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Linear Reservoirs in Series Model for Unit Hydrograph of Finite Duration

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Abstract

The unit hydrograph of a river basin can be estimated from the records of rainfall and streamflow by various methods. In this paper, a linear reservoirs in series model is used to derive the unit hydrograph of a certain duration. This approach is similar to that proposed by Nash for the instantaneous unit hydrograph. Expressions are obtained for the unit hydrograph ordinates of the model as function of model parameters, number of reservoirs and storage coefficient. A computer program is developed to simulate the conversion of the rainfall excess to the direct runoff by the model and to optimize the model parameters according to a selected criterion. Application of the method is illustrated in an example.

Key words: Unit hydrograph, Linear reservoirs, Reservoirs in series.

Introduction

The derivation of the flow hydrograph resulting from a certain rainfall over a river basin is an important problem in hydrology. A black-box model called the unit hydrograph is commonly used for this purpose. The unit hydrograph, proposed by Sherman in 1932 (Sherman, 1932), is defined as the hydrograph of direct runoff resulting from unit (1 cm) depth of effective rainfall falling over the basin area at a uniform rate during a specified period of time. It is assumed here that the hydrographs resulting from rainfalls of different depths will be similar, with magnitudes proportional to the depth of rainfall. The basin system transforming the effective rainfall to the direct runoff is assumed to be linear. Although this assumption is not quite correct, the unit hydrograph theory gives results that are acceptable for practical purposes.

Various methods have been used to derive the unit hydrograph of a basin from given records of rainfall and streamflow. It is relatively easy to determine the unit hydrograph when these records are available for an isolated storm distributed uniformly over the basin, with approximately constant intensity, of short duration, separated from other storms. Otherwise, the hydrograph of composite storms can be used in deriving the unit hydrograph, applying the superposition principle for linear systems (Bayazıt, 2001).

Having obtained the unit hydrograph of a basin for a certain duration of the effective rainfall, the unit hydrograph for another duration can be easily derived, again using the superposition principle. A hypothetical concept facilitating the theoretical analysis is that of the instantaneous unit hydrograph (IUH) obtained when the duration of the effective rainfall is infinitesimally small. The IUH is the transfer (impulse response) function of the basin system converting the effective rainfall to direct runoff. Several mathematical models have been developed for its determination from the rainfall and runoff records.

The outflow from the system (direct runoff), y, can be computed from the known inflow (effective rainfall), x, by means of the IUH, u, using the linear system theory (Chow *et al.*, 1988):

$$y(t) = \int_{o}^{t} u\,(t-\tau)\,x\,(\tau)\,d\tau \qquad (1)$$

where t(s) is the time, and τ(s) is a dummy variable. In Eq. (1) x and y have the same units (such as m^3/s or cm/min), and u has the unit 1/s.

Nash (1957) proposed a conceptual model for the IUH consisting of n identical linear reservoirs in series. A linear reservoir has an outflow, y, proportional to the amount of water stored in it, S:

$$S = K\,y \qquad (2)$$

where K is the storage coefficient of the reservoir. S and K have, respectively, the dimension of volume (m^3) and time (s). Taking the derivative of Eq. (2) with respect to time t

$$\frac{dS}{dt} = K\frac{dy}{dt} \qquad (3)$$

The equation of continuity for the reservoir can be written as

$$\frac{dS}{dt} = x - y \qquad (4)$$

where x is the inflow. Combining Eq. (3) with Eq. (4):

$$x = y + K\frac{dy}{dt} \qquad (5)$$

Integrating with the initial condition y(0) = 0

$$y(t) = \int_{0}^{t} x(\tau)\frac{1}{K}\,e^{-(t-\tau)/K}\,d\tau \qquad (6)$$

Comparing Eq. (6) with Eq. (1), it is seen that the linear reservoir has the IUH:

$$u(t) = \frac{1}{K}\,e^{-t/K} \qquad (7)$$

In the reservoirs in series model, the output of a reservoir constitutes the input into the downstream reservoir. The IUH of the model can be derived as follows. When unit rainfall excess enters the first reservoir at the instant of time t = 0, the outflow will be given by Eq. (7), which is also the input to the second reservoir.

