A Comparative Study of Multiobjective Optimization Methods in Structural Design

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Abstract

The computational algorithms of different multiobjective optimization techniques and their applications to structural systems are presented. The weighting, ϵ-constraint, goal programming and modified game theory methods are described along with a comparative study of the results. The conflicting nature of the objective functions is studied through two multiobjective optimization problems. Specifically, the design of a 25-bar space truss and that of a satellite with flexible appendages are considered in numerical studies. The results from the multiobjective optimization methods are evaluated in terms of a supercriterion. It is concluded that the results obtained using the goal programming and modified game theory/goal programming approaches are properly balanced yielding the best compromise in the presence of conflicting objectives.

Key Words: Multiobjective optimization, objective function, structural design, supercriterion.

Yapısal Tasarımda Çok Objektifli Optimizasyon Tekniklerinin Mukayeseli Bir Çalışması

Özet

Değişik çok objektifli optimizasyon teknikleri ve yapısal sistemlere uygulamaları sunulmaktadır. Çok objektifli optimizasyon tekniklerinden ağırlık, ϵ-sınırlama, küresel kriter, amaç programlaması ve değiştirilmiş oyun teorisi teknikleri mukayeseli sonuçlarla açıklanmıştır. Objektif fonksiyonlarının birbirleriyle çelişen karakterleri iki örnek problem üzerinde incelenmiştir. Bu problemler 25 çubuktan oluşan bir kafes kriş yapısının ve esnek kolları olan bir uydunun tasarımını kapsamaktadır. Bu problemler üzerinde çok objektifli optimizasyon tekniklerinden alınan sonuçlar bir süperkriter yardımıyla değerlendirilmiştir. Amaç programlaması ve amaç programlaması ile birleştirilen değiştirilmiş oyun teorisi tekniklerinin çok objektifli optimizasyon teknikleri arasında birbirleriyle çelişen objektif fonksiyonlarına rağmen en uzlaşımlı sonuçları verdikleri anlaşılmıştır.

Anahtar Sözcükler: Çok objektifli optimizasyon, objektif fonksiyonu, yapısal tasarım, süperkriter

1. Introduction

Many optimum design problems involve several objective functions that are generally conflicting in nature. The optimization methods dealing simultaneously with several objective functions, multiobjective or multicriteria optimization methods, play a very important role in these cases. This is particularly true in the optimum modeling and design of large space structures where the optimization of structural weight and that of fundamental natural frequency of vibrations are of primary concern to the designer.

Several studies dealing with multiobjective optimization techniques have been reported over the past several years since the appearance of the paper by Kuhn and Tucker (1951). Pareto optimality serves as the basic criterion in most of the multiobjective literature. The concept of multicriteria optimization in the context of engineering applications was introduced by Zadeh (1963). The survey paper by Stadler (1984) summarizes the applications of multicriteria optimization in mechanics.

The most extensive use of multiobjective optimization has been made in optimal structural modeling and design. Commonly used single objectives in multiobjective structural optimization include the weight, fundamental natural frequency, deflection and velocity of the structure. The control energy is also used by Sunar and Rao (1995) as an objective function for actively controlled structures.

This paper presents a comparative study of the multiobjective optimization techniques in the context of structural modeling and design.

2. Multiobjective Optimization Techniques

Multiobjective optimization schemes usually collapse multiple objective functions into a single objective function in some manner and the resulting problem is solved as a single optimization problem. All the multiobjective optimization techniques involve, to a certain extent, a trade-off between different objectives and thus some engineering judgment may be needed to set up the priorities for the various objective functions. At the same time, the fact that one method, in some cases, may prove to be superior to others in keeping a good balance in the simultaneous optimization of all the objective functions should not be overlooked.

A standard vector optimization problem can be stated as follows:

$$
\begin{aligned}
\text{minimize} \quad & f(\mathbf{X}) \\
\text{subject to} \quad & g_j(\mathbf{X}) \le 0, \quad j = 1, 2, \ldots, m \\
& h_j(\mathrm{X}) = 0, \quad j = 1, 2, \ldots, p
\end{aligned} \tag{1}
$$

where $\mathbf{f} = [f_1 \ f_2 \ \ldots \ f_r]^T$ is the vector of objective functions, $\mathbf{X} = [x_1 \ x_2 \ \ldots \ x_n]^T$ is the vector of design variables, g_j is the jth inequality constraint function and h_j is the jth equality constraint function. Note that any maximization objective can be converted into a minimization objective by simply changing its sign.

