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The Automated Fairing of Ship Hull Lines Using Formal Optimisation Methods

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Abstract

The problem of creating fair ship design curves is of major importance in Computer Aided Ship Design environment. The fairness of these curves is generally considered a subjective notion depending on the judgement of the designer (eg., visually pleasing, minimum variation of curvature, devoid of unnecessary bumps or wiggles, satisfying certain continuity requirements). Thus an automated fairing process based on objective criteria is clearly desirable. This paper presents an automated fairing algorithm for ship curves to satisfy objective geometric constraints. This procedure is based on the use of optimisation tools and cubic B-spline functions. The aim is to produce curves with a more gradual variation of curvature without deteriorating initial shapes. The optimisation based fairing procedure is applied to a variety of plane ship sections to demonstrate the capability and flexibility of the methodology. The resulting curves, with their corresponding curvature plots indicate that, provided that the designer can specify his objectives and constraints clearly, the procedure will generate fair ship definition curves within the constrained design space.

Key words: Fairing, Optimisation, Ship Lines, Curvature, B-splines.

Introduction

Fairing of the geometry of ship hull forms is a principal design requirement of the ship design process. The fairness of the hull form will be required to improve hydrodynamic performance and producibility characteristics as well as aesthetic properties. The solution to this rather complex problem is generally achieved by reducing the problem of fairing a 3-dimensional body to a series of 2-dimensional problems. For convenience, the surface of the ship is traditionally described by a mesh of intersecting curves called ship lines. These curves are mainly planar and represent sections along the length, depth, and breadth (i.e., body sections, waterlines, and buttock lines). Thus, the problem can be reduced to fairing these planar design curves, which together form the 3-dimensional body.

Fairing of ship hull forms is, traditionally, achieved by flexible battens and weights. This method was introduced in the 18^{th} century, and has been successfully used until modern numerical techniques and powerful digital computers became available. This method is based on the successive fairing of ship lines on 3 different planes (i.e. body sections, waterlines and buttocks) in an iterative manner. Provided that the designer has sufficient experience and time, the resultant form should have acceptable 3-dimensional fairing properties. The process has no objective measures for geometric fairness, and the fairing characteristics of the resulting hull form geometry greatly depend on the designer's ability and experience. Thus, giving the fairing of a set of lines defining a ship's hull form to 'n' different designers/loftsmen, all working independently of each other, will result in 'n' different solutions.

Fairing is a part of most Computer Aided Ship Design (CASD) packages commercially available today. In the earlier cases the fairing procedures were based on interactive routines where the designer visually observes the design curves and interactively modifies it until satisfactory fairness is achieved (Snaith and Parker, 1972). Alternatively, the designer is presented with curvature plots which help to identify the regions of unfairness (Horsham, 1988). These procedures can be seen as the computerised version of the traditional manual fairing method achieved by flexible battens and weights. Hence, the procedure has no objective criteria and suffers the same drawbacks of the manual fairing method, i.e. the need for excessive time and suitably trained and experienced personnel.