The output of the second reservoir can be computed by Eq. (6) taking u(t) given by Eq. (7) as its input x(τ), with the result

$$y_2(t) = \int_{0}^{t} \frac{1}{K}\,e^{-t/K}\,\frac{1}{K}\,e^{-(t-\tau)/K}d\tau = \frac{t}{K^2}\,e^{-t/K} \qquad (8)$$

Similarly, the output of the n-th reservoir will be

$$y_n(t) = \frac{1}{(n-1)!K}\left(\frac{t}{K}\right)^{n-1} e^{-t/K} \qquad (9)$$

which is the IUH of the reservoirs in series model. This model has two parameters: n, the number of reservoirs, and K, the storage coefficient.

Unit Hydrograph of Finite Duration

In this study, the linear reservoirs in series model developed for the derivation of the IUH is applied to the unit hydrograph of finite duration Δt. Figure 1 shows the model configuration, in which the index i refers to the reservoir number and j refers to the time interval.

The ordinates of the unit hydrograph at time interval Δt can be determined as follows. Consider the input x = 1 into the first reservoir (i = 1) at the first time interval Δt. Initially all the reservoirs are empty. The storage in the first reservoir is $S_{1,1} = 1$, and output is

$$y_{1,1} = 1/K = C \qquad (10)$$

which is the input to the second reservoir (i = 2). Its output is

$$y_{2,1} = C.C = C^2 \qquad (11)$$

Similarly, the output of the reservoir i = n at the time interval Δt is

$$y_{n,1} = C^n \qquad (12)$$

In the second time interval t = 2Δt, there is no input into the first reservoir. Its storage is $S_{1,2}$ = 1-C, and its output is

$$y_{1,2} = C(1-C) \tag{13}$$

The second reservoir has an initial storage C-C^2 = C(1-C), to which $y_{1,2}$ is added. Its output at the time interval 2Δt is

$$y_{2,2} = 2\,C^2\,(1-C) \tag{14}$$

The output of the final reservoir i = n at the time interval 2 Δt is

$$y_{n,2} = n\,C^n(1-C) \tag{15}$$

In a similar manner, the output of the reservoir i = n at the time interval j Δt is

$$y_{n,j} = a_{n,j}\,C^n\,(1-C)^{j-1} \tag{16}$$

where the coefficient $a_{n,j}$ can be computed iteratively by the expression

$$a_{n,j} = a_{n-1,j} + a_{n,j-1} \tag{17}$$

with $a_{1,1} = a_{1,2} \ldots = a_{1,n} = 1$, and $a_{1,1} = a_{2,1} = \ldots = a_{j,1} = 1$. Table 1 gives the $a_{n,j}$ values for n = 1,2, ... ,20 and j = 1,2, ... ,10. For any value of n, the sum of the $y_{n,j}$values is equal to one:

$$\sum_{j=1}^{\infty} y_{n,j} = 1 \tag{18}$$

Eq. (16) gives the ordinates of the unit hydrograph of duration Δt of the linear reservoirs in series model when the input (effective rainfall) and the output (direct runoff) are expressed in same the units.

Usually Δt is expressed in hours, flows $U_{n,j}$ in m^3/s, $y_{n,j}$ in cm, and the basin area A in km^2. In this case the unit hydrograph ordinates are computed by the expression

$$U_{n,j} = 2.778\,\frac{1}{\Delta t}\,y_{n,j}\,A \tag{19}$$

where 2.778 is a unit conversion factor.

Simulation of the Model

The conversion of the effective rainfall hyetograph to the direct runoff hydrograph by means of the linear reservoirs in series model is simulated on the computer. The simulation program derives the hydrograph ordinates for a certain hyetograph when n and K values are given. For each reservoir in the system, finite difference forms of Eq. (2) and Eq. (4) are applied at successive time intervals Δt to obtain the outputs of the final reservoir (i = n), the hydrograph ordinates.