In the numerical study, the objective functions are normalized such that

$$
C = k_1 f_1(\mathbf{X}^0) = \cdots = k_r f_r(\mathbf{X}^0) \tag{2}
$$

where C is any convenient number, $k'_i\text{s}(i = 1, 2, \ldots, r)$ are constants and $\mathbf{X}^0$ is the starting design vector. Thus the new objective functions are given by

$$
F_i(\mathbf{X}) = k_i f_i(\mathbf{X}), \quad i, 1, 2, \ldots, r. \tag{3}
$$

A different normalization scheme involves solving the single objective optimization problems:

$$
\begin{aligned}
\text{minimize} \quad & f_i(\mathbf{X}), \quad i = 1, 2, \ldots, r \\
\text{subject to} \quad & g_j(\mathbf{X}) \le 0, \quad j = 1, 2, \ldots, m \\
& h_j(\mathbf{X}) = 0, \quad j = 1, 2, \ldots, p.
\end{aligned} \tag{4}
$$

The minimum value of $f_i(\mathbf{X})$ is called the best value of $f_i(\mathbf{X})$, which is shown as $f_i(\mathbf{X}_i^*)$ where $\mathbf{X}_i^*$ is the optimal design vector obtained when only f_i is minimized. The worst value f_{iw} of f_i is defined as the maximum value of f_i on the set $\{\mathbf{X}_1^*, \mathbf{X}_2^*, \ldots, \mathbf{X}_r^*\}$ and is determined from $f_{iw} = \max\limits_{j=1,2,\ldots,r} f_i(\mathbf{X}_j^*)$. Then the following normalization procedure, which gives 0 as the minimum value and 1 as the maximum value of the ith objective function, can be used:

$$
f_{ni}(\mathbf{X}) = \frac{f_i(\mathbf{X}) - f_i(\mathbf{X}_i^*)}{f_{iw} - f_i(\mathbf{X}_i^*)} \tag{5}
$$

where $f_i(\mathbf{X})$ is the value of the ith objective function at any design. This leads to the vector of normalized functions: $\mathbf{f}_n = [f_{n1} \ f_{n2} \ \ldots \ f_{nr}]^T$.

For the purpose of comparing the relative efficiencies of the various multiobjective optimization techniques, a supercriterion (S), also known as the bargaining model, is constructed as follows:

$$
S = \prod_{i=1}^{r} [f_{iw} - f_i(\mathbf{X})]. \tag{6}
$$

The supercriterion gives an indication as to how far an objective function is from its worst value at any design. Thus the higher the value of S, the better the modeling and design in terms of a compromise solution. The various multiobjective optimization techniques considered in the comparative analysis are summarized below.

2.1. Weighting Method

In this method, the objective functions are made scalars in a suitable manner. Two methods are used in this study.

In the first method, the problem is posed as follows:

$$
\begin{aligned}
\text{minimize} \quad & F(X) = \sum_{i=1}^{r} c_i F_i(\mathbf{X}) \\
\text{subject to} \quad & g_j(\mathbf{X}) \leq 0, \qquad j = 1, 2, \ldots, m \\
& h_j(\mathbf{X}) = 0, \qquad j = 1, 2, \ldots, p
\end{aligned}
\tag{7}
$$

where c_i is a constant indicating the weight (and hence importance) assigned to F_i. By giving a relatively large value to c_i it is possible to favor F_i over other objective functions. Note that the condition $\sum_{i=1}^{r} c_i = 1$ can be posed by the designer in Eq. (7).

The following problem is solved in the second method:

$$
\begin{aligned}
\text{minimize} \quad & F(\mathbf{X}) = \prod_{i=1}^{r} F_i^{c_i}(\mathbf{X}) \\
\text{subject to} \quad & g_j(\mathbf{X}) \leq 0, \qquad j = 1, 2, \ldots, m \\
& h_j(\mathbf{X}) = 0, \qquad j = 1, 2, \ldots, p
\end{aligned}
\tag{8}
$$

where it is assumed that all the objective functions are to be minimized. An objective function can be maximized in this method by switching the sign of the exponent of the objective function from plus to minus.

2.2. ϵ-Constraint Method

The method optimizes one of the objective functions while the others are required to have specified upper bounds. In other words, it minimizes one objective function and simultaneously maintains the maximum acceptable levels for the other objective functions. The formulation adopted in this work is as follows:

$$
\begin{aligned}
\text{minimize} \quad & F_i(\mathbf{X}), \qquad i = 1, 2, \ldots, r \\
\text{subject to} \quad & g_j(\mathbf{X}) \leq 0, \qquad j = 1, 2, \ldots, m \\
& h_j(\mathbf{X}) = 0, \qquad j = 1, 2, \ldots, p \\
& F_j(\mathbf{X}) \leq \epsilon_j, \qquad j = 1, 2, \ldots, \text{ and } j \neq i.
\end{aligned}
\tag{9}
$$

