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Rigid Parabolic Stamp on a Nonlocal Elastic Half Plane

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Abstract

The solution is given for the problem of a frictionless rigid parabolic stamp in nonlocal elasticity. Interestingly enough, none of the classical singularities exist in the nonlocal solutions. Classical and nonlocal elasticity solutions are compared with each other. The superiority of nonlocal theory is depicted.

Key Words: Nonlocal Elasticity, Punch, Contact Problem, Stamp, Elastic Half Plane

Yerel Olmayan Yarım Düzlemde Rijit Parabolik Zımba

Özet

Bu çalışmada sürtünmesiz rijit parabolik zımba problemi yerel olmayan elastisite teorisi çerçevesinde incelenmiştir. Bu amaçla önce yerel olmayan teorinin temel denklemleri verilmiştir. Klasik elastisite teorisinin tekillik gösterdiği noktalarda, yerel olmayan teorinin sonlu değerler verdiği gösterilmiş ve yerel olmayan teorinin üstünlükleri vurgulanmıştır.

Anahtar Sözcükler: Yerel Olmayan Elastisite, Zımba, Temas Problemi, Elastik Yarım Düzlem.

Introduction

Nonlocal effects are especially important in dealing with microscopic phenomena, for example, crack initiation, fracture at sharp geometrical discontinuities and concentrated forces. Nonlocal theory brings to the surface the importance of the scale effect in the applicability region of various mathematical models. This is borne out by the physics of matter. Classical theories lack such a scale. As a result, many critical phenomena cannot be explained and predicted by means of these classical theories. Conventional elasticity fails to predict a reasonable solution to certain problems. For example, the stress field in a medium weakened by a line crack predicted by conventional elasticity goes to infinity like $1/\sqrt{(r)}$. Another example would be dislocation problems. The stress field and the elastic energy in a medium with a single dislocation goes to infinity in the core of the dislocation. And finally, conventional elasticity predicts no dispersion for wave propagation which is shown to be incorrect in various ways. The nonlocal elastic solutions of these problems not only eliminate such physically unacceptable predictions, but also give results which are in excellent agreement with the results predicted by atomic theories and experiments (See Artan 1996a, Artan 1996b, Eringen 1972, Eringen and Balta 1979). In nonlocal elasticity, the stress at a point is regarded as functional of the strain tensor. For linear homogeneous solids, this introduces material moduli which are functions of distance (Eringen and Edelen 1972, Eringen 1987). The nonlocal theory of elasticity is also of recent origin, and differs from the local one in fundamental hypotheses. As is well known, in the classical theory of elasticity, the balance law is postulated to be valid only on the whole of the body. As a result of this approach, the constitutive equations of nonlocal elasticity appear as integral equations in terms of the strain tensor

$$t_{ij}(x) = \int_V \alpha(|\mathbf{x}' - \mathbf{x}|)\{\lambda e_{kk}(\mathbf{x}', t)\delta_{ij} + 2\mu\epsilon_{ij}(\mathbf{x}', t)\}dv(\mathbf{x}') \quad (1)$$

for linear, isotropic and homogeneous materials. The advantage of nonlocal elasticity over the conventional one has been indicated by the solution of certain problems (see, for example, Artan 1996a, Artan 1997, Eringen 1976). The essentials of nonlocal theory were established by Eringen, Edelen Kunin and Kroner (For a brief introduction to the subject see Eringen 1972, Eringen 1974, Eringen 1967, Kröner 1967, Kunin 1967). The problem of a frictionless parabolic stamp on a linearly elastic half plane was solved by Muskhelishvili in 1963, and the solution displays the unexplainable infinite stresses at the ends of the stamp. In this article, the problem is remodeled using nonlocal constitutive law for a linear elastic medium. Nonlocal theory remedies this defect of classical theory. The programs Mathematica, Derive and Latex are used throughout.

