Turkish J. Eng. Env. Sci.
28 (2004) , 369 – 376.

Development of Universal Flight Trajectory Calculation Method for Unguided Projectiles

Mehmet AKÇAY
Technical and Project Management Department of Turkish Land Forces Command, Ankara-TURKEY

Received 28.04.2004

Abstract

A universal external ballistic trajectory model for unguided projectiles is developed in this study. The mathematical model is based on the Maclaurin Series. The Maclaurin Series coefficients are predicted empirically. The method not only considers the aerodynamics of the projectile but also the effect of the height of the location of the gun above sea level. The results of the application of this method for several different calibers of projectiles are compared with the trajectory radar measurements and the modified point mass trajectory model results.

Key words: Flight trajectory, Unguided projectiles, Trajectory radar measurements.

Introduction

The flight characteristics of projectiles have been very well studied in the literature (Molitz and Strobel, 1963; Lieske and Reiter, 1966; Akçay, 1986, 1991; Akçay and Erim, 1992; Lieske and Amoruso, 2003). The main difficulties are due to the need for precise calculations of the aerodynamics and flight mechanics parameters of the projectiles. Some of these parameters, such as drag and moment coefficients, have to be measured in wind tunnels or gun tunnel tests. The results of the solution of complete external ballistics equations have to be satisfied with very expensive and laborious live firings in test ranges in order to obtain correction factors (Akçay, 1986). For each trajectory, special sets of inputs have to be used and a set of external ballistic equations is solved in order to obtain necessary ballistic parameters.

In this study, a universal trajectory calculation method was developed and applied for an 81 mm mortar projectile,and 155 mm M107 and 155 mm M864 projectiles. Calculated trajectory parameters for the 81 mm mortar projectile were compared with the results of a modified point mass trajectory method and experimental firing data. The trajectories obtained for the 155 mm M107 and 155 mm M865 projectiles were also compared with the trajectory radar measurements. All the results are quite satisfactory for the application of the proposed method.

The method is applicable to predict the velocity of the projectile,and the x and y position, time of flight and angle of inclination at any point at the trajectory of a projectile. The maximum height of the projectile and its maximum range can also be calculated with this method. Wind effect was not considered in this study.

The difference between this study and the conventional approach is that the present method utilizes one set of data (composed of initial velocity, angle of fire and point of fall) in order to predict only one empirical value for each charge valid for low and high firing angles. The developed method is applicable for every caliber of projectiles from 0 to ∞ and subsonic, transonic and supersonic flight regimes.

This method gives us a very quick and very cheap prediction opportunity for the trajectory parameter calculations of any unguided projectile.

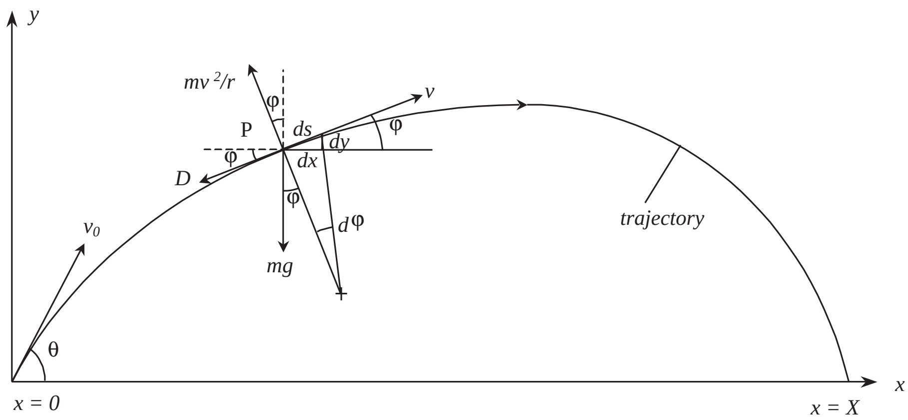

Figure 1. The motion of a free flight projectile.

Mathematical model

The motion of a projectile was studied according to a coordinate system fixed to the firing point seen in Figure 1.