The selections of $F_i(\mathbf{X})$ and ϵ_j of this method are not straightforward and depend on the particular problem under consideration. As shown in the above formulation, the optimization problem (Eq. (9)) can be solved for all $F_i(\mathbf{X})$'s ($i = 1, 2, \ldots, r$) and the optimum solution that is best suited to the problem can be chosen among the r solutions. But this insolves much computational effort. In this work, ϵ_j is chosen as $\epsilon_j = 1.1 \, F(\mathbf{X}^0)$. Note that this choice of ϵ_j is arbitrary and any other choice, such as $1.2 \, F_j(\mathbf{X}^0)$, may also be used depending upon the problem. But the basic philosophy of this method does not alter with different selections of ϵ_j. In general, the higher values of ϵ_j's mean a wider feasible region for the single objective optimization problem and this may in turn give a more improved solution for $F_i(\mathbf{X})$ at the expense of the other objective functions.

2.3. Goal Programming Method

In this method, the designer sets goals to be attained for each objective and a measure of the deviations of the objective functions from their respective goals is minimized. The generalized goal programming method proposed by Ignizio (1976) is adapted to non-linear problems as follows:

$$
\begin{aligned}
\text{minimize} \quad & F(\mathbf{X}) = \left\{ \sum_{i=1}^{r} c_i [d_i^+ + d_i^-]^q \right\}^{1/q}, \quad q \geq 1 \\
\text{subject to} \quad & F_i(\mathbf{X}) - d_i^+ + d_i^- = T_i, \quad i = 1, 2, \ldots, r \\
& g_i(\mathbf{X}) \leq 0, \qquad j = 1, 2, \ldots, m \\
& h_j(\mathbf{X}) = 0, \qquad j = 1, 2, \ldots, p
\end{aligned}
\tag{10}
$$

where d_i^+ and d_i^- are, respectively, the under-achievement and over-achievement of the *ith* goal; T_i is the goal (or target) set by the designer for the *ith* objective function (Rao et al. (1988)). In this work, the goal of the *ith* objective function is taken as its single optimum value, i.e., $T_i = F_i(\mathbf{X}_i^*)$. Furthermore, it is assumed that over-achievement of the

goals is not possible and hence d_i^- need not be de-

fined. Thus the problem stated in Eq. (10) becomes

$$
\begin{aligned}
\text{minimize} \quad & F(\mathbf{X}) = \left\{ \sum_{i=1}^{r} c_i [F_i(\mathbf{X}) - F_i(\mathbf{X}_i^*)]^q \right\}^{1/q} , \quad q \geq 1 \\
\text{subject to} \quad & F_i(\mathbf{X}) - F_i(\mathbf{X}_i^*) \geq 0, \quad i = 1, 2, \ldots, r \\
& g_i(\mathbf{X}) \leq 0, \qquad\qquad j = 1, 2, \ldots, m \\
& h_j(\mathbf{X}) = 0, \qquad\qquad j = 1, 2, \ldots, p.
\end{aligned}
\tag{11}
$$

2.4. Modified Game Theory

In game theory, each player is associated with an objective function so that he tries to maximize his profit at the expense of the profits of the other players. When adopting this theory into a multiobjective optimization problem, the profit of a player is viewed as a negative objective function relative to the profits of the other players. A bargaining model or supercriterion is constructed and the final optimal solution (Pareto optimal solution) is obtained by maximizing the supercriterion. According to the game theory, the Pareto optimal solution is determined by solving the following problem (Rao and Hati (1979)):

$$
\begin{aligned}
\text{minimize} \quad & f_c(\mathbf{c}, \mathbf{X}) = \sum_{i=1}^{r} c_i f_i(\mathbf{X}) \\
\text{subject no} \quad & 0 \leq c_i \leq 1, \quad i = 1, 2, \ldots, r, \text{ and } \sum_{i=1}^{r} c_i = 1 \\
& g_i(\mathbf{X}) \leq 0, \quad j = 1, 2, \ldots, m \\
& h_i(\mathbf{X}) = 0, \quad j = 1, 2, \ldots, p
\end{aligned}
\tag{12}
$$

where $\mathbf{c} = [c_1 \ c_2 \ \ldots \ c_r]^T$. The equality constraint in Eq. (12) can be eliminated by substituting $\left(1 - \sum_{i=1}^{r-1} c_i \right)$ for c_r. The supercriterion S of Eq. (6) is maximized so that the resulting solution gives the optimal combination of the objective functions ($\mathbf{c}^*$) and the final optimum solution ($\mathbf{X}^* = \mathbf{X}_c^*$). Hence the minimization of the problem given by Eq. (12) and the maximization of the supercriterion S must be simultaneously accomplished in the game theory with $\mathbf{c}$ and $\mathbf{X}$ being optimization variables, which is numerically cumbersome.

