbitrary integration cell. It is possible to regularise the strong singular integral using Gauss theorem without consulting special coordinate transformation and the representation of Taylor series expansion (see Dallner and Kuhn [1993]). The advantage of this approach is that it is applicable to any arbitrary integration cell and location of the singular point in which the load point (source point) is located.

It can be concluded that the use of the integral identities gives more accurate solutions for the interior stress and strain rates, but it is obviously more tedious and consumes more computation time. It is possible to circumvent the strongly singular integrals by differentiating the displacement rates via the shape functions in order to obtain the strain and the stress rates or by using the particular integral approach.

In the first approach, which is well known and may be called the classical approach, the discretisation of the entire solution domain is not compulsory, because the stress rates at interior points and at boundary nodes can be treated separately. In the particular integral approach, the effect of the plasticity in the solution domain is treated as a special kind of body force and can be put into boundary variables over the global shape functions without performing any domain integration. In this method the internal (fictitious nodes) should be consistent with the boundary discretisation. Therefore, the entire solution domain has to be discretised by using both boundary nodes and fictitious nodes. In order to avoid the discretisation, whole domain sub-region formulation must be implemented.

Numerical Implementation of the Integral Equation

It is obvious from the elasto-plastic BE formulation discussed above that both boundary elements and domain cells (internal cells) are necessary in order to perform the integrals arising in the BE formulation. Both the boundary elements and the domain cells used in two-dimensional elasto-plastic BE analysis are illustrated in Figure 1. In a manner similar to the elastostatic BE analysis, the boundary is represented as a collection of boundary elements. The zones, where the plastic deformation is expected, in the solution domain are discretised into domain cells in order to perform domain integrals.

The elasto-plastic BE equation in the initial strain approach (without considering body forces) in discretised form, can be written as follows:

$$\begin{aligned}
 C_{ij}(P)\dot{u}_j(P) + \sum_{m=1}^M \sum_{c=1}^3 \dot{u}_j(Q) \int_{-1}^{+1} T_{ij}(P, Q) N_c(\xi) J(\xi) d\xi &= \sum_{m=1}^M \sum_{c=1}^3 \dot{t}_j(Q) \int_{-1}^{+1} U_{ij}(P, Q) N_c(\xi) j(\xi) d\xi \\
 + \sum_{m=1}^D \sum_{c=1}^8 \dot{\epsilon}_{ij}^p(q) \int_{-1}^{+1} \int_{-1}^{+1} W_{kij}(P, q) N_c(\xi_1, \xi_2) j(\xi_1, \xi_2) d\xi_1 d\xi_2 & \quad (10)
 \end{aligned}$$

where P denotes the node where the integration is performed, Q indicates the c^{th} node of the m^{th} boundary element and q indicates the c^{th} node of the m^{th} domain cell. M is the total number of boundary elements and D is the total number of domain cells. $N_c(\chi)$ is the quadratic shape function and $J(\chi)$ is the jacobian of transformation. The integration process and evaluation of equation coefficients are explained in detail by Becker (1997).

The Displacement Gradient Approach

By differentiating the interior displacements via the shape functions over each domain cell, and using the jacobian of transformation, the displacement differentials with respect to the local co-ordinates χ_1 and χ_2 can be obtained. To obtain the total strain rates, the strain-displacement relationship can be used as follows:

$$\dot{\epsilon}_{ij} = \frac{1}{2} \left(\frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i} \right) \tag{11}$$

In the plane stress problems the total strain rate in the third direction can be obtained by using $\dot{\sigma}_{33}=0$, i.e.,

$$\dot{\epsilon}_{zz} = \frac{-\nu}{1-\nu} \frac{-\nu}{1-\nu} (\dot{\epsilon}_{xx} + \dot{\epsilon}_{yy}) - \frac{1-2\nu}{1-\nu} (\dot{\epsilon}_{xx}^p + \dot{\epsilon}_{yy}^p) \tag{12}$$

Using Hookes law, the cell stress rates can be written as follows:

$$\dot{\sigma}_{ij} = 2\mu(\dot{\epsilon}_{ij} - \dot{\epsilon}_{ij}^p) + \frac{2\mu\nu}{1-2\nu} \delta_{ij} \dot{\epsilon}_{kk} \tag{13}$$

It should be noted that the stresses at the boundaries are obtained from the local displacement gradients and traction rates. Full details of the evaluation of the stress and strain rates at the boundary and interior nodes are presented by Gun (1997).