If the trajectory is represented by a continuous function $y = f(x)$, the Maclaurin Series expansion of this function at point x is

$$y = f(x) = f(0) + \tfrac{1}{1!}f'(0)\cdot x + \tfrac{1}{2!}f''(0)\cdot x^2 + \tfrac{1}{3!}f'''(0)\cdot x^3 + \cdots + \tfrac{1}{n!}f^{(n)}(0)\cdot x^n + R \tag{1}$$

where R represents the residue expressed by

$$R = \tfrac{1}{(n+1)!}f^{(n+1)}(0)\cdot x^{n+1}\ ;$$
$$0 \le x \le X\ ; \qquad \lim_{n\to\infty} R = 0$$

The details of this subject are well explained by Kaplan (1963) and Mollitz and Strobel(1963).

The first and second derivatives of function $f(x)$ at any point x of the trajectory are, respectively

$$f'(x) = \frac{df}{dx} = \tan\varphi \tag{2}$$

$$f''(x) = \frac{1}{\cos^2\varphi}\frac{d\varphi}{dx} \tag{3}$$

where φ is the angle of inclination of the trajectory. By taking the radius of curvature of the trajectory r to be positive

$$r\,d\varphi = -ds \tag{4}$$

then

$$\frac{d\varphi}{dx} = -\frac{1}{r\cos\varphi} \tag{5}$$

The application of Newton's Second Law on the normal direction of the trajectory gives

$$m\frac{v^2}{r} = mg\cos\varphi \tag{6}$$

where g is the gravitational acceleration, and m and v are the mass and the velocity of the projectile, respectively. Combining of Eq. (6) with Eq.(5) gives

$$\frac{d\varphi}{dx} = -\frac{g}{v^2} \tag{7}$$

By introducing Eq. (7) in to Eq. (3)

$$f''(x) = -\frac{g}{v^2\cos^2\varphi} \tag{8}$$

is obtained.

Using the definition of the horizontal velocity component

$$v_x = \frac{dx}{dt} = v\,\cos\varphi \tag{9}$$

into Eq. (8)

$$f''(x) = -\frac{g}{v_x^2} \tag{10}$$

and taking its derivative gives

$$f'''(x) = -\frac{1}{v_x^2}\frac{dg}{dx} + 2\frac{g}{v_x^3}\frac{dv_x}{dx} \tag{11}$$

The change in gravity g with height can be written as

$$g \cong g_{SL}\left(1 - 2\frac{h+y}{R_e}\right) \tag{12}$$

where g_{SL} is the standard gravitation at sea level, h is the height of the firing point above sea level, and R_e is mean radius of the world. Standard gravitation at sea level is g_{SL}= *9.80662* m/s^2. With the derivation of Eq. (12)

$$\frac{dg}{dx} = -2\frac{g_{SL}}{R_e}\tan\varphi \tag{13}$$

is obtained. By applying Newton's Second Law in the x direction

$$-m\frac{d\,v_x}{dt} = D\cos\varphi \tag{14}$$

and for the change in horizontal velocity by dx the following relation is obtained

$$\frac{d\,v_x}{dx} = -\frac{(D/m)\cos\varphi}{v_x} \tag{15}$$

where D is the aerodynamic drag force of the projectile. The third derivative of the trajectory function obtained with the combination of Eq. (9), Eq. (13), Eq. (15) and Eq. (11) is

$$f'''(x) = 2\frac{g_{SL}}{R_e}\frac{\sin\varphi}{v^2\cos^3\varphi} - 2\frac{g\,(D/m)}{v^4\cos^3\varphi} \tag{16}$$

The initial values of the derivatives of trajectory function equations (2), (8) and (16) at the firing point $x = 0$ are the following:

$$f'(0) = \tan\theta \tag{17a}$$

$$f''(0) = -\frac{g_0}{v_0^2\cos^2\theta} \tag{17b}$$

$$f'''(0) = 2\frac{g_{SL}}{R_e}\frac{\sin\theta}{v_0^2\cos^3\theta} - 2\frac{g_0\,(D_0/m)}{v_0^4\cos^3\theta} \tag{17c}$$

where v_0 is the initial velocity of the projectile, θ is the angle of fire, g_0 is the local gravitation at the firing location and D_0 is the drag of projectile at the initial velocity. By putting Eqs.(17a), (17b) and (17c) into Eq. (1) given for the Maclaurin Series

$$y = \tan\theta\cdot x - \frac{g_0}{2v_0^2\cos^2\theta}\cdot x^2 + \frac{g_0}{3v_0^2\cos^3\theta}\left[\frac{g_{SL}}{g_0}\frac{1}{R_e}\sin\theta - \frac{(D_0/m)}{v_0^2}\right]\cdot x^3 + \cdots \tag{18}$$