The Classical Elasticity Solution a Parabolic Stamp

The stress distribution under a frictionless stamp can be found as

$$\sigma(t) = \Phi^+(t) - \Phi^-(t) \quad (2)$$

where

$$\Phi(z) = \frac{2\mu}{\pi(\kappa+1)\sqrt{(z-a)(b-z)}} \int_a^b \frac{\sqrt{(t-a)(b-t)}f'(t)\,dt}{t-z} + \frac{D}{\sqrt{(z-a)(b-z)}} \quad (3)$$

where $a = -l$ and $b = l$, $2l$ is the width of the base, and P_0 is the given magnitude of the forces applied to the stamp. The solution will be physically possible if

$$P_0 \geq \frac{2\pi\mu}{R\,(\kappa+1)}\,l^2 \quad (4)$$

$f(x)$ is the profile of the stamp (see Figure 1). For parobolical stamp $f(t)$ becomes

$$f(t) = \frac{t^2}{2R} \quad (5)$$

where the radius R is very large.

$$\kappa = \frac{\lambda+3\mu}{\lambda+\mu}; \quad \lambda \geq 0,\ \mu \geq 0, \quad 1 \leq \kappa \leq 3 \quad (6)$$

where λ and μ are Lamé constants. The constant D is determined by the condition

$$\int_a^b \sigma(t)\,dt = P_0 \quad (7)$$

The stress distribution under the frictionless stamp becomes (see for full details Muskhelishvili 1963)

$$\sigma(t) = \frac{2\mu(l^2 - 2t^2)}{R(\kappa+1)\sqrt{l^2-t^2}} + \frac{P_0}{\pi\sqrt{l^2-t^2}} \quad (8)$$

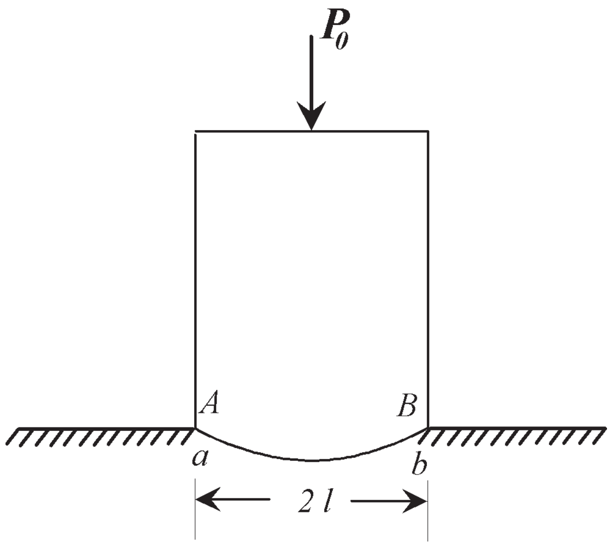

Figure 1. Parobolical stamp on an elastic half plane

The Basic Equations of Nonlocal Elasticity

The governing equations of the nonlocal theory of elasticity are

$$t_{kl,k} = 0 \tag{9}$$

$$t_{kl} = \int_V \{\lambda'(|\mathbf{x}' - \mathbf{x}|)\mathbf{e}'_{jj}(\mathbf{x}')\delta_{kl} + 2\mu'(|\mathbf{x}' - \mathbf{x}|)\mathbf{e}'_{kl}(\mathbf{x}')\, dv(\mathbf{x}') \tag{10}$$

$$\mathbf{e}'_{kl} = \frac{1}{2}(\mathbf{u}'_{k,l} + \mathbf{u}'_{l,k}); \quad \mathbf{u}'_k = \mathbf{u}_k(\mathbf{x}') \tag{11}$$

where t_{kl} is the nonlocal stress tensor, $\mathbf{u_k}$ is the displacement vector, $\mathbf{e}_{kl}$ is the strain tensor and the comma as a subscript denoting the partial derivative, that is

$$\mathbf{t}_{kl,m} = \frac{\partial \mathbf{t}_{kl}}{\partial x_m}; \quad \mathbf{u}'_{k,l} = \frac{\partial \mathbf{u}_k{}'}{\partial x_l{}'} \tag{12}$$