Incremental/Iterative Solution Procedure

In the elastoplastic FE analysis, depending on the formulation of the stiffness matrix, either the tangential stiffness technique or the initial stiffness technique can be employed. However, because of the nature of the elastoplastic BE formulation, neither of these approaches can be used. Instead, the concepts of the initial strain and the initial stress techniques, used in the FE approach, can be modified in order to be applicable to the elastoplastic BE analysis.

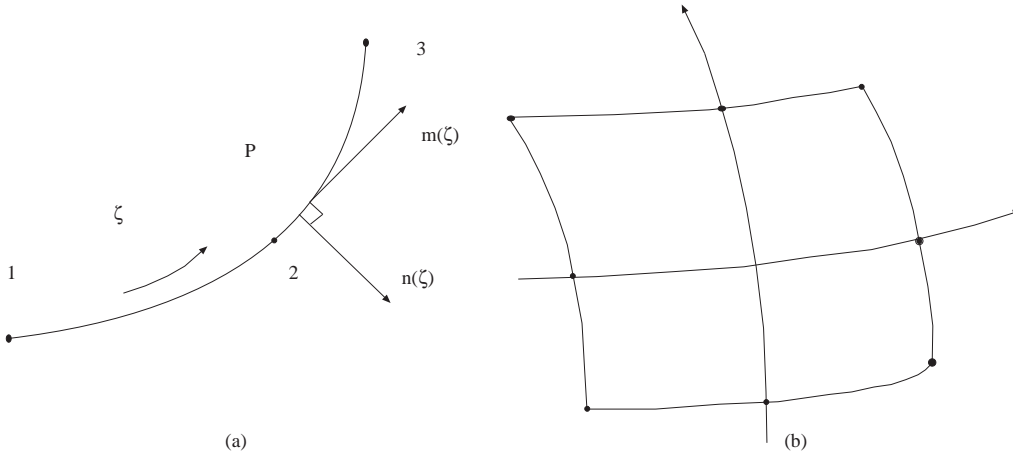


Figure 1. Isoparametric quadratic elements used in elastoplastic BE analysis (a) Three-noded boundary element (b) Eight-noded quadrilateral domain cell

In the elastoplastic BE analysis, the plastic strain increments can be calculated by using the flow rule expressions in equations (2a) and (2b). In equation (2a) the actual stress increments are required in order to obtain the initial strain increments. The initial stresses can be obtained by using equation (2b),

in which the total strain increments are assumed to be known. In the initial strain approach it is possible to obtain the plastic strain increments by using equation (2b), which can handle the perfectly plastic material behaviour. In BE formulations, there is no significant difference between the initial stress approach

and the initial strain approach, because the integral equations in both approaches include the effect of plasticity. The initial strain formulation is considered more suitable for traction-controlled problems, because the first approximations for the stress increments are usually reasonably accurate.

Both the flow rules and integral equations are given in rate form. Therefore, at the end of the iteration process for each load step, a suitable pseudo-time step is chosen and multiplied by the rates in order to determine the actual incremental variables. The time step can be chosen as unity and the prescribed boundary conditions, defined in an incremental form, are treated in the same manner.

In order to define the yield surface, it is necessary to use a scalar factor R to multiply the stress increment, as follows:

$$f\sigma_{ij} + Rd\sigma_{ij}^e + Rd\sigma_{ij}^e - Y = 0 \quad (14)$$

In this expression, $d\sigma_{ij}^e$ is the incremental estimated elastic stress vector and σ_{ij} is the stress state at the beginning of the load step. More refined methods can be used for the evaluation of R , e.g., using quadratic interpolation (e.g., Bicanic (1989)).

During the iteration process, by using the yield criterion (surface) f^n in the n^{th} iteration and the value of the yield criterion f^{n+1} in the $(n+1)$ iteration number, the following four possibilities have to be examined:

i) $f^n < 0$ and $f^{n+1} < 0$, i.e., there is no plastic deformation and the solution is purely elastic.

ii) $f^n < 0$ and $f^{n+1} \geq 0$, i.e., transition occurs from elastic to plastic conditions. Therefore, the evaluation of the elastic stress increments is required in order to bring the stress state to where the onset of yield commences. The plastic strains are obtained by using the flow rule and the remaining part, $(1-R)d\sigma_{ij}^e$, which occurs beyond the yield point.

iii) $f^n = 0$ and $f^{n+1} \geq 0$, i.e., the stress increments occur completely beyond the yield point. Hence it requires only computation of the plastic strain increment by using either equation (2a) or equation (2b).

iv) $f^n = 0$ and $f^{n+1} < 0$, i.e., unloading takes place and it is assumed that there is no plastic deformation for the node being considered.