is obtained. In the parentheses, the first term is very small in comparison with the second . This makes the cancellation of the first term possible. The rearrangement of Eq. (18) according to the third power of x gives

$$y \cong \tan\theta\cdot x - \frac{g_0 x^2}{2v_0^2\cos^2\theta}\cdot[1 + K\cdot x] \tag{19}$$

where

$$K = \frac{2}{3}\frac{(D_0/m)}{v_0^2\cos\theta} \tag{20}$$

If we remember the classical definition of the aerodynamic drag

$$D_0 = C_D\frac{1}{2}\rho_0\,v_0^2\,S \tag{21}$$

then

$$K = K_0\rho_0 \tag{22}$$

where ρ_0 is air density at the firing location, C_D and S are the drag coefficient and the characteristic area of the projectile, respectively, and K_0 is a constant calculated as

$$K_0 = \frac{C_D\,S}{3m\cos\theta} \tag{23}$$

Equation (19) becomes, finally,

$$y \cong \tan\theta \cdot x - \frac{g_0 x^2}{2v_0^2\cos^2\theta} \cdot [1 + K_0\rho_0 x] \qquad (24)$$

By using the condition at the falling point of the projectile

$$x = X\ , \qquad y = 0 \qquad (25)$$

in Eq. (24), one obtains

$$K_0 = \frac{1}{\rho_0 X}\left(\frac{v_0^2 \sin 2\theta}{g_0 X} - 1\right) \qquad (26)$$

According to Eq. (26), during the firing test of any projectile, if the initial velocity, firing angle and the distance between the firing point and falling point of the projectile (range) are measured, with the use of air density and gravitation at the height of firing point above sea level, K_0 can be calculated easily. The gravitation at the firing point from Eq. (12) can be obtained as

$$g_0 \cong g_{SL}\left(1 - 2\frac{h}{R_e}\right) \qquad (27)$$

The air density depending on altitude h can be written as

$$\rho_0 \cong \rho_{SL}\left(1 - \frac{\lambda\, h}{T_0}\right)^{\frac{g}{\lambda R}} \qquad (28)$$

where λ is the temperature gradient in the atmosphere, R is the gas constant of the air,and T_0 and ρ_{SL} are the standard air temperature and density at sea level, respectively.

The external ballistic parameters of a projectile cover velocity v, flight duration t, maximum height of the trajectory y_{max},the place where y_{max}occurs x_s, the angle of fall of the projectile ϕ and range x. These parameters were derived by means of the basic formulations explained above. The results of these calculations are given below.

The duration of flight t and velocity v at any point on the trajectory are

$$t = \frac{2}{9v_0\cos\theta}\frac{1}{\rho_0 \mathrm{K}_0}\left[(1 + 3\rho_0\mathrm{K}_0 x)^{3/2} - 1\right] \qquad (29)$$

$$v = v_0\frac{\cos\theta}{\cos\phi}\,(1 + 3\rho_0\mathrm{K}_0 x)^{-1/2} \qquad (30)$$

Maximum height of the trajectory, y_{max}, and the x-coordinate of this location, x_s, is given by

$$y_{\max} = x_s \tan\theta\frac{1 + 2\rho_0\mathrm{K}_0 x_s}{2 + 3K_0 x_s} \qquad (31)$$

$$x_s = \frac{1}{3\rho_0\mathrm{K}_0}\left[(\ 1 + 3\rho_0\mathrm{K}_0\mathrm{X}\,(1 + \mathrm{K}_0\rho_0\mathrm{X})\ )^{1/2} - 1\right] \qquad (32)$$

and

$$\tan\phi = \tan\theta - \frac{g_0 X}{2v_0^2\cos^2\theta}\,(2 + 3\rho_0\mathrm{K}_0 X) \qquad (33)$$

is obtained (Molitz and Strobel, 1963). Here, ϕ is the angle of fall of the projectile.