Due to the computational complexities involved with the original game theory, a modification to the method was suggested (Rao and Freiheit (1991)). The solution given by the modified game theory (MGT) is expected to be near the optimal solution obtained by the original game theory. The algorithm for the MGT is as follows:

a) Formulate the normalized supercretiron, S_n, as

$$
S_n = \prod_{i=1}^{r} [1 - f_{ni}(\mathbf{X})].
\tag{13}
$$

Note that due to the normalization used in Eq. (5), S_n will always have a value between 0 and .

b) Formulate a Pareto optimal objective function FC in terms of the normalized objective functions. In this work, FC is formulated using the weighting method as

$$
FC = \sum_{i=1}^{r} c_i f_{ni}(\mathbf{X})
\tag{14}
$$

with $\sum_{i=1}^{r} c_i = 1$.

FC is also formulated using the goal programming method as

$$
FC = \left\{ \sum_{i=1}^{r} [f_{ni}(\mathbf{X})]^q \right\}^{1/q}, \quad q \geq 2.
\tag{15}
$$

c) The new optimization problem is posed as

$$
\begin{aligned}
\text{minimize} \quad & F(\mathbf{X}) = FC - S_n \\
\text{subject to} \quad & f_i(\mathbf{X}_i^*) \leq f_i(\mathbf{X}) \leq f_{iw}, \quad i = 1, 2, \ldots, r \\
& g_j(\mathbf{X}) \leq 0, \qquad\qquad\quad j = 1, 2, \ldots, m \\
& h_j(\mathbf{X}) = 0, \qquad\qquad\quad j = 1, 2, \ldots, p.
\end{aligned} \tag{16}
$$

It is important to account for the constraints on the objective functions to guarantee that S_n remains between 0 and 1.

3. Numerical Results

The comparative study of the multiobjective optimization techniques is made using two example problems. In both problems, the multiobjective optimization problems are collapsed into single objective optimization problems using the multiobjective optimization schemes as discussed above and the resulting single objective optimization problems are solved via the method of feasible directions (Arora (1989)). The basic idea of this method is to move from one feasible design to an improved feasible design to an improved feasible design. Hence two properties are desired in this method. The first one is the feasibility of the new design and the second one is that the new objective function is smaller than the old objective function. The iterative process is repeated until no more improvement in the objective function is practically possible within the feasible region.

3.1. Three-Objective Design of a 25-Bar Truss

The design of the truss shown in Figure 1 is considered under two load conditions (Table 1). The truss is subject to constraints on the member stresses and Euler buckling. The allowable stresses for all the members are assumed to be the same in tension and compression and is denoted as σ_a. The Young's modulus is as sumed to be $E = 6.9 \times 10^{10}$ Pa and the material density to be $\rho = 2770$ kg/m³. The members are assumed to be tubular and the ratio of a nominal diameter to thickness is taken as 100. Therefore the buckling stress in any ith member becomes

$$
\sigma_{bi} = \frac{-100.01\pi EA}{8\ell_i^2}, \quad i = 1, 2, \ldots, 25 \tag{17}
$$

where A_i and ℓ_i are the cross-sectional area and length, respectively, of the ith member. The cross-sectional areas of members of the truss are taken as design variables and are arranged into eight different groups such that $A_1, A_2 = A_3 = A_4 = A_5, A_6 = A_7 = A_8 = A_9, A_{10} = A_{11}, A_{12} = A_{13}, A_{14} = A_{15} = A_{16} = A_{17}, A_{18} = A_{19} = A_{20} = A_{21}$ and

$A_{22} = A_{23} = A_{24} = A_{25}$. Thus there are eight design variables for this problem. Upper and lower bounds are imposed on the design variables as $x_i^\ell \leq x_i \leq x_i^u, i = 1, 2, \ldots, 8$. The upper and lower bounds x_i^u and x_i^ℓ on all the members are taken as 3.2258×10^{-3} m² and 6.45×10^{-5} m², respectively. Three objective functions are consideed: minimization of weight, minimization of displacement of node 1 of the truss and maximization of the fundamental natural frequency of the truss. The objective functions can be expressed as
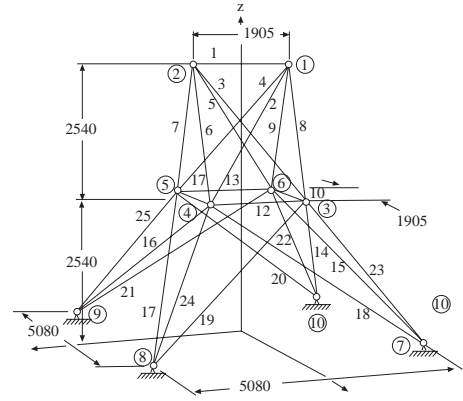