One of the main goals of the fairing process is to automate the process and hence minimise subjective human intervention, which can lead to many inconsistencies in the resulting hull form geometry. The development of automated procedures in which the fairness is defined in an objective manner and achieved within the boundaries set for the design problem is clearly desirable. During the past decade there have been numerous attempts to produce automated fairing procedures for curves and surfaces to be employed in both the CAD and CASD environment. Pramila (1978) used a linearised fairness functional which minimises strain energy for ship hull surfaces. Maccallum and Zhang (1986) described an automatic smoothing algorithm based on B-splines' curvature behaviour property and applied this to some curve forms used in ship design. Nowacki *et al.* (1989) described a surface approximation scheme based on minimisation of the sum of the strain energy of mesh lines and the potential energy of springs attached to the data points. Rogers and Fog (1989) applied their constrained B-spline curve/surface fitting algorithm to ship hull forms to generate defining polygons for curves and defining polygonal nets for surfaces. Sapidis and Farin (1990) proposed an automatic fairing algorithm for B-spline curves. The algorithm is based on removing and reinserting knots of the spline. Liu *et al.* (1991) introduced constrained smoothing B-spline curve fitting for mesh curves of ships by minimising an energy functional as a fairness measure. Huanzong *et al.* (1991) proposed a fairing method by minimising the elastic strain energy of mesh curves of hull surfaces. Moreton and Sequin (1992) applied non-linear optimisation techniques to minimise a fairness functional based on variation of curvature. Nowacki and Lü (1994) proposed procedures for developing fair curves under constraints in which the fairness criterion is based on the linear combination of the square of the second and the third derivative norm and the constraints apply to approximation conditions, end conditions and an integral condition pertaining to the area under the curve, while Pigounakis and Kaklis (1996a) developed a 2 stage automatic algorithm for fairing cubic parametric B-splines under convexity, tolerance and end constraints. An iterative knot removal and reinsertion technique is employed which adopts the curvature-slope discontinuity as the fairness measure. Pigounakis *et al.* (1996b) proposed 3 algorithms for fairing spatial B-spline curves: local fairing by knot removal and local/global fairing based on energy minimisation. Hahmann (1998) proposed an automatic and local fairing algorithm for bi-cubic B-spline surfaces. In the proposed method a local fairness criterion selects the knot where the spline needs to be faired and the control net is modified by a constrained least squares approximation. Poliakoff *et al.* (1999) presented an automated curve fairing algorithm for cubic B-spline curves based on an extension of Kjellander's (1983) algorithm, which is based on finding and correcting the offending data point. The point to be faired is chosen by calculating for each point the distance to be moved and then choosing the one for which the distance is greatest. Kantorowitz *et al* (2000) described a method for fairing ship hull lines which determines the suitable number of control points to produce the required shape of the body sections. Yang and Wang (2001) presented a method for planar curve fairing by minimal energy arc splines where as a first step the optimal tangents for curve interpolation are computed and the point positions are adjusted by smoothing discrete curvatures.

The 3-dimensional ship hull fairing problem is generally reduced to the fairing of 2-dimensional ship hull design curves, namely sections, waterlines and buttocks. The main reason behind this is the complexity of the 3-dimensional fairing problem. In addition a robust theory for the fairness of curves would greatly contribute to surface fairing methodologies. Indeed, most of the existing surface fairing methodologies either borrow or adapt ideas from curve fairing. Thus, the curve fairing problem often arises in ship hull design and deserves attention. Therefore, in this paper an automated fairing algorithm for generating fair ship lines is presented. This algorithm is based on a variational scheme in which all the numerical details of the fairing process are hidden from the designer. The fairing problem is specified in terms of functional optimisation in which the desired form of the curve has a minimum measure of shape quality. The measure of shape quality is selected to provide as uniform a distribution of curvature as possible. The resulting ship hull form should not deviate significantly from the original form in order not to degrade hydrostatic and hydrodynamic performance characteristics already obtained. Therefore, a closeness constraint is imposed to ensure that the deviations between the original and faired lines are not excessive. Hence, the main goal of this fairing procedure can simply be defined as the reduction in curvature variation while retaining the overall curve characteristics.

The presentation of this paper begins with a description of the curvature concept, which is used as a basic indicator of fairness of curves and surfaces. Then the fairing problem is formulated as a non-linear optimisation problem in which the objective is to produce ship definition lines with reduced curvature distribution. The fairing method is applied to typical cubic B-spline ship curves and the results are presented in order to demonstrate the effectiveness of the numerical procedure.

Curvature concept and definition of fairness for ship hull lines

A mathematical fairing process would require an objective fairing criterion, which may be defined in terms of the distribution of curvature or local radius of curvature (=evolute) along the curve.

The curvature $\kappa(t)$ of planar curves $r(t) = [x(t), y(t)]$ has a positive or negative sign depending on whether it curves to the left or right. Thus, this signed curvature is highly desirable detecting inflection points as well as convex and concave regions of a curve. Hence, the signed curvature can be expressed as follows (Farin and Sapidis, 1989):

$$\kappa(t) = \frac{\ddot{x}(t)\dot{y}(t) - \ddot{y}(t)\dot{x}(t)}{\left[\dot{x}(t)^2 + \dot{y}(t)^2\right]^{3/2}} \tag{1}$$

Curvature plots are typically displayed as curvature vs. arc length or directly along the curve as an offset curve with the distance proportional to the curvature values. The curvature plot consists of segments normal to the curve emerging from a number of points on the curve and whose lengths are proportional to the magnitude of curvature at the associated point. The characteristics of a curve are evidenced by the undulations of its curvature plot. Inflection points occur when the curvature plot crosses the curve (sign change), flat regions produce zero curvature values, bulging tendencies produce locally increased values, and flattening tendencies produce locally reduced curvature values.