In order to obtain more precise trajectory parameters, instead of one firing, K_0 can be calculated by a least square fit using different firing angles for a given charge number. In this study, K_0 is obtained by a fourth order least square fit, e.g., for a given charge number and n different firing angles resulting in n different K_0 values, a fit for K_0 can be obtained in the form ($0 < \theta_c < \frac{\pi}{2} \quad x = \mathrm{X}_i$):

$$\begin{array}{llll} \theta_1 & x_1 & \rightarrow & \mathrm{K}_{01} \\ \theta_2 & x_2 & \rightarrow & \mathrm{K}_{02} \quad \text{then } \mathrm{K}_0 = a_0 + a_1\theta + \\ \theta_3 & x_3 & \rightarrow & \mathrm{K}_{03} \quad a_2\theta^2 + a_3\theta^3 + a_4\theta^4 \\ : & : & : & \quad \text{where } a_0 ... a_4 \text{ are the least} \\ \theta_n & x_n & \rightarrow & \mathrm{K}_{0n} \quad \text{square coefficients} \end{array}$$

Results and Discussion

An 81 mm MKE MOD 214 projectile was very well studied experimentally at Karapınar Firing Range. MOD 214 is a high explosive mortar projectile fired from an 81 mm UT1 smooth bore mortar. This mortar is used with a 6 charge system. The projectile has a maximum range of 5825 m in standard atmospheric conditions with the charge 6.

The calculation of parameter K_{0i} from Eq.(28) by means of experimental results given in Akçay (1986) yields the following:

Charge

1	$K_0 = 0.01036080 - 3.98774 \cdot 10^{-5} \cdot \theta + 5.77946 \cdot 10^{-8} \cdot \theta^2 - 3.7208 \cdot 10^{-11} \cdot \theta^3 + 9.02138 \cdot 10^{-15}\theta^4$
2	$K_0 = 0.00350224 - 1.36667 \cdot 10^{-5} \cdot \theta + 2.09674 \cdot 10^{-8} \cdot \theta^2 - 1.42654 \cdot 10^{-11} \cdot \theta^3 + 3.68442 \cdot 10^{-15}\theta^4$
3	$K_0 = 0.00371785 - 1.50379 \cdot 10^{-5} \cdot \theta + 2.32350 \cdot 10^{-8} \cdot \theta^2 - 1.59441 \cdot 10^{-11} \cdot \theta^3 + 4.1033 \cdot 10^{-15}\theta^4$
4	$K_0 = 0.00337945 - 1.31117 \cdot 10^{-5} \cdot \theta + 1.95794 \cdot 10^{-8} \cdot \theta^2 - 1.29790 \cdot 10^{-11} \cdot \theta^3 + 3.25309 \cdot 10^{-15}\theta^4$
5	$K_0 = 0.00301452 - 1.18366 \cdot 10^{-5} \cdot \theta + 1.79710 \cdot 10^{-8} \cdot \theta^2 - 1.21042 \cdot 10^{-11} \cdot \theta^3 + 3.08271 \cdot 10^{-15}\theta^4$
6	$K_0 = 0.00361498 - 1.43986 \cdot 10^{-5} \cdot \theta + 2.21129 \cdot 10^{-8} \cdot \theta^2 - 1.50543 \cdot 10^{-11} \cdot \theta^3 + 3.86986 \cdot 10^{-15}\theta^4$

Just as an example, the set of trajectories for charge 6 is obtained for standard atmospheric conditions with Eq. (24) and shown in Figure 2. The maximum range reaches 5825 m for an elevation of 800 mils.

Table 1 shows the differences among the flight parameters calculated with the modified point mass trajectory model (Lieske and Reiter, 1966) and with the method given in this study. The given method overpredicts the maximum range by 0.3%, the maximum height of trajectories by 2% and the flight duration by 0.3% with respect to the modified point mass trajectory model calculations. The experimental results obtained at the Karapınar firing range, located 1000 m above sea level, are also compared with the results calculated with the given method. The proposed method overpredicts the maximum range by 1% and underpredicts the flight duration by 1.8% compared with the experimental data.

Within the scope of the Fırtına Self Propelled Howitzer program, many 155 mm projectiles such as standard type conventional M107 projectiles and submunition type M864 projectiles were fired at the Karapınar Firing Range in order to predict their aerodynamics and flight mechanics parameters. During these tests, trajectory radar was used to follow the projectile's motion. Trajectory radar has the capability of recording the x, y and z coordinates, velocity, acceleration and duration of flight of the projectile at any point of the trajectory. These sets of trajectory data obtained for the 155 mm M107 projectile and one set of trajectory data for the 155 mm M864 projectile were used to compare the calculated results obtained by the method given in this study. The calculated and measured trajectories of M107 and M864 projectiles for the same firing conditions are given in Figure 3.