Correction Factor

It is necessary to use a correction factor for plane strain cases, because the stress increment in the third direction, σ_{33}^P , is computed indirectly and employed in the flow rule. Therefore, it may be assumed that

the new values obtained using the flow rules will not agree with the old values. Lee (1983) presented correction factors for traction or displacement control problems, in order to consistently employ the flow rules during the iterations. For problems with prescribed traction boundary conditions, the correction factor, λ , is given by

$$\lambda = \frac{(\dot{S}_{kl}\dot{\sigma}_{kl})^{n-1} + E\dot{S}_{33}(\epsilon_{33}^P)^n}{(\dot{S}_{kl}\dot{\sigma}_{kl})^{n-1} + E\dot{S}_{33}(\epsilon_{33}^P)^{n-1}} \quad (15)$$

In addition to the above correction factor, an acceleration factor, α , ranging from 1.2 to 1.7 can be used to accelerate convergence, as follows:

$$\dot{\epsilon}_{ij}^p = (\dot{\epsilon}_{ij}^p)^n + \alpha [(\dot{\epsilon}_{ij}^p)^{n-1} - (\dot{\epsilon}_{ij}^p)^n] \quad (16)$$

Converge Criteria

In order to ensure that a prescribed standard accuracy is satisfied in the iterative process, a convergence criterion must be used. There are four practical convergence criteria:

(i) The norm of changes in the primary unknown vector $[\dot{x}]$,

(ii) The norm of the changes in the plastic strain increments,

(iii) The magnitude of changes in the primary unknowns vector, $[\dot{x}]$, or

(iv) The magnitude of the changes in the plastic strain increments which are considered as secondary unknown.

The first and second criteria are based on the square root of the sum of the components of either the primary unknowns or the secondary unknowns. For this type of convergence criterion, typical values of percentage changes range between 0.001% and 0.01%. The third and fourth criteria are based on the absolute values of the largest term in either the primary unknowns or the secondary unknowns to the largest percentage change in each term of either the primary unknowns or the secondary unknowns. For this type of convergence criterion, the convergence values for the percentage changes range between 0.1% and 1%. It is worth mentioning that the present algorithm allows the user to attempt to reduce load increment to a minimum and to use average values for the stress and strain rates during iterations, which is more practical, efficient and suitable for more complex plasticity applications, such as contact problems. The full details of the present algorithm can be found in the work of Gun (1997).

Numerical Results

The present initial strain formulation, in which the displacement rates, obtained from the integral equations for displacement rates, are differentiated via the shape functions in order to obtain the stress and strain increments at interior points, is applied to some classical test problems and the results are compared with either analytical solutions or the corresponding FE solutions. In order to evaluate the integrals, six Gaussian points for integrals over boundary and interior elements are employed.

For the FE analysis, the commercially available finite element package ABAQUS [1995] was used. ABAQUS is a general purpose FE package with many non-linear capabilities. The ABAQUS input file is prepared in terms of elements, nodes' material properties, boundary conditions, load steps and output control. In modelling elasto-plastic problems using ABAQUS, the user can either define the number of load steps and incrementations within each step or let the program use automatic incrementation by dividing the total applied load into small steps and analysing each load-step. Isoparametric quadratic quadrilateral 8-noded elements and 4-noded elements with reduced (2x2) integration points were used in all the applications.

Uniaxial Test Problems (plane stress)

This test problem consists of a square plate subjected to uniform tension in the x-direction. The geometry and loading conditions are presented in

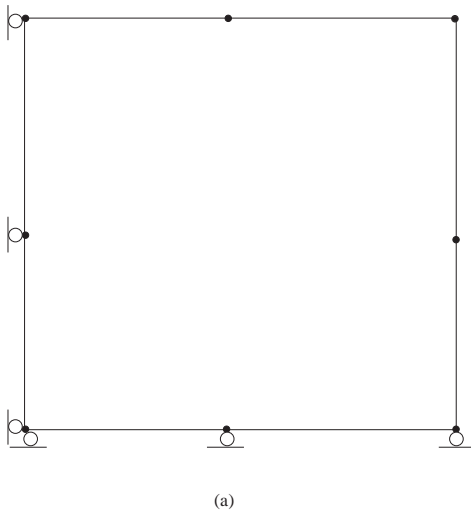


Figure 2. The BE discretisation is shown in Figure 3 where two meshes were used: mesh A (4 boundary elements with no internal points) and mesh B (8 boundary elements and 4 internal cells). The material is to be loaded in tension up to 587 N/mm² and it is to be solved using 24 increments. The following properties are assumed for the linear strain hardening material

