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Buckling of Slender Prismatic Columns with a Single Edge Crack under Concentric Vertical Loads

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Abstract

The investigation of the stability behavior of slender columns with cracks is an important problem and finds applications in structural, mechanical and aerospace engineering. This study investigates the buckling of slender prismatic columns with a single nonpropagating edge crack subjected to concentrated vertical loads. The transfer matrix method and fundamental solutions of intact columns (columns without any cracks) are combined for determining the buckling loads of cracked columns. The cracked section is modeled by a massless rotational spring whose flexibility depends on the local flexibility induced by the crack. Eigenvalue equations are obtained explicitly for columns with various end conditions, from second order determinants. Numerical examples show that the effects of a crack on the buckling load of a column depend on the depth and the location of the crack. As expected, buckling load decreases conspicuously as the crack depth increases. For a constant crack depth, a crack located in the section of the maximum bending moment causes the largest decrease in the buckling load. On the other hand, if the crack is located just in the inflexion point at the corresponding intact column, it has no effect on the buckling load. The study showed that the transfer matrix method is a simple and efficient method with which to analyze cracked columns.

Key words: Buckling, Stability, Slender prismatic columns, Crack.

Introduction

Stability represents one of the main problems in solid mechanics, and must be controlled to ensure the safety of structures against collapse. It has a crucial importance, especially for structural, aerospace, mechanical, nuclear, offshore and ocean engineering (Bažant and Cedolin, 1991; Bažant, 2000).

In classical stability analysis, an elastic column is said to be stable if for any arbitrarily small displacement from its equilibrium position the column either returns to its original undisturbed position or acquires an adjoined stable position when left to itself (Aristizábal-Ochoa, 2004). Buckling is one of the fundamental forms of instability of column structures. Buckling of a column is defined as the change of its equilibrium state from one configuration to another at a critical compressive load. The concept of the critical load of an elastic column was introduced by Euler in 1744. The solutions for the elastic buckling analysis of columns under various loading, restraint and boundary conditions are well documented in the literature (Timoshenko and Gere, 1961; Handbook of Structural Stability, 1971).

Columns and compression members may contain various imperfections. For example, columns may be subjected to unintended small lateral loads, they may be initially curved rather than perfectly straight, the axial load may be slightly eccentric, or disturbing moments and shear forces may be applied at column ends. Unlike beams subjected to transverse loads and small axial forces, columns are quite sensitive to imperfections (Bažant and Cedolin, 1991). Imperfections have been recognized for a long time and their effects on structural stability have been well investigated. Columns and other structural elements may also have real damage such as cracks. The cracks may develop from flaws due to applied cyclic loads, mechanical vibrations, aerodynamic loads etc. (Kishen and Kumar, 2004). It is obvious that cracks lower structural integrity and should be certainly taken into account in the stability, safety and vibration analyses of structures.

The study of the stability behavior of cracked columns is an important problem and finds applications in structural, mechanical and aerospace engineering. Research performed to date on the stability of cracked columns lags far behind that performed on uncracked ones. Here, some previous studies directly related to the present study are briefly reviewed.

Liebowitz *et al.* (1967) carried out experimental studies on the axial load carrying capacity of notched and unnotched columns. For notched columns, Liebowitz and Claus (1968) proposed a theoretical failure criterion based on the stress intensity factor and fracture toughness. Okamura *et al.* (1969) identified the compliance of a cracked column to a bending moment to study the load carrying capacity and fracture load of a slender column with a single crack. The buckling of cracked columns subjected to follower and vertical loads was investigated by Anifantis and Dimarogonas (1983). Nikpour (1990) studied the buckling of cracked composite columns. Li (2001) investigated the buckling of multi-step columns with an arbitrary number of cracks, taking into account the effects of shear deformations.

In this paper, using the transfer matrix method and fundamental solutions of intact columns (columns without any cracks), buckling analysis of slender prismatic columns of rectangular cross section, with a single nonpropagating edge crack, is performed. The transfer matrix method is an efficient and attractive tool for the solution