The fairness of a curve is intimately related to the distribution of curvature over the form, favouring gradual transitions and avoiding abrupt changes. Hence, a variety of definitions can be found in the literature for fairness criteria mostly associated with the curvature properties of curves. The widely accepted ones are the following:

- A curve is fair if its curvature plot consists of relatively few monotone pieces (Farin and Sapidis, 1989)
- A curve is characterised as fair if its curvature plot is continuous, has the appropriate sign (if the convexity of the curve is prescribed), and it is as close as possible to a piecewise monotone function with as few monotone pieces as possible (Sapidis and Farin, 1990)
- A C^2 curve is considered fair if it minimises the integral of the squared curvature with respect to arc length (Roulier and Rando, 1994)
- A fair curvature plot should be free of any unnecessary variation, i.e. the distribution of curvature on a fair curve must be as uniform as possible (Pigounakis *et al.*, 1996b)

Thus, based on the above definitions it is required of a fair ship line to have the following characteristics:

- Devoid of unintended noise (erratically distributed high frequency, high amplitude undulations)
- Devoid of unintended flat regions, and flattening/bulging tendencies
- Continuous first and second derivatives
- Free of unnecessary variation, i.e. limited and specified inflection points,
- Curvature as uniformly distributed as possible, i.e. monotonically increasing or decreasing

- The deviation of offset points should be minimal, i.e. within the boundaries defined by the designer
- Shape preservation

All curves satisfying the conditions stated above will look fair or pleasing to the eye of an experienced designer, and these shape requirements should be translated into mathematical terms in order to implement them into a fairing algorithm.

Formulation of ship hull lines fairing as an optimisation problem

Ship definition curves are generally obtained by inspecting corresponding curvature plots in an interactive fairing process. Due to the aforementioned drawbacks of this process, researchers are in a constant search for alternative and objective methods. The main goal is to develop automated fairing procedures to generate ship curves that have uniformly distributed (smooth) curvature plots.

Formulation of the ship lines fairing process as an optimisation problem can be set as a typical engineering problem. It is formed of 3 basic elements, namely the design variables, objective function and constraints.

Design variables: $X(X_1, X_2, \ldots, X_n)$

Objective function: $F(X) = f(X_1, X_2, \ldots, X_n)$

Constraints: $g_j(X) > 0$, $j = 1, 2, \ldots, n$

The task is to minimise an objective function defined in terms of design variables subject to given constraints. The design variables for the ship lines fairing problem are the predefined 2-D offset points for the curve in consideration. An initial curve, which satisfies the required performance characteristics, is assumed to be available. The objective is to eliminate undesirable shape features in order to produce a curve fairer than the original one. While removing the flaws of the curve (e.g., unintended inflections, noise), the general shape of the curve must be preserved in order not to degrade specific performance characteristics. This is mainly achieved by including the constraints into the problem. The structure of the optimisation problem is illustrated in Figure 1. The following must be available in order to formulate and solve the fairing problem, which are discussed in detail in the following sections:

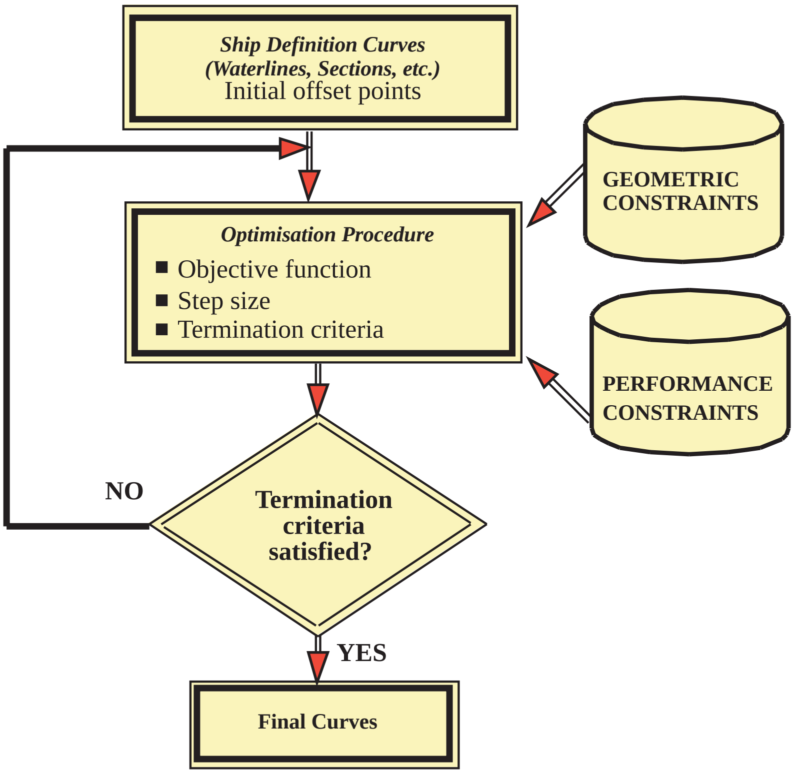

Figure 1. Structure of the optimisation based ship lines fairing procedure.

- A set of initial offset points
- A fairness metric described in terms of design variables
- Definition of constraining boundaries
- Selection of a suitable non-linear optimisation algorithm

The fairness metric

A formal optimisation problem will require the definition of an objective function to be minimised, which is a function of some free geometric design variables. Thus, in an optimisation based fairing procedure, the objective function is formulated in terms of a fairness measure. The selection of the fairness metric is the first step in the automatic generation of fair ship lines.

The choice of the fairness functional is crucial for the effectiveness of the procedure and the quality of the curve is highly dependent on the type of the fairness functional. There is no unique mathematical criterion measuring the fairness of a given curve because it is generally the designer's decision to accept a curve as fair enough or not. Nevertheless, most fairness functionals are motivated by physical reasoning describing a certain energy over the form or have geometric meaning. The non-linear curve modelling a thin elastic beam is known as the minimum energy curve, and is characterised by bending least while passing through a given set of points. It is generally considered to be an excellent criterion for producing smooth curves (Horn, 1983). Hence, in this paper the minimum energy curve functional is adopted as the objective function of the problem to be minimised. This criterion is mainly selected for its simplicity and effectiveness, which tends to level the curvature while keeping it on average as low as possible (Roulier and Rando, 1994). This will be later proved by various ship curve applications. The functional representing the strain energy of the curve can be expressed as the arc length integral of the curvature squared:

$$E = \int \kappa(s)^2 ds \tag{2}$$

where s denotes the arc length of the curve.

This functional is regarded as the fairness measure of the problem and is computed for the curve to obtain the quantity which characterises the desirability of the product curve under that metric. Hence, the required shape will obviously be the one that satisfies the predefined geometric constraints while optimising this quantity.

Constraints

One of the fundamental requirements of the ship lines fairing process is that the faired lines must be as close as possible to the initial lines in order to satisfy certain geometric performance characteristics. Therefore, geometric constraints should be imposed.

Constraints of this particular fairing problem are set in terms of the control polygon of the B-spline curve (e.g., end conditions, deviation form original offset points, integral constraints). The coordinates of each design variable are not to change more than a specified tolerance and some control points are fixed to guarantee continuity (first and last points of the curve). Hence, the deviation of a faired curve from the original one would be within the defined boundary domain.

The solution method: Hooke and Jeeves algorithm

The success of the optimisation process will mainly depend on the algorithm used. The optimisation algorithm used is the non-linear direct search method of Hooke and Jeeves (1961). This robust and efficient numerical solution method is one of the most widely used numerical solution algorithms, and has been found to work well for the problem under discussion. It attempts in a simple though ingenious way to find the most profitable search directions.

This method is specifically developed for non-constrained problems. Hence, for the purpose of the numerical treatment of the constraints of this problem, an internal penalty function technique is applied which transforms the problem into an unconstrained optimisation problem where the objective function can be taken as

$$F(X, r_k) = f(X) + r_k \sum_{i=1}^{N} \frac{1}{g_i(X)}, \qquad r_k > 0 \tag{3}$$

where f(X) refers to the objective function of the constrained optimisation problem, g_i(X) the constraints of the problem and N the number of constraints. r_k is an internal penalty function parameter where the value of the penalty parameter is chosen to have a large value (approx. 1000) to add a penalty to the objective function as soon as one or more constraints g(X) are violated.

The Hooke and Jeeves method is based on 2 types of step-by-step searches alternating in turn: first a *local search*, which is a unidirectional variation of each design variable resulting in the direction of steepest descent, and a *pattern move,* and which represents a rotation of the search direction which accelerates the search by the aid of increasing the step widths. The search routine can only proceed in a feasible space because outside that space the penalty functions are not defined.