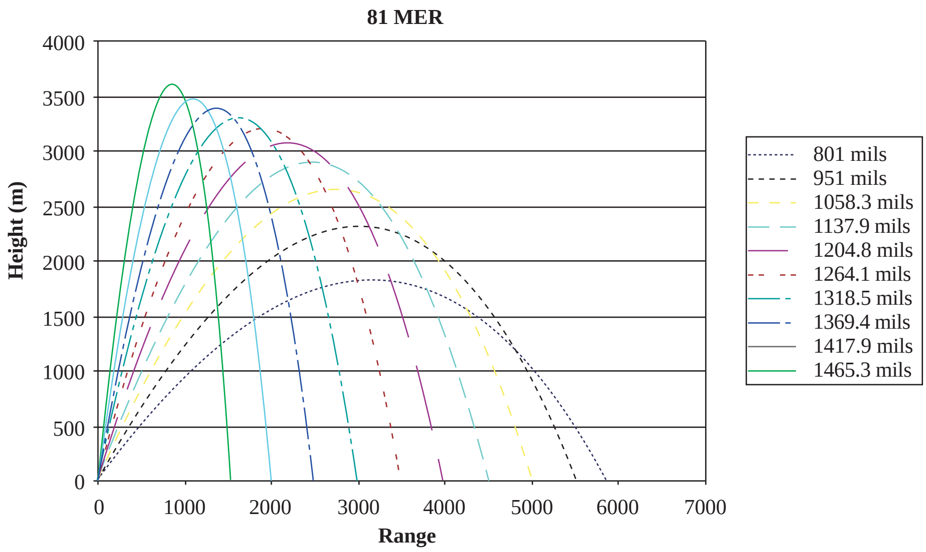

Figure 2. The set of trajectories of 81 mm MKEK MOD 214 mortar projectiles for charge 6 ($v_0 = 330$m/s).

Table 1. Comparison of the ballistic parameters calculated with the method given in this study and the modified point mass trajectory method (MMTM) and live firing tests performed at the Karapınar Firing Range.

Charge	θ (mils)	h (m)	Range (m)	$y_{\max}$ (m)	x_s (m)	Flight duration (s)	Ref.
		0	500	674	255	23.4	MMTM
		0	506	696	264	23.8	Method
1	1400						
		1000	516	710	268	24.36	Method
		1000	547			24.52	Exp.
		0	900	1255	467	32	MMTM
		0	906	1297	480	23.3	Method
2	1400						
		1000	929	1339	490	33.51	Method
		1000	954			33.73	Exp.
		0	2900	1570	1543	35.8	MMTM
		0	2908	1581	1550	35.6	Method
3	1100	1000					
			2997	1605	1650	36	Method
		1000	3036			36.9	Exp.
		0	2460	2378	1304	44.1	MMTM
		0	2464	2426	1325	43.9	Method
4	1300					45	
		1000	2557	2626	1475	46.95	Method
		1000	2480				Exp.
		0	2910	2883	1556	48.5	MMTM
		0	2928	2947	1600	48.2	Method
5	1300						
		1000	3055	3277	1800	49.72	Method
		1000	2942			51	Exp.
		0	3160	3219	1707	51.3	MMTM
		0	3160	3268	1725	50.6	Method
6	1300						
		1000	3318	3745	2050	51.88	Method
		1000	3287			52.75	Exp.

The errors obtained in maximum ranges calculated for M107 projectiles for 3 different angles of fire lie within the limits $^{+0.45}_{-0.126}$% of experimental values. The errors for maximum height of the trajectories are within the limits $^{+1.45}_{+5}$% compared with the experimental data. The proposed method underestimates the maximum range of M864 by 0.039% and overestimates the maximum height of the trajectory of M864 by 0.19% .

The evaluation shows that the precision of the calculation increases with the increasing initial velocity of the projectiles. The trajectory of the M864 projectile is predicted precisely by the current method. The discrepancies among the results obtained with this method, the measurements and the modified point mass trajectory model are basically due to the truncation error that occurred in the Maclaurin Series and the atmospheric conditions which are not completely simulated in the proposed method. Only change in gravity, air density and the height of the firing post can be considered.