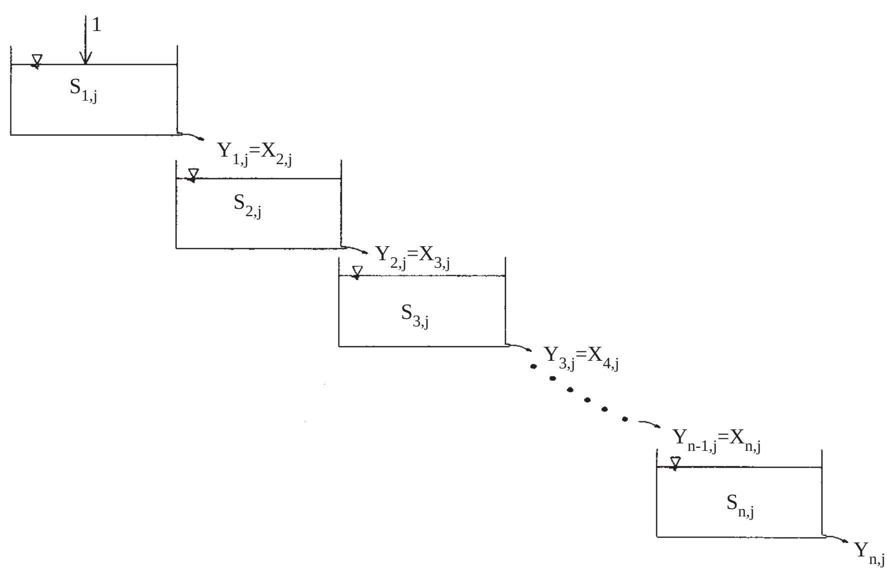

Figure 1. Configuration of the linear reservoirs in series model for unit hydrograph estimation.

Table 1. $a_{n,j}$ coefficients of the model.

n \ j	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9	10
3	1	3	6	10	15	21	28	36	45	55
4	1	4	10	20	35	56	84	120	165	220
5	1	5	15	35	70	126	210	330	495	715
6	1	6	21	56	126	252	462	792	1287	2002
7	1	7	28	84	210	462	924	1716	3003	5005
8	1	8	36	120	330	792	1716	3432	6435	11440
9	1	9	45	165	495	1287	3003	6435	12870	24310
10	1	10	55	220	715	2002	5005	11440	24310	48620
11	1	11	66	286	1001	3003	8008	19448	43758	92378
12	1	12	78	364	1365	4368	12376	31824	75582	167960
13	1	13	91	455	1820	6188	18564	50388	125970	293930
14	1	14	105	560	2380	8568	27132	77520	203490	497420
15	1	15	120	680	3060	11628	38760	116280	319770	817190
16	1	16	136	816	3876	15504	54264	170544	490314	1307504
17	1	17	153	969	4845	20349	74613	245157	735471	2042975
18	1	18	171	1140	5985	26334	100947	346104	1081575	3124550
19	1	19	190	1330	7315	33649	134596	480700	1562275	4686825
20	1	20	210	1540	8855	42504	177100	657800	2220075	6906900

The simulation program can be used to determine the optimal values of the parameters n and K of the linear reservoirs in series model for the unit hydrograph of duration Δt, when the direct runoff and rainfall excess ordinates are given, as explained below.