Figure 1. Twenty Five-Bar Truss

$$
f_1(\mathbf{X}) = 9.81 \sum_{i=1}^{25} \rho A_i \ell_i, \quad f_2(\mathbf{X})
$$

$$
= \sum_{i=1}^{2} (\delta_{xi}^2 + \delta_{yi}^2 + \delta_{zi}^2)^{1/2}, \quad f_3(\mathbf{X}) = \omega_n \tag{18}
$$

where δ_{xi}, δ_{yi} and δ_{zi} are the x, y and z components of displacement of node 1 for load condition $i(i = 1, 2)$, and ω_n is the fundamental natural frequency of vibrations of the truss. The following constraints are considered:

$$
\begin{aligned}
|\sigma_{ij}(\mathbf{X})| \leq \sigma_a, \qquad & i = 1, 2, \ldots, 25, \quad j = 1, 2 \\
-\sigma_{ij}(\mathbf{X}) \leq -\sigma_{bi} \qquad & i = 1, 2, \ldots, 25, \quad j = 1, 2
\end{aligned} \tag{19}
$$

where $\sigma_{ij}(\mathbf{X})$ is the stress in member i in load condition j. The allowable stress σ_a is assumed to be 2.76×10^8 Pa for all the members.

Table 1. Loads Acting on 25-Bar Truss

		Joint		
	1	2	3	6
		Load Condition	1 (N)	
F_x	0	0	0	0
F_y	88,960	-88,960	0	0
F_z	-22,240	-22,240	0	0
		Load Condition	2 (N)	
F_x	4,448	0	2,224	2,224
F_y	44,480	44,480	0	0
F_z	-22,240	-22,240	0	0

The single objective optimization results and the results obtained by various multiobjective optimization techniques are given in Table 2. The starting design variables are taken as $x_i^0 = 2.54 \times 10^{-2}$ m ($i = 1, 2, \ldots, 8$), which correspond to objective function values of $f_1(\mathbf{X}^0) = 1475$ N, $f_2(\mathbf{X}^0) = 3.92 \times 10^{-2}$ m and $f_3(\mathbf{X}^0) = 68.866$ Hz. The objective functions are scaled so that $f_1(\mathbf{X}^0) = F_2(\mathbf{X}^0) = F_3(\mathbf{X}^0) = 500$. The individual objective function optimizations yielded the best and worst values of the objective functions as $f_1(\mathbf{X}_1^*) = 1037N$, $f_{1w} = 7025.5N$, $f_2(\mathbf{X}_2^*) = 7.83 \times 10^{-3}$ m, $f_{2w} = 4.9 \times 10^{-2}$ m, $f_3(\mathbf{X}_3^*) = 113.511$ Hz, and $f_{3w} = 72.288$ Hz.

Table 2. Results for 25-Bar Truss

Method	f_1(N)	f_2(m)	f_3(Hz)	Iter.	S
Single Min. of f_1	1037.04	0.0490	73.331		
Single Min. of f_2	7025.53	0.0078	72.288		
Single Min. of f_3	3934.39	0.0328	113.511		
Weighting (1)	1462.79	0.0359	90.986	16	12002
Weighting (2)	1700.55	0.0286	85.753	13	12912
ϵ-Constraint	1622.01	0.0394	96.993	27	11270
Goal Programming ($q = 2$)	1636.42	0.0302	87.070	15	13197
Goal Programming ($q = 3$)	1675.89	0.0289	85.364	14	12419
MGT (Weighting)	1503.87	0.0365	90.392	19	11052
MGT (Goal Prog. $q = 2$)	3033.65	0.0232	103.141	11	28066

The first weighting method is applied with $c_1 = c_2 = c_3 = 1/3$ and the second with $c_1 = c_2 = c_3 = 1$. The ϵ-constraint method is formulated as follows:

$$\begin{array}{ll} \text{minimize} & -F_3(\mathbf{X}) \\ \text{subject to} & g_j(\mathbf{X}) \leq 0, \qquad j = 1, 2, \ldots, 100 \\ & F_1(\mathbf{X}) \leq 550 \\ & F_2(\mathbf{X}) \leq 550. \end{array} \qquad (20)$$

The results listed in Table 2 for different optimization schemes indicate that the improvement in one objective function yields deterioration in the others. The results of the weighting method are compromise solutions and are not necessarily Pareto optimal. The ϵ-constraint method can also be formulated so that $F_1(\mathbf{X})$ and $F_2(\mathbf{X})$ are individually taken as objective functions, but this involves additional computational effort. In addition to the stress and buckling constraints, the constraints imposed on $F_1(\mathbf{X})$ and $F_2(\mathbf{X})$ in this method (Eq. (20)) are found to be active. These new constraints determine the level of compromise for the method. In the goal programming method, it is assumed that $c_1 = c_2 = c_3 = 1$. The method is implemented with $q = 2$ and 3 and the results are found to be almost identical.