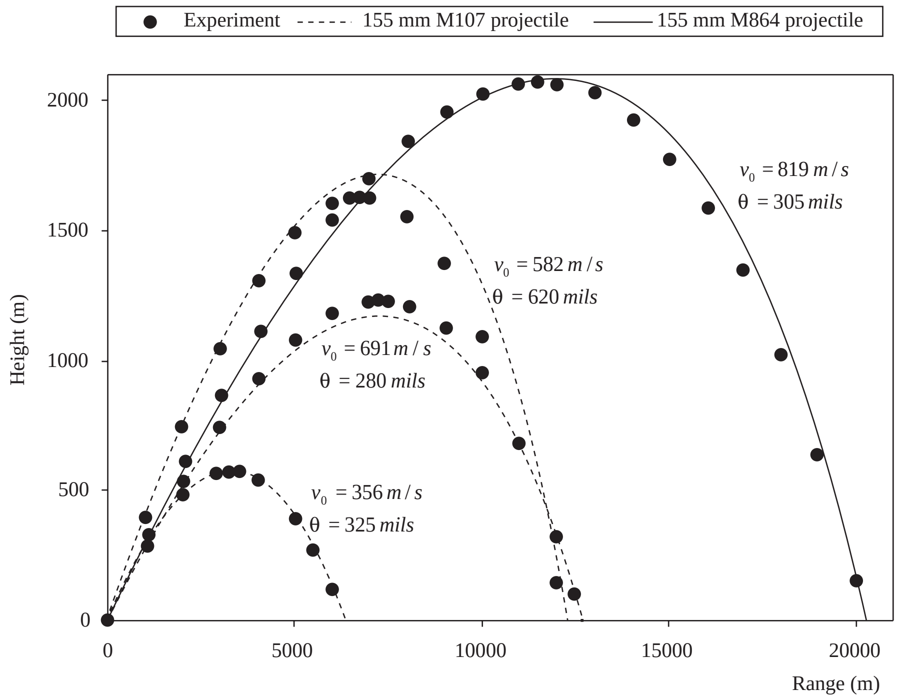

Figure 3. The predicted and measured trajectories of 155 mm M107 and M864 projectiles. Firings and trajectory radar measurements were carried out at Karapınar, 1000 m above sea level.

With a given projectile if we have limited firing range experimental data for 4 or 5 different angles of elevation we can theoretically obtain an infinite number of trajectories for $0 < \theta_c < \frac{\pi}{2}$.This method is a very simple, very fast and very cheap way of obtaining acomplete set of ballistic parameters without using an expensive set-up including trajectory radars,a large experimental team, ammunition and a gun.

This method is also very useful for predicting the initial firing conditions for a given target (Akçay, 1991). Another application of the method may be to find out if it is possible to hit a given target behind an obstacle like a hill or mountain. In this case, a 3- dimensional digital map is advised to be used with this method.

Nomenclature

a	constant parameter for K
K	parameter for a given projectile
R_e	radius of the sphere, approximately 6,356,766 m depending on the location
ρ	air density
t	duration of flight
θ	angle of fire
ν	muzzle velocity of a projectile at any point of a trajectory
v_0	initial muzzle velocity of a projectile
x, y	Cartesian coordinates
X	maximum range of a projectile
x_s	x location where $y_{\max}$ occurred
$y_{\max}$	maximum height of a projectile on its trajectory
mil	unit of angular measurement used in external ballistics. (6400 mils = 360°)

References

Akçay, M., "Calculation of Optimum External Ballistic Parameters for Firing on a Target", İTÜ Dergisi (in Turkish), 49, 37-43, 1991.

Akçay, M., "Theoretical and Experimental Studies of a Type of Projectile", İTÜ Institute for Engineering Sciences, PhD Thesis, İstanbul, 1986.

Akçay, M. and Erim, M.Z., "A Study on External Ballistic Characteristics of a Projectile", Bulletin of the Technical University of İstanbul, 45, 515-526, 1992.

Kaplan, W., Advanced Calculus, Addison Wesley Publishing Company, London, 355-356, 1963.

Lieske, R.F. and Amoruso, M. J., "Mid-Course Guidance for Modified Point Mass Guided Munitions Trajectory Model", BRL Report No: 71, November 2003.

Lieske, R.F. and Reiter, M.L.R., "Equation of Motion for a Modified Point Mass Trajectory", BRL Report No:1314, March, 1966.

Molitz, H. and Strobel, R., "Aussere Ballistik", Springer Verlag, Göttingen, 272-273, 1963.