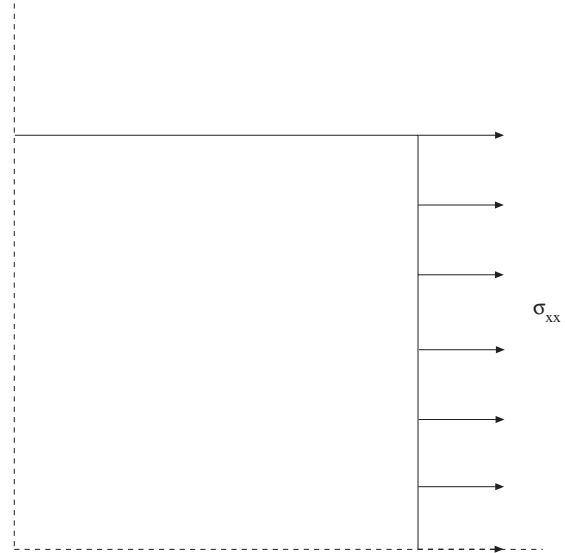


Figure 2. Geometry and loading condition for uniaxial test problem

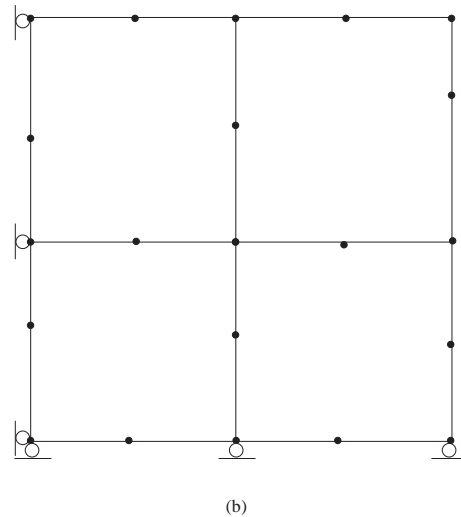


Figure 3. Boundary element discretisation for uniaxial test problem

$$\begin{aligned} \sigma_y &= 483 \text{ N/mm}^2 \\ E &= 200 \times 10^3 \text{ N/mm}^2 \\ \nu &= 0.3 \\ H &= 4223.8267 \text{ N/mm}^2 \end{aligned}$$

In the FE model, this problem is represented by 4 quadratic plane stress elements with reduced (2x2) integration. As depicted in Figure 4(a), the stress-strain curve given by mesh A, which has no internal points, is in excellent agreement with the corresponding FE results. Figure 4(b) shows a comparison of the two BE meshes, which indicates that there are no

significant changes in stress levels either at interior or boundary nodes used in mesh B.

Thick Cylinder Under Internal Pressure

The geometry and loading conditions of this test problem are given in Figure 5. The radius ratio b/a is taken to be 2. The analytical solution of this problem was presented by Hodge and White [1950]. The material is assumed to be elastic-perfectly plastic with the following material properties:

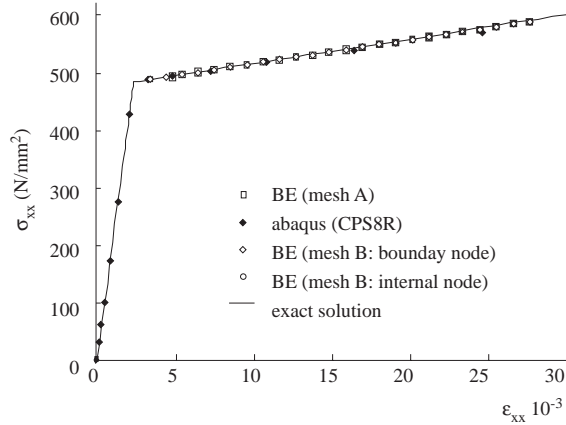
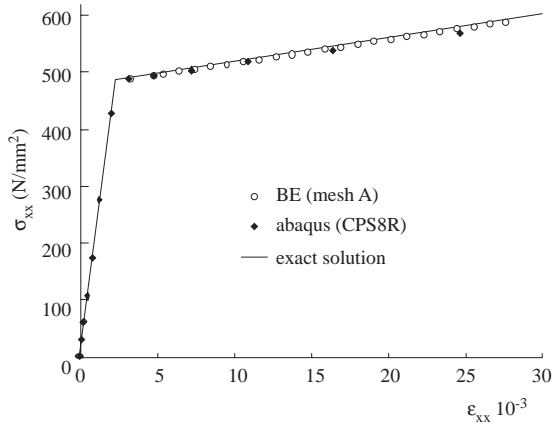


Figure 4. (a) : Results for uniaxial plane stress problem (BE, using mesh A, and ABAQUS)
 (b) : Results for uniaxial plane stress problem (BE, using mesh A and mesh B, and ABAQUS).