Obviously it is not possible to obtain a perfect fit of the simulated hydrograph ordinates to those of the observed hydrograph. A best fit is sought on the basis of a chosen criterion. The values of the parameters n and K are optimized so that the relative error is minimized. The relative error is defined as

$$e = w_1 e_1 + w_2 e_2 \tag{20}$$

where w_1 and w_2 are the weight coefficients, and e_1 and e_2 are the relative errors with respect to the total hydrograph ordinates and the hydrograph peak, respectively.

$$\begin{aligned} e_1 &= \sum_{i=1}^{m} \left(\frac{y_{io}-y_{ic}}{y_{io}}\right)^2 \\ e_2 &= \left(\frac{y_{po}-y_{pc}}{y_{po}}\right)^2 \end{aligned} \tag{21}$$

where m is the total number of hydrograph ordinates, y_i are the hydrograph ordinates (i = 1,2, ... ,m), and y_p is the hydrograph peak ordinate. The subscripts o and c refer to the observed and computed hydrographs, respectively. The optimization by Eq. (20) considers both the total hydrograph ordinates and the hydrograph peak. If $w_1 = 0$, $w_2 = 1$, then only the hydrograph peak ordinate is considered in the optimization.

Another alternative is to consider the time and magnitude of the hydrograph peak:

$$e' = w_1' \; e_1' \; + \; w_2' \; e_2' \tag{22}$$

where

$$\begin{aligned} e_1' &= \left(\frac{t_{po}-t_{pc}}{t_{po}}\right)^2 \\ e_2' &= e_2 \end{aligned} \tag{23}$$

where t_p is the time to the peak

Once the optimization criterion is chosen and the weights are determined, optimal values of n and K are found by minimizing the relative error:

$$\min\ e \qquad \text{or} \qquad \min\ e' \tag{24}$$

Table 2. Observed rainfall excess and direct runoff.

Time (min)	Rainfall Excess (cm)	Direct Runoff (m^3/s)
20	0.562	0
40	1.968	456
60	0.526	3990
80	0.083	12275
100		13166
120		9268
140		6717
160		5682
180		3632
200		1881
220		1445
240		820

Time (min)	Direct Runoff (m^3/s)
260	638
280	508
300	299
320	215
340	189
360	137
380	91
400	78
420	65
440	59
460	46
480	39