We use the Einstein summation convention for repeated indices. (9) and (11) are the same, both in local and nonlocal elasticities. (10) expresses the fact that the stress at arbitrary point $\mathbf{x}$ depends on the strains at all the points $\mathbf{x}'$ of the body. λ' and μ' are Lamé constants of the nonlocal medium and they depend on the distance between $\mathbf{x}$ and $\mathbf{x}'$. They can be taken as

$$\lambda' = \alpha(|\mathbf{x}' - \mathbf{x}|)\lambda; \quad \mu' = \alpha(|\mathbf{x}' - \mathbf{x}|)\mu \tag{13}$$

where λ and μ are the Lamé constants of the local case. $\alpha(|\mathbf{x}' - \mathbf{x}|)$ is called the kernel function and is the measure of the effect of the strain at $\mathbf{x}'$ on the stress at $\mathbf{x}$. It can be easily shown that (See Eringen 1976)

$$t_{kl}(\mathbf{x}) = \int_V \alpha(|\mathbf{x}' - \mathbf{x}|)\, \sigma_{kl}(\mathbf{x}')\, d\mathbf{x}' \tag{14}$$

where $\sigma_{kl}(\mathbf{x}')$ is a local stress field.

The Solution of a Parabolic Stamp on a Nonlocal Elastic Half Plane

In this article, the kernel function will be chosen as (see Figure 2)

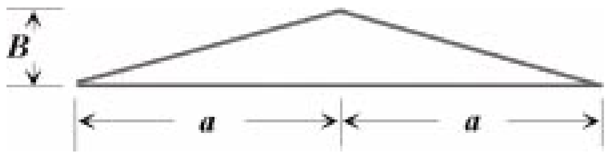

Figure 2. Kernel function

$$\alpha(|\mathbf{x}' - \mathbf{x}|) = \left\{ \begin{array}{ll} B\left\{1 - \dfrac{|\mathbf{x}' - \mathbf{x}|}{a}\right\} & |\mathbf{x}' - \mathbf{x}| < a \\ 0 & |\mathbf{x}' - \mathbf{x}| > a \end{array} \right\} \tag{15}$$

where

$$B = \frac{1}{a}; \quad a = 0.00000004cm \tag{16}$$

a is called atomic distance (See Artan 1996a, Artan 1996c, Artan 1997).

The nonlocal stress field under a parabolic stamp can be calculated by using (14).

$$t(x) = \int_{x-a}^{x+a} \left(1 - \frac{|x - x'|}{a}\right) \left(\frac{2\mu(l^2 - 2x^2)}{R(\kappa+1)\sqrt{l^2 - x^2}} + \frac{P_0}{\pi\sqrt{l^2 - x^2}}\right) dx' \tag{17}$$

After tedious calculations the nonlocal stress field becomes

$$\begin{aligned}
t(x) \quad &= \quad \frac{(\frac{P_0}{a} - \frac{P_0\,x}{a^2})\ \arcsin(\frac{a-x}{l})}{\pi} + \frac{(\frac{P_0}{a} + \frac{P_0\,x}{a^2})\ \arcsin(\frac{a+x}{l})}{\pi} \\
&- \quad \frac{2\,P_0\,x\ \arcsin(\frac{x}{l})}{a^2\,\pi} - (-3\,(1+\kappa)\,P_0\,R\,(-2\,\sqrt{l^2-x^2} \\
&+ \quad \sqrt{-a^2+l^2-2\,a\,x-x^2} + \sqrt{-a^2+l^2+2\,a\,x-x^2}) \\
&+ \quad 2\,\mu\,\pi\,(-2\,(l^2-x^2)^{\frac{3}{2}} + (-a^2+l^2-2\,a\,x-x^2)^{\frac{3}{2}} \\
&+ \quad (-a^2+l^2+2\,a\,x-x^2)^{\frac{3}{2}}))/(3\,a^2\,(1+\kappa)\,\pi\,R); \\
& \quad -l+a \le x \le l-a
\end{aligned} \tag{18}$$