$$\begin{aligned} \sigma_y &= 200 \text{ N/mm}^2 \\ E &= 200 \times 10^3 \text{ N/mm}^2 \\ \nu &= 0.33 \end{aligned}$$

The boundary element discretisation is shown in Figure 6, in which two meshes are used: mesh A with 10 boundary elements and 4 cells occupying a 15° sector and mesh B occupying a 50° sector with the same number of elements. For the FE analysis, four 8-noded axisymmetric elements with reduced (2x2) integration are employed. The variation in the load factor P/σ_y with the non-dimensionalised displacement ($4\mu\mu_b/\sigma_y a$) at the outer radius for both BE meshes and the corresponding FE solutions are depicted in Figure 7. In the BE analysis 5 load increments were used to reach the load factor, P/σ_y , value of 0.79. The BE solutions show very good agreement with the analytical and FE results. Figure 8 shows the hoop stress distribution along the radius for the load factor, P/σ_y , value of 0.76. The BE results, computed using 5 load increments are again in good agreement with the analytical and FE results.

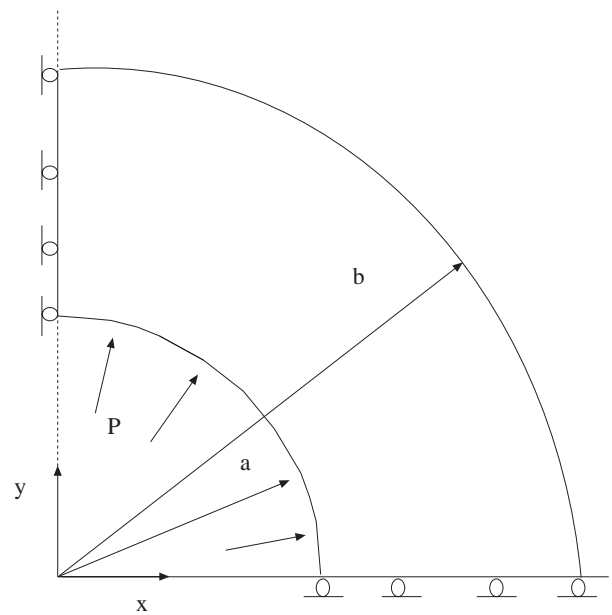


Figure 5. Thick cylinder subjected to internal pressure

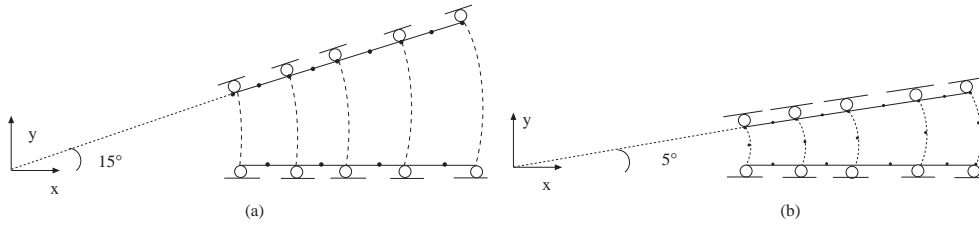


Figure 6. BE discretisation for internally pressurised cylinder (a) Mesh design for mesh A (b) Mesh design for mesh B

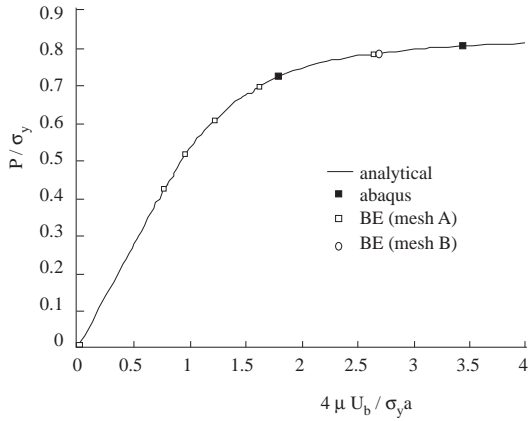


Figure 7. Non-dimensionalised radial displacement at outer radius of thick cylinder for BE, using mesh A and B, ABAQUS and analytical results from Hodge and White [1950].