The MGT is conceptually different from the other methods in trying to obtain a near Pareto optimal solution to the multiobjective optimization problem. Table 2 shows that the optimum design of the MGT with the goal programming technique somewhat resembles the result produced by the single objective optimization of $f_3(\mathbf{X})$, but a close examination of both results reveals that the result of the MGT with

the goal programming techniques is well-balanced between different objectives. This conclusion is also supported by the largest value of S obtained by the MGT with the goal programming technique.

3.2. Four-Objective Design of a Satellite with Flexible Appendages

The appendages of the satellite illustrated in Figure 2 are modeled as cantilever beams of circular cross-section (Starkey and Kelecy (1988)). Consider the problem of rotating the satellite about the z-axis from some initial angle θ_i to some final angle θ_f. If it is assumed that the appendages are flexible and lightly damped, the motion of the appendages will continue even after the rotation has ceased at θ_f. Since such oscillations may corrupt the mission of the whole craft, the goal is to generate a control law
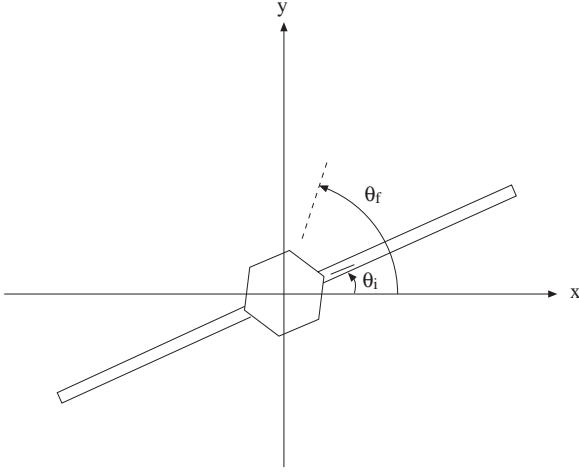
for the torque T, and structurally modify the appendage so that they come to rest shortly after the craft reaches the angular position θ_f.

The satellite is modeled as illustrated in Figure 3. The main body of the satellite is assumed to be cylindrical with a radius of r_0 and is modeled as a rigid mass with a mass moment of inertia of J. The appendages are modeled by dividing each appendage into three segmets of length ℓ. The segments are modeled as point masses m_1, m_2 and m_3 connected by massless, girid links of length ℓ each. The links are joined with torsional springs of stiffnesses k_1, k_2 and k_3 at the joints. Symmetry is assumed between the appendages.

The satellite is actively controlled by a torque applied about the z-axis of the main body. The torque T is given by
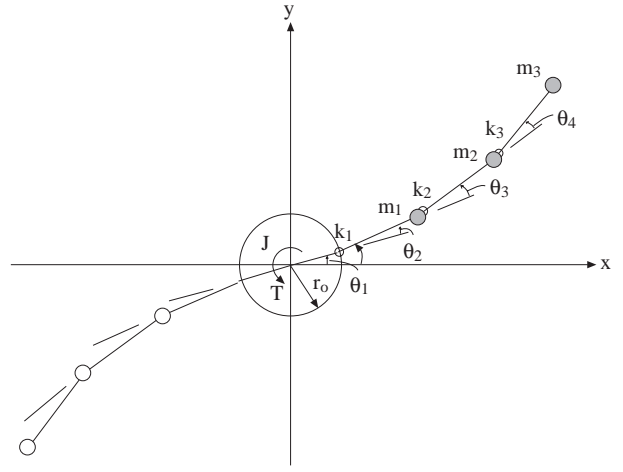

Figure 2. Satellite with Flexible Appendages

Figure 3. Model of Satellite

$$T = g_1(\theta_f - \theta_1) - g_2\theta_2 - g_3\theta_3 - g_4\theta_4 - g_5\dot{\theta}_1 - g_6\dot{\theta}_2 - g_7\dot{\theta}_3 - g_8\dot{\theta}_4 \qquad (21)$$

where g_1 through g_8 are the feedback gains of the controller.