If we consider the problem of minimising $f(X_1, X_2, \ldots, X_n)$, the general procedure, can be described as follows:

- Start with an arbitrarily chosen initial base point $(b_1, b_2, \ldots, b_n)$ and step lengths $(h_1, h_2, \ldots, h_n)$ for the respective variables $(X_1, X_2, \ldots, X_n)$.
- The method proceeds by a sequence of exploratory and pattern moves. The procedure for an exploratory move about the point $(b_1, b_2, \ldots, b_n)$ is as follows:
- Evaluate f $(b_i + h_i)$. If the move from b_i to $b_i + h_i$ is a success, replace the base point b_i by $b_i + h_i$. If it is a failure, evaluate $f(b_i - h_i)$. If this move is a success, replace b_i by $b_i - h_i$. If it is another failure, retain the original base point b_i.
- Repeat the above procedure for each variable in turn finally arriving at a new base point after (2n+1) function evaluations at most.
- If the new variable value b_i^* is equal to the original base point value b_i, halve each of the step lengths h_i and return to the first step. The calculations terminate when the step lengths have been reduced to some prescribed level. If $b_i^* \neq b_i$, make a pattern move from b_i^*.
- A pattern move attempts to speed up the search by using information already acquired about $f(X_1, X_2, \ldots, X_n)$. It is invariably followed by a sequence of exploratory moves, with a view to finding an improved direction of search in which to make another move. The procedure for a pattern move from b_i^* is as follows:
- It seems sensible to move from b_i^* in the direction $(b_i^* - b_i)$, since a move in this direction has already led to a decrease in the value of $f(X_1, X_2, \ldots, X_n)$. Therefore, move from b_i^* to $(2b_i^* - b_i)$ and continue with a new sequence of exploratory moves about $(2b_i^* - b_i)$.
- If the lowest function value obtained during the pattern and exploratory moves of $(2b_i^* - b_i)$ is less than b_i^*, then a new base point b_i^{**} has been reached. In this case, return to $(2b_i^* - b_i)$ with all suffices increased by unity. Otherwise abandon the pattern move from b_i^* and continue with a new sequence of exploratory moves about b_i^*.

Application and evaluation of the algorithm for actual ship lines

The automated fairing procedure is applied to 3 typical ship definition curves. These examples represent offsets taken at transverse sections along the ship length, called body sections. The small dots displayed in the figures represent the control points of the curves.

The first 2-D curve is from a mathematical hull form, the well-known Wigley form, selected to demonstrate the performance of the fairing procedure (Wigley, 1934). The offset values y(x,z) of the hull surface, which is symmetrical in both fore-aft and port-starboard, is defined by the following equation:

$$y(x,z) = \frac{B}{2}\left[1 - \left(\frac{2x}{L}\right)^2\right]\left[1 - \left(\frac{z}{D}\right)^2\right] \tag{4}$$

where

x : distance from amidships, positive forward,

z : distance from baseline, positive downwards,

L, B, D : length, breadth and depth of the hull, respectively.

The mathematical hull form has an intrinsically fair surface and hence fair waterlines, buttocks and body sections. The mid-section curve of this hull form is selected as the first application of this fairing procedure and is displayed in Figure 2a. The offset points of this section are then randomly distorted by adding erratically distributed undulations into the data, which is shown in Figure 2b. The offset points of the disturbed curve serve as the initial points of the optimisation. After approximately 1500 iterations the faired curve is obtained, which is displayed in Figure 2c. The curvature plot is drawn along the curve to make all the undulations and unfair regions visible. It is clear from the figures that the faired curve is radically improved in terms of fairness. The percentage improvement in terms of the objective function is 95%, and this reduction in the objective function value is mathematical proof of the improvement in the optimised curve. The curvature plot of the resulting curve has an evenly distributed smooth shape like the original Wigley section. It should be noted that for a fairing algorithm to be considered a useful tool, the deviation between the original and the faired curves should be within the acceptable design limits. For this application the difference of the sectional area of the resulting curve from the original Wigley mid-section is 1.2% and therefore the difference is within the tolerated limits. A simple algorithm using divided differences principle (Renz, 1982; Narlı and Sarıöz, 1998) can eliminate high frequency errors of the distorted curve but these errors are deliberately left to show the effectiveness of the optimisation algorithm.