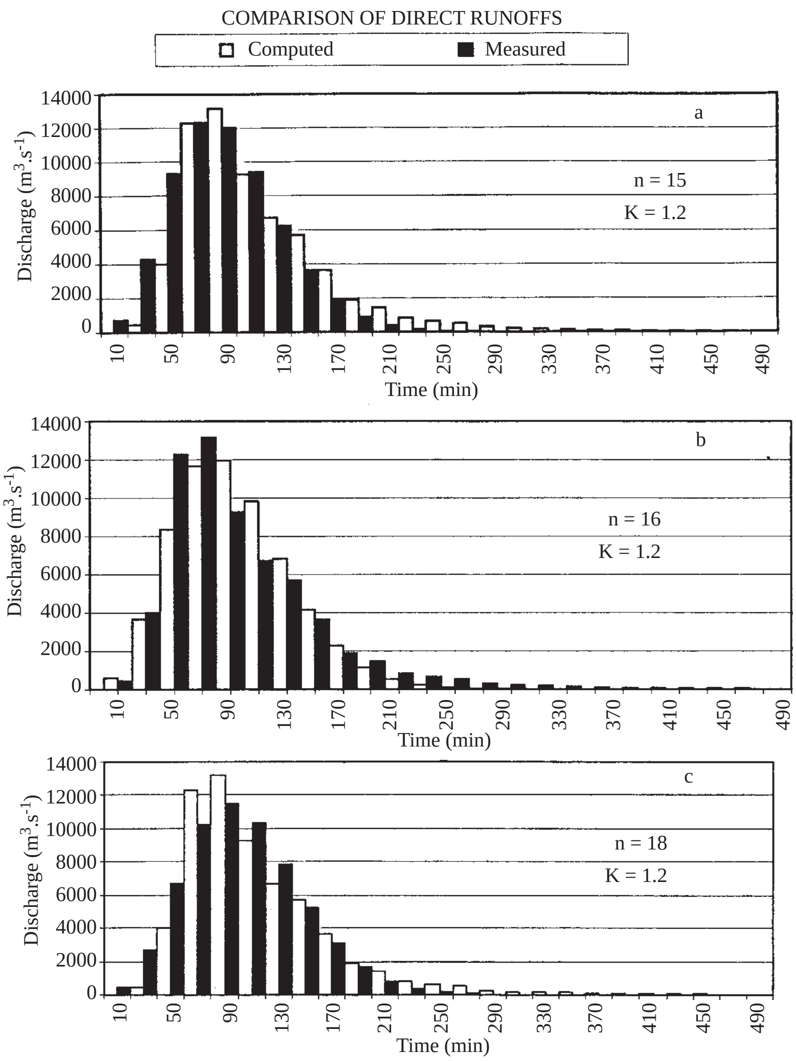

Figure 2. Comparison of the measured and simulated direct runoff hydrographs: (a) n = 15, K = 1.2, (b) n = 16, K = 1.2, (c) n = 18, K = 1.2.

Application

Unit hydrograph derivation by the linear reservoirs in series model is illustrated in an example. Singh (1992) gives the rainfall excess (computed from the observed hyetograph) and direct runoff ordinates for a basin of area A = 2393 km^2 (Table 2).

The linear reservoirs in series model is fitted to the observed hydrograph by different criteria and different weight coefficients. The results are shown in Table 3.

Measured direct runoff hydrograph is compared with the hydrographs estimated by the linear reservoirs in series model for different values of n and K (Figures 2a, b, c). Unit (1 cm depth) hydrographs of the linear reservoirs in series model are compared with the unit hydrograph of the basin estimated by the Collins method (Singh, 1992), which is a trial-and-error procedure, as shown in Figures 3a, b, c.

A better fit to the observed flows is achieved with the parameter values n = 6, K = 1.4 found by trial and error (Figure 4), in which case the estimated unit hydrograph is quite close to that found by Singh (1992). This hydrograph corresponds to certain values of the weight coefficients that were not considered in the optimization.

Table 3. Optimization results.

Criterion	$w_1(w'_1)$	$w_2(w'_2)$	n	K
min e	0	1	15	1.2
mine'	1	0	16	1.2
min e	0.5	0.5	18	1.2