In the limit $a = 0$, the nonlocal stress field reverts to the classical field. That is

$$\lim_{a\to 0} t(x) = \frac{2\mu(l^2-2x^2)}{R(\kappa+1)\sqrt{l^2-x^2}} + \frac{P_0}{\pi\sqrt{l^2-x^2}} \tag{19}$$

In the boundaries, the stress field is calculated as

$$\begin{aligned}
t(x) \quad &= \quad \int\limits_{x-a}^{l} \left(1 - \frac{|x-x'|}{a}\right)\left(\frac{2\mu(l^2-2x^2)}{R(\kappa+1)\sqrt{l^2-x^2}} + \frac{P_0}{\pi\sqrt{l^2-x^2}}\right) dx' \\
&= \quad \frac{P_0\,(a+x)}{2a^2} + \frac{(\frac{P_0}{a} - \frac{P_0\,x}{a^2})\ \arcsin(\frac{a-x}{l})}{\pi} \\
&- \quad \frac{2\,P_0\,x\ \arcsin(\frac{x}{l})}{a^2\,\pi} - (-3\,(1+\kappa)\,P_0\,R\,(-2\,\sqrt{l^2-x^2} \\
&+ \quad \sqrt{-a^2+l^2+2\,a\,x-x^2}) + 2\,\mu\,\pi\,(-2\,(l^2-x^2)^{\frac{3}{2}} \\
&+ \quad (-a^2+l^2+2\,a\,x-x^2)^{\frac{3}{2}}))/(3\,a^2\,(1+\kappa)\,\pi\,R); \\
& \quad (l-a) \le x \le l
\end{aligned} \tag{20}$$

$$\begin{aligned}
t(x) \quad &= \quad \int\limits_{-l}^{x+a} \left(1 - \frac{|x-x'|}{a}\right)\left(\frac{2\mu(l^2-2x^2)}{R(\kappa+1)\sqrt{l^2-x^2}} + \frac{P_0}{\pi\sqrt{l^2-x^2}}\right) dx' \\
&= \quad \frac{(\frac{P_0}{a} + \frac{P_0\,x}{a^2})\ \arcsin(\frac{a+x}{l})}{\pi} - \frac{2\,P_0\,x\ \arcsin(\frac{x}{l})}{a^2\,\pi} \\
&- \quad (-6\,(1+\kappa)\,P_0\,R\,(-2\,\sqrt{l^2-x^2} + \sqrt{-a^2+l^2-2\,a\,x-x^2}) \\
&+ \quad \pi\,(-3\,a\,(1+\beta)\,P_0\,R + 3\,P_0\,R\,x + 3\,\beta\,P_0\,R\,x - 8\,\mu\,(l^2-x^2)^{\frac{3}{2}} \\
&+ \quad 4\,\mu\,(-a^2+l^2-2\,a\,x-x^2)^{\frac{3}{2}}))/(6\,a^2\,(1+\beta)\,\pi\,R) \\
& \quad -l \le x \le (-l+a)
\end{aligned} \tag{21}$$

The stresses at the boundaries become

$$\begin{aligned}
t(l) = t(-l) \quad &= \quad \frac{l\,P_0}{a^2} + \frac{\left(\frac{P_0}{a} - \frac{l\,P_0}{a^2}\right)\ \arcsin(\frac{a-l}{l})}{\pi} \\
&- \quad \Big(-6\ (1+\kappa)\ \sqrt{-(a\ (a-2\,l))}\,P_0\,R \\
&+ \quad \pi\ \Big(4\,(-(a\ (a-2\,l)))^{\frac{3}{2}}\,\mu - 3\ (1+\kappa)\ (a+l)\ P_0\,R\Big)\Big) \\
&/ \quad (6\,a^2\ (1+\kappa)\ \pi\,R)
\end{aligned} \tag{22}$$