The equations of motion of the satellite can be derived using Lanrange's method as

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}(K + \xi G_1) & -M^{-1}\xi G_2 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \qquad (22)$$

where I represents the 4×4 identity matrix and

$$M = \frac{1}{\ell^2} \begin{bmatrix} J + 2\ell^2 a_1 + 2r_0^2 a_2 + 4r_0\ell a_3 & 2\ell^2 a_1 + 2r_0\ell a_3 & 4\ell^2 a_4 + 2r_0\ell a_5 & 2m_3(3\ell^2 + r_0\ell) \\ 2\ell^2 a_1 + 2r_0\ell a_3 & 2\ell^2 a_1 & 4\ell^2 a_4 & 6\ell^2 m_3 \\ 4\ell^2 a_4 + 2r_0\ell a_5 & 4\ell^2 a_4 & 2\ell^2(m_2 + 4m_3) & 4\ell^2 m_3 \\ 2m_3(3\ell^2 + r_0\ell) & 6\ell^2 m_3 & 4\ell^2 m_3 & 2\ell^2 m_3 \end{bmatrix}$$

where

$$a_1 = m_1 + 4m_2 + 9m_3, a_2 = m_1 + m_2 + m_3, a_3 = m_1 + 2m_2 + 3m_3, a_4 = m_2 + 3m_3, a_5 = m_2 + 2m_3$$

and

$$K = \frac{1}{\ell^2} \begin{bmatrix} 0 & 0 & 0 & \\ 0 & k_1 & 0 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & 0 & 0 & k_3 \end{bmatrix}, \xi^T = \frac{1}{\ell^2}[1 \ 0 \ 0 \ 0], \ \ G_1 = [g_1 \ g_2 \ g_3 \ g_4], \ \ G_2 = [g_5 \ g_6 \ g_7 \ g_8]$$

where

$$k_i = \frac{\pi E d_i^4}{32\ell}, \quad i = 1, 2, 3$$

in which d_i represents the diameter of the ith segment and E is the Young's modulus of the material. The mass of the ith segments is

$$m_i = \frac{\pi}{4}\rho\ell d_i^2, \quad i = 1, 2, 3$$

where ρ is the mass density of the material and the moment of inertia of the satellite body is chosen as

$$J = 15m_0\ell^2$$

where m_0 is the initial value adopted for the mass of an appendage segment. The following properties are assumed in the numerical calculations: $\rho = 2770$ kg/m^3, $E = 6.9 \times 10^9$ Pa, $d_0 = 7.62 \times 10^{-2}$ m, r_0

(radius of the satellite's main body) $= 0.3048$ m, $\ell = 0.6096$ m, $G_0 = [g_{01} \ g_{02} \ldots g_{08}] = 0$, where d_0 is the initial diameter assumed to be the same for all the segments and G_0 is the initial feedback gain of the controller.

The design variable vector $\mathbf{X} = [x_1 \ x_2 \ \ldots \ x_{11}]$ is such that x_i for $i = 1, 2, 3$ corresponds to the structural change of the ith segment from its initial diameter d_0 (i.e., $d_i = d_0 + x_i$) and x_i for $i = 4, 5, \ldots, 11$ corresponds to the feedback gains of the controller. The following four objective functions are considered for minimization: weight of the satellite, control cost (taken as the sum of the squares of the gains), maximum real part of the eigenvalues of the satellite, and tangent of the largest angle that the eigenvalues make with the negative real axis. Thus

$$f_1(\mathbf{X}) = 9.81\sum_{i=1}^{3} m_i, \quad f_2(\mathbf{X}) = \sum_{i=4}^{11} x_i^2, \quad f_3(\mathbf{X}) = \max(\sigma_i), \quad f_4(\mathbf{X}) = \max\left(\frac{\omega_i}{\sigma_i}\right) \tag{23}$$

where σ_i and ω_i are the real and imaginary parts of the ith eigenvalue, respectively. The real parts of the eigenvalues of the satellite are required to be -0.3 or less. The contraints are stated as

$$\sigma_i + 0.3 \le 0, \quad i = 1, 2, \ldots, 8 \tag{24}$$

The results of the optimization are shown in Table 3. When applying the multiobjective optimization techniques, the normalization indicated by Eq. (5) was used. The initial design is chosen as (Starkey and Kelecy (1988)): $x_i = 5.77 \times 10^{-3}m$, $x_2 = -4.49 \times 10^{-3}m$, $x_3 = -2.9 \times 10^{-2}m$, $x_4 = 8933.2N \cdot m$, $x_5 = 106411.1N \cdot m$, $x_6 = -48835.2N \cdot m$, $x_7 = -27174.2N \cdot m$, $x_8 = 32252.7N \cdot m \cdot s$,