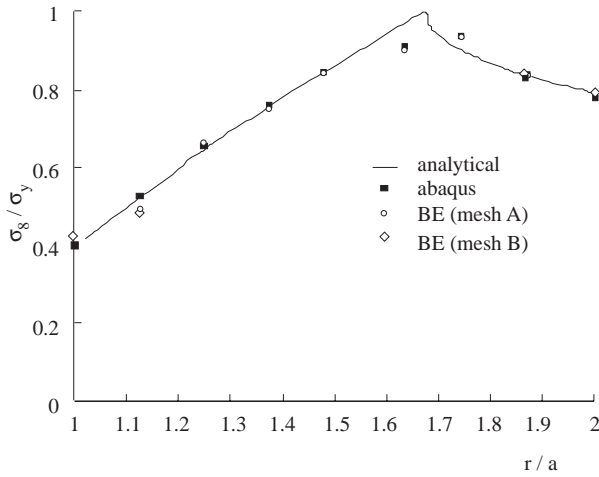


Figure 8. Hoop stresses with radius at the load factor $P/\sigma_y = 0.76$ for BE, using mesh A and B, ABAQUS and analytical results from Hodge and White [1950].

Perforated Plate in Tension

This problem is included to demonstrate the applicability of the BE formulation in a stress concentration situation in which a sharp stress gradient occurs near the plastic region. This plane stress problem was also investigated experimentally by Theocaris and Marketos [1964]. The geometry and loading conditions are given in Figure 9. The following material properties are assumed for the linear strain hardening material:

$$\begin{aligned} \sigma_y &= 24.3 \text{ N/mm}^2 \\ E &= 700 \text{ N/mm}^2 \\ H &= 224 \text{ N/mm}^2 \\ \nu &= 0.2 \end{aligned}$$

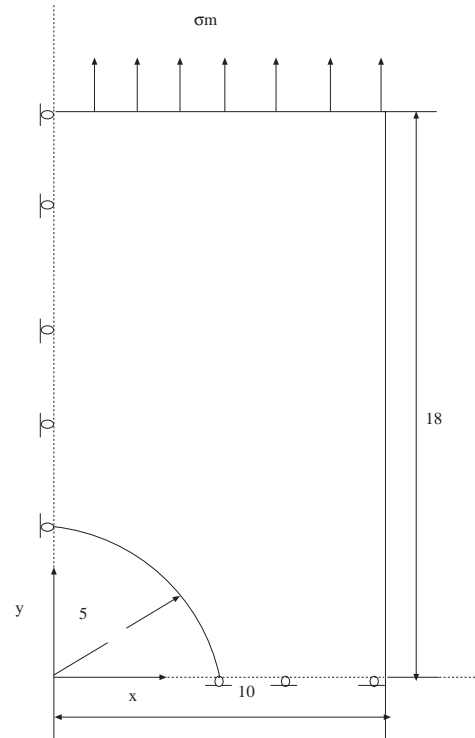


Figure 9. Perforated plate problem

For the BE analysis, three meshes were employed, as represented in Figure 10: mesh A with 12 cells and 19 boundary elements, mesh B with 16 cells and 22 boundary elements and mesh C with 20 cells and 25 boundary elements. All BE meshes employ partial rather than full interior modelling in order to demon-

strate this capability of the initial strain approach. For the FE analysis, two meshes are also used: mesh A with 120 quadratic plane stress elements and 905 nodes, and mesh B with 500 first-order plane stress elements and 546 nodes.