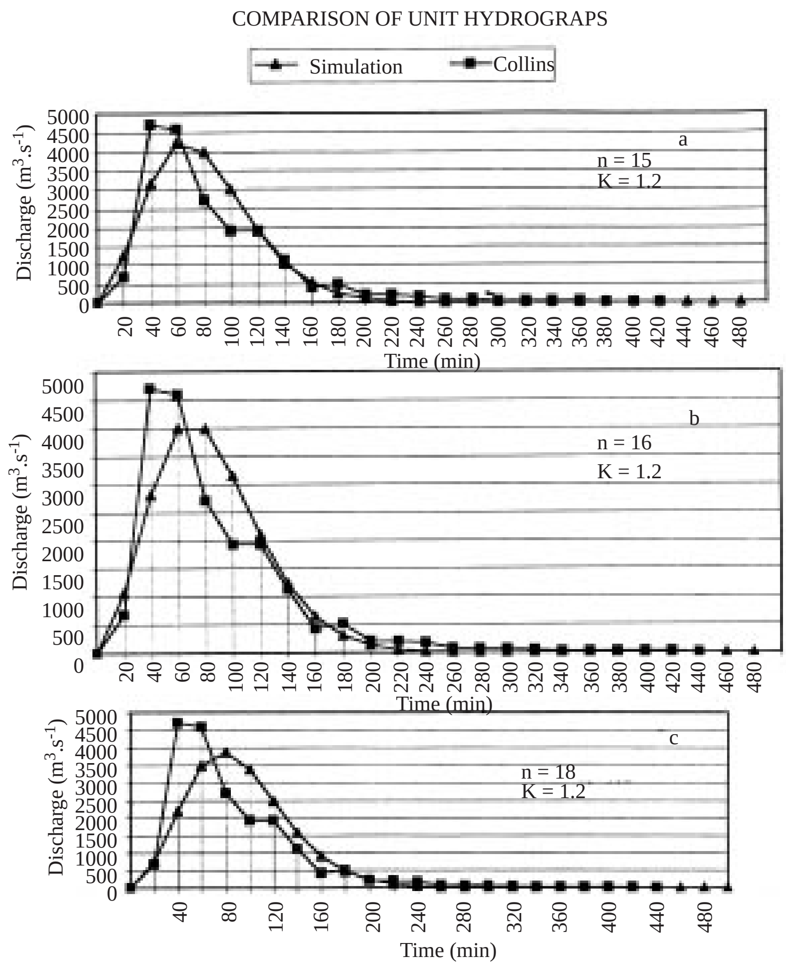

Figure 3. Comparison of the unit hydrograph of the model with that estimated by the Collins method: (a) n = 15, K = 1.2, (b) n = 16, K = 1.2, (c) n = 18, K = 1.2.

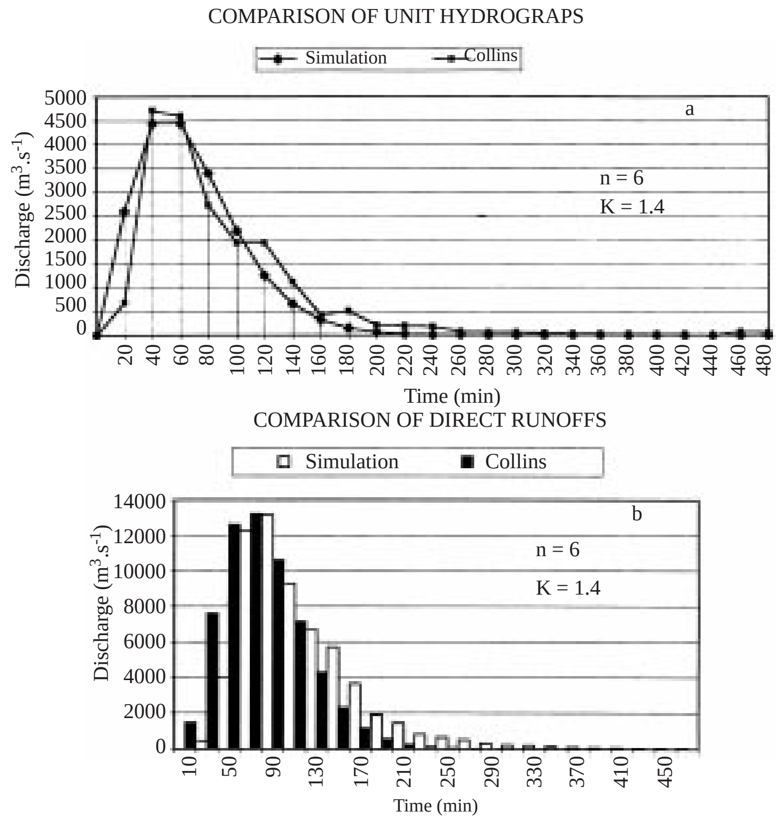

Figure 4. (a) Comparison of the unit hydrograph of the model with that estimated by the Collins method, (b) Comparison of the measured and simulation direct runoff hydrographs. n = 6 and K = 1.4.

Conclusions

The unit hydrograph of a basin can be estimated from the given rainfall excess hyetograph and direct runoff hydrograph of a river basin by a linear reservoirs in series model that was originally developed for the instantaneous unit hydrograph. The model has two parameters: n, number of reservoirs, and K, storage coefficient.

An expression is derived that gives the unit hydrograph ordinates of the model for a given pair of n and K values. A computer program is prepared that simulates the conversion of the rainfall to the runoff using the model. The parameters of the model can be optimized on the basis of a selected criterion expressing the fit of the estimated direct runoff hydrograph to the observed hydrograph. An example shows that satisfactory agreement can be achieved when the model parameters are selected appropriately.

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