$x_9 = -219929.3N \cdot m \cdot s$, $x_{10} = 259359.8N \cdot m \cdot s$, $x_{11} = -38628.9N \cdot m \cdot s$. This corresponds to $f_1(\mathbf{X}^0) = 183.35N$, $f_2(\mathbf{X}^0) = 1.32695 \times 10^{11}$, $f_3(\mathbf{X}^0) = -0.44622$, and $f_4(\mathbf{X}^0) = 9.6023$. The single objective optimization of each problem is performed such that $F_1(\mathbf{X}^0) = F_2(\mathbf{X}^0) = F_3(\mathbf{X}^0) = F_4(\mathbf{X}^0) = 50,000$ and the results are listed in the first four rows of Table 3. Note that the values of S are calculated without regard to $f_2(\mathbf{X})$. It can easily be detected by inspection of the first four rows of the table and by the information given above that $f_1(\mathbf{X}_1^*) = 58.03N$, $f_{1w} = 175.09N$, $f_2(\mathbf{X}_2^*) = 2.054 \times 10^{10}$, $f_{2w} = 1.327 \times 10^{11}$, $f_3(\mathbf{X}_3^*) = -0.65$, $f_{3w} = -0.3$, $f_4(\mathbf{X}_4^*) = 3.56$, $f_{4w} = 14.00$.

Table 3. Results for Satellite

Method	$f_1(\mathrm{N})$	$f_2(-)$	$f_3(-)$	$f_4(-)$	Iter.	S
Single Min. of f_1	58.03	1.327×10^{11}	-0.30	5.99		
Single Min. of f_2	130.33	2.054×10^{10}	-0.30	14.00		
Single Min. of f_3	175.09	1.327×10^{11}	-0.65	7.61		
Single Min. of f_4	162.23	1.327×10^{11}	-0.34	3.56		
Weighting (1)	172.11	1.327×10^{11}	-0.53	5.39	100	513
Weighting (2)	175.93	1.327×10^{11}	-0.39	3.82	13	-66
ϵ-Constraint	124.01	1.327×10^{11}	-0.43	8.89	23	2851
Goal Programming ($q = 2$)	146.93	1.327×10^{11}	-0.49	7.27	395	3151
Goal Programming ($q = 3$)	137.51	1.327×10^{11}	- 0.47	8.15	141	3254
MGT (Weighting)	165.42	1.327×10^{11}	-0.36	3.55	500	487
MGT (Goal Prog. $q = 2$)	146.91	1.327×10^{11}	-0.49	7.28	395	3151

The large differences in the nature and magnitude of the objective functions made the problem very sensitive to changes in the gains of the controller. Most of the methods converged to the same gains. It was observed that the feedback gains needed for the control of the satellite were high in magnitude. Furthermore, the minimum value that the maximum of the real parts of system eigenvalues can take was found to be -0.6453. This observation together with the fact that the least slope obtained was 3.56 shows that the system is difficult to control.

As Table 3 indicates, the MGT/goal programming and goal programming methods yield large values of the supercriterion relative to the other methods. Hence, based on this observation, the solutions of the MGT/goal programming and goal programming methods may be preferred over the others.

4. Conclusions

A comparative study of several methods of multiobjective optimization has been carried out using two structural design optimization problems. As mentioned before, an engineering judgment often needs to be brought into the picture when a multicriteria optimization problem is to be solved. An improvement in one objective might necessitate some deterioration in other objectives. A multiobjective optimization method is evaluated based on such criteria as its reliability and robustness in reaching Pareto optimal design from a starting point, its efficiency of convergence inrelatively small number of iterations and the ease with which it can be applied to general design problems.

The weighting method is easy to implement, but its final design is not guaranteed to be Pareto optimal. The ϵ-constraint method requires additional constraints to be satisfied. These constraints serve as levels of trade-off between the objective functions. Different ways may be adopted in formulating this method and its application to a design problem may be somewhat troublesome.

The goal programming method was tested with values of q being 2 and 3. Both values of q caused the method to approximately converge to the same final points. The method attempts to produce a good balance between the individual objective functions by trying to keep them close to their optimum values and hence aims at reaching a Pareto optimal design.

The MGT is theoretically designed to reach a near Pareto optimal design and is introduced so that the game theory can be practically applied without much deviation from its original form. Although the final design is hoped to be near Pareto optimum, the use of the MGT requires some additional work from the designer.

The MGT/goal programming and goal programming methods, in general, give higher values of S than the other multiobjective optimization methods. Therefore, the final designs obtained by these two methods may be concluded to be properly balanced with the best compromise in the presence of conflicting objectives.

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