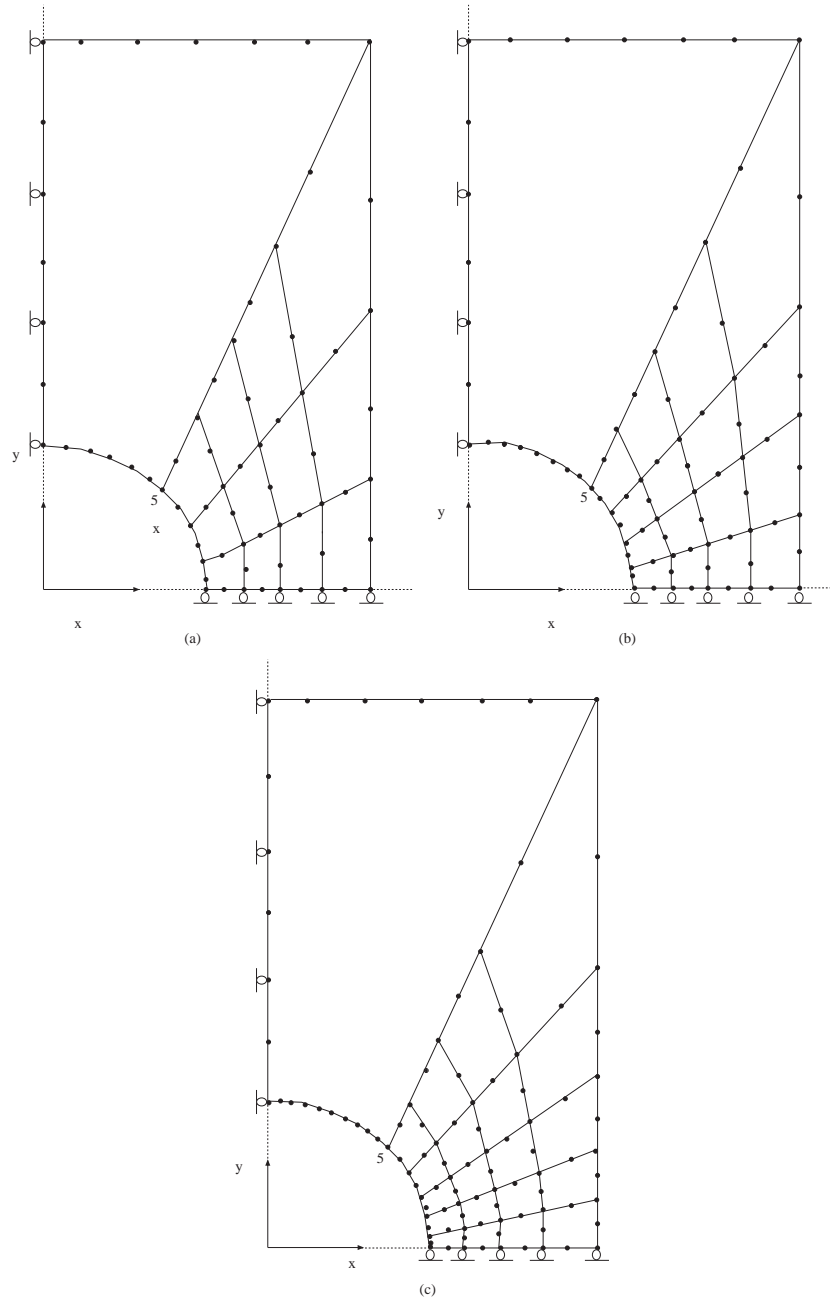


Figure 10. BE discretisation of the perforated plate (a) Mesh design for mesh A (b) Mesh design for mesh B (c) Mesh design for mesh C

Figure 11 shows the development of maximum strain at the first yielding point with the mean stresses, σ_m , at the root of the plate, for the BE, FE and previously published experimental results. The BE results have been computed to reach the non-dimensionalised mean stresses, $2\sigma_m/\sigma_y$, values of 0.9897. Both the BE and FE solutions are a little lower than experimental results. All three meshes used in the BE analysis gave consistent results until near the collapse load point. At the collapse load point mesh A did not give the correct approximation because the boundary element discretisations were not enough to cover the plastic response of the structure. Meshes B and C gave slightly better results than ABAQUS at the collapse load point. Figure 12 shows that the BE and FE results are in a satisfactory agreement on the variation of non-dimensionalised stresses σ_{yy}/σ_y at the root of the plate near the collapse load in which the non-dimensionalised stresses, σ_{yy}/σ_y are equal to 0.91.

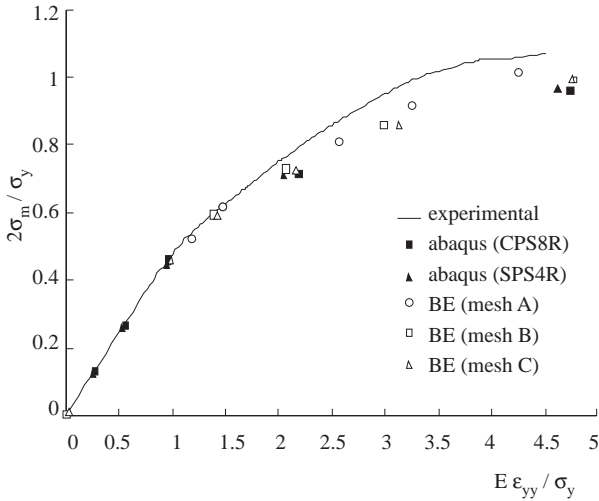


Figure 11. Stress-strain response at the first yielding point BE, using mesh A, B and C, ABAQUS and experimental results from Theocaris and Marketos [1964].

Figures 13 and 14 show comparisons of the BE solutions using perfectly plastic and linear strain hardening material models with the experimental results, for the mean non-dimensionalised stresses $2\sigma_m/\sigma_y$ and σ_{yy}/σ_y , respectively. Good agreement is reached between the BE solutions from meshes B and C and the experimental results. For the BE analysis, 6, 4 and 4 load increments were employed for meshes A, B and C, respectively. For the perfectly-plastic BE

analysis, 4 and 3 load increments were employed for meshes B and C respectively.