Conclusion

It has been clearly indicated by the founders of nonlocal continuum mechanics that nonlocal theories of continua take place between discrete theories (such as quantum mechanics, or lattice dynamics) and conventional continuum theories. More precisely, nonlocal theories are continuum theories which take into account the discrete nature of the matter via far-reaching interactive internal forces. Therefore, a characteristic length which can be interpreted as the quantification of microstructure, plays an important role in nonlocal theories. For an isotropic solid, the characteristic length is naturally the distance between atoms (in this paper, this parameter is called the nonlocality parameter). It is also well known that for phenomena with a large enough (large enough compared to the microstructural characteristic length), characteristic length (let's say, the wave length in a wave propagation problem), conventional theories and nonlocal theories will yield quite close results. On the other hand, solutions to phenomena having a small characteristic length, especially those with severe strain gradients, will be remarkably different in both theories. The solution will especially differ around the region where severe strain gradients occur. As a natural consequence of this qualitative discussion on nonlocal theories, the solutions for the displacement and the stress field in a half space loaded by a parabolic stamp will be very close if R (which plays the role of characteristic length for external effects) is large. In other words, if the stamp is "blunt" then the nonlocal theory of elasticity will not provide new insight in to the problem. But, if the stamp is "sharp" enough, in other words, if R is small compared to the microstructural characteristic length, then the difference between solutions obtained in both theories cannot be ignored. Considering the fact that nanoindenters, indenters with a "very sharp tip" (it is not rare lately to see studies performed with an indenter having a tip radius of a few hundered nanometers) have been becoming more popular, it is clear that this study will help researchers interpret their work.

The following results are observed (see Figure 3 and Figure 4)

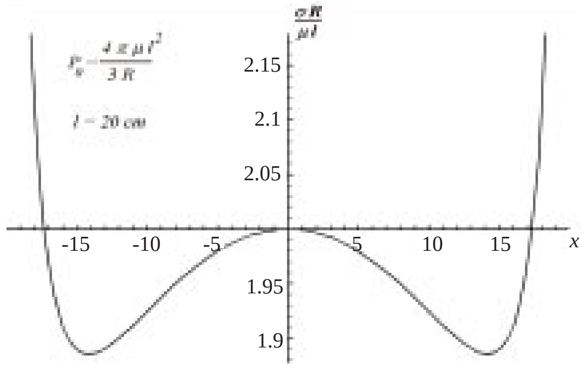

Figure 3. Local and nonlocal stresses far from the ends

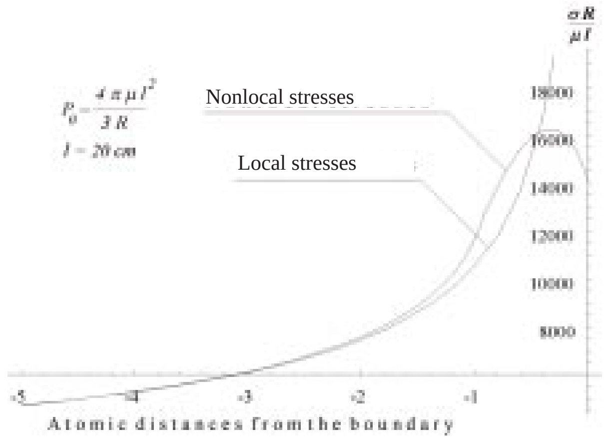

Figure 4. Local and nonlocal stresses at the front end

a) The nonlocal stress field is finite for all the points.
b) In the limit $a \to 0$, the nonlocal stress field reverts to the classical stress field.
c) The nonlocal stress field has a maximum, but the maximum stress does not occur at the boundary but further down.
Similar results have already been obtained in other papers (See Artan 1996a, Artan 1997)

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