A bulbous bow section, shown in Figure 3, is selected as a second example. The initial value of the objective function is reduced by 45% after approximately 1000 iterations. It is clear from the figure that the variation of curvature is more gradual in the resulting curve. The deviation between the geometric characteristics of the original and resulting curves is within acceptable design limits. Table 1a shows the offset points of original and faired curves together with corresponding percentage of deviation from the original offset values of the curve. Table 1b displays the number of iterations, the value of the objective function, and the percentage of improvement in terms of the objective function.

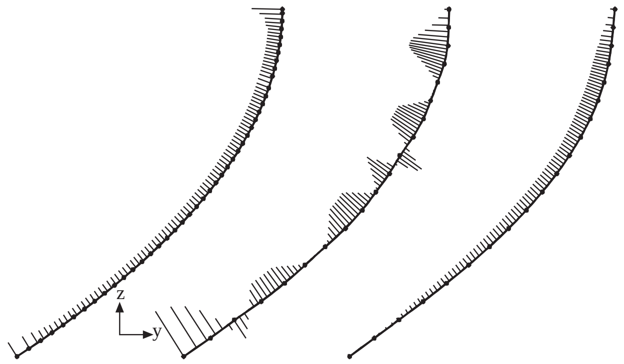

Figure 2. **(a)** Original, **(b)** distorted, **(c)** faired Wigley hull mid-section (1500 iterations, 95% of improvement in the objective function).

Table 1a. Comparison of original and optimised offset values of the bulbous bow section.

No. of offsets	z values	Original Offsets (y)	Optimised Offsets (y)	Percentage of deviation (%)
1	0	0.0000	0.0000	0% (defined as a constraint)
2	0.05	0.1000	0.1100	10%
3	0.1	0.2500	0.2250	10%
4	0.2	0.3500	0.3150	10%
5	0.3	0.3400	0.3076	9.5%
6	0.4	0.2700	0.2430	10%
7	0.5	0.1000	0.1100	10%
8	0.6	0.0200	0.0220	10%
9	0.7	0.0400	0.0360	10%
10	0.8	0.1100	0.1005	8.6%
11	0.9	0.2200	0.2008	8.7%
12	1.0	0.3000	0.3000	0% (defined as a constraint

Table 1b. Number of iterations, the value of the objective function, and the percentage of improvement in terms of the objective function.

	No. of Iterations	Objective Function	Improvement of objective function (%)
Initial Curve	0	3.671	0
	⋮	⋮	⋮
	250	3.049	16.9
	⋮	⋮	⋮
	500	2.580	29.7
	⋮	⋮	⋮
	750	2.276	38
	⋮	⋮	⋮
Optimised Curve	952	2.024	45

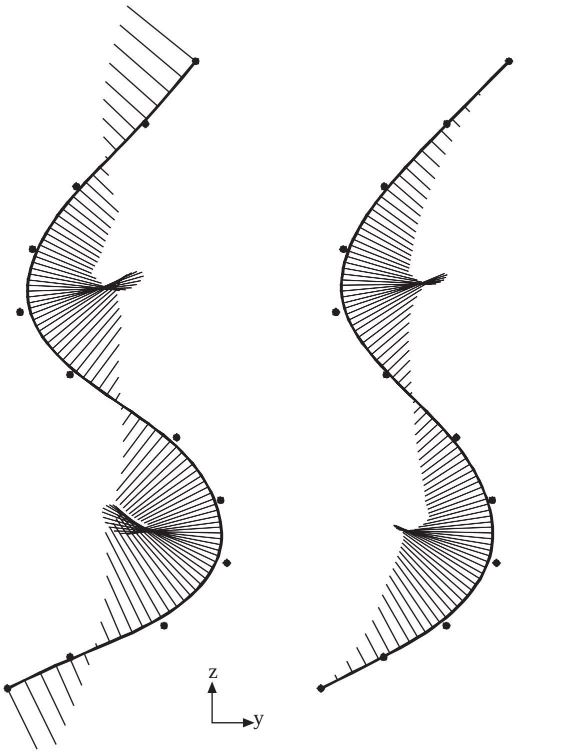

Figure 3. **(a)** Initial and **(b)** faired bulbous bow section (952 iterations, 45% of improvement in the objective function).