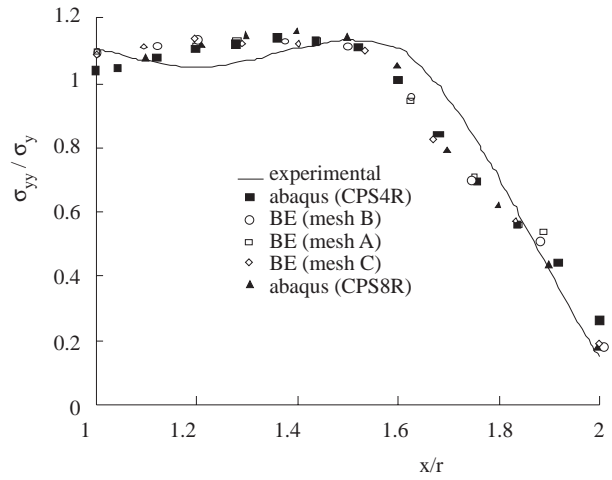


Figure 12. Stress distribution at the root of the plate near the collapse load for BE, using mesh A, B and C, ABAQUS and experimental results from Theocaris and Marketos [1964].

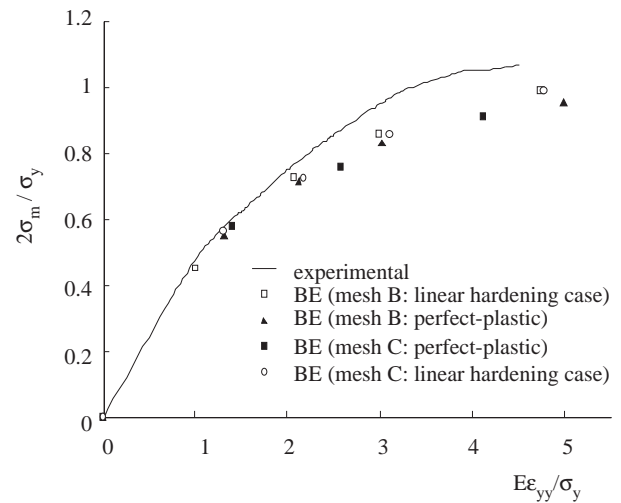


Figure 13. Stress-strain response at the first yielding for BE, using mesh B and C, and assuming linear strain hardening and perfect-plastic cases, and experimental results from Theocaris and Marketos [1964].

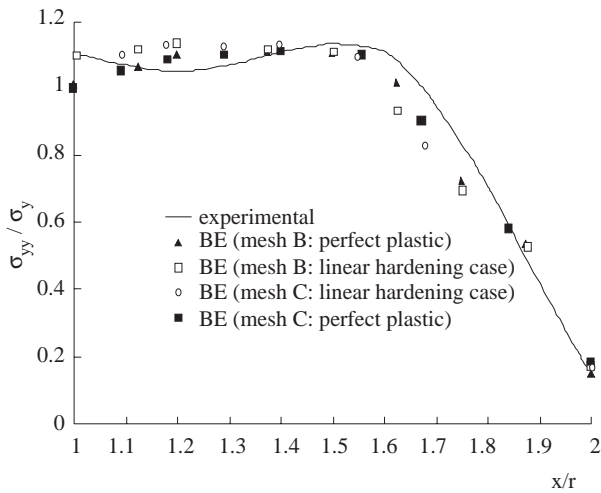


Figure 14. Stress distribution at the root of the plate point near the collapse load for BE, using mesh B and C, and assuming linear hardening and perfect-plastic cases and experimental results from Theocaris and Marketos [1964].

Conclusion

The initial strain displacement gradient BE formulation was implemented in a Fortran computer program, BEPLAST, which was written with em-

phasis on clarity. The program listing contains full details of all the numerical algorithms and carefully explains the book-keeping adopted in the incremental-iterative algorithms.

The computer program BEPLAST, capable of handling mixed hardening material behaviour, was applied to several classical test problems in order to assess its accuracy and reliability. The applications included uniaxial tensile loading, a pressurised thick cylinder and perforated a plate under tension. The BE solutions were shown to be in very good agreement with the corresponding analytical and FE solutions provided by the commercially available FE package, ABAQUS. The range of applicability of the present BE formulation can be extended to cover multi-domain and contact mechanics problems. This would require a carefully designed robust numerical algorithm to incorporate nested iterations and load increments that are capable of monitoring contact development as well as marching the solution along the elasto-plastic material path. Introducing frictional stick-slip behaviour would further complicate the numerical algorithms. This is the subject of the further research work.

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