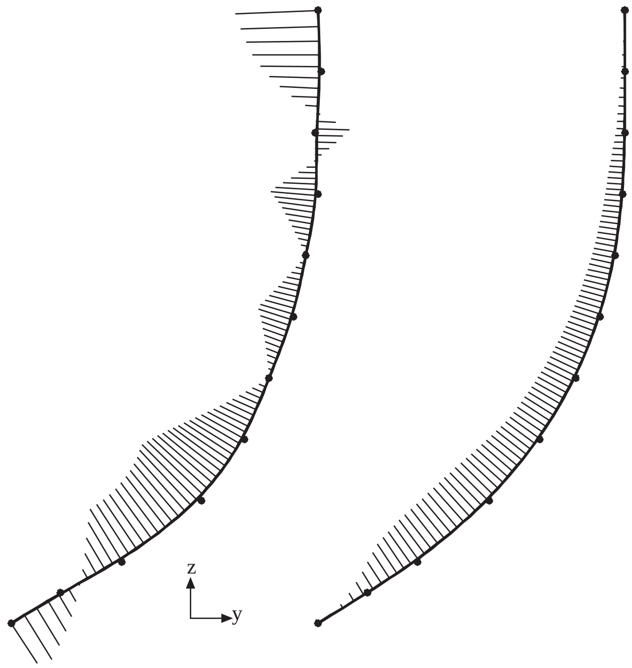

Figure 4. **(a)** Initial and **(b)** final V-shaped section (1050 iterations, 43% of improvement in the objective function).

A V-shaped ship section is the last application of this fairing procedure. The shape of this curve is also deliberately distorted to demonstrate the effectiveness of the developed methodology. The unwanted inflection points can clearly be seen from Figure 4a, which is exaggerated by the curvature plot of the curve. The optimisation process starts by taking the offset points of the distorted curve as the initial values of the design variables. At the end of the optimisation process a rather improved form of the initial curve is obtained. This curve and its curvature plot are displayed in Figure 4b. The final curve is devoid of unnecessary wiggles, favouring gradual transition and avoiding abrupt changes. The unintended inflection points are eliminated throughout the optimisation process. The degree of fairness of this curve is evidenced by the value of the objective function, which is 43% lower than the initial value. The sectional area of the final curve is only changed 4% from the original one, which is considered to be within the feasible design space.

The common feature of all applications is that the strong variation of the curvature almost disappears in the final curves, while the distribution of curvature along the curve becomes smoother. Furthermore, the value of the fairness measure decreases with an increasing number of iterations, which is a clear indication of generating a curve fairer than the original one.

Conclusions

The main objective of this research is to fair planar ship definition curves represented by cubic B-splines within the specified distance constraints. In this prospect, an automated fairing methodology for ship design curves is proposed. A fairness criterion is used (minimising energy functional) as the objective function of the optimisation algorithm to be minimised. Geometric constraints are imposed (e.g., end points are fixed) to obtain the optimum curve in terms of both fairness and closeness to the original curve. Typical ship definition curves along with a mathematical curve taken from a mathematical hull form are selected as applications. The offset points of these curves are randomly distorted to demonstrate the effectiveness of the method. The curvature plots of these curves are drawn along with the curves to display the improvement achieved by the fairing method. Both visual and numerical results demonstrate the effectiveness of the method as an effective tool for correcting the curvature plots of ship definition curves and the main advantage of the method is that it is an automated process which would not require human intervention.

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Nomenclature

r(t)	parametric curve
t	curve parameter
x(t)	x-coordinates
y(t)	y-coordinates
$\dot{x}(t)$	first derivative of x with respect to parameter t
$\dot{y}(t)$	first derivative of y with respect to parameter t
$\ddot{x}(t)$	second derivative of x with respect to parameter t
$\ddot{y}(t)$	second derivative of y with respect to parameter t
L	length of the hull
B	breadth of the hull
D	depth of the hull
z	distance from baseline, positive downwards
x	distance from amidships, positive forward
y(x,z)	offset values of the hull surface
X_i	design variables
f(X)	objective function of the constrained optimisation problem
$g_i(X)$	the constraints of the problem
N	the number of constraints
r_k	penalty parameter
b_i	initial base points
b_i^*, b_i^{**}	new base points for design variables X_i
h_i	step length
κ (t)	curvature
E	strain energy of the curve
s	arc length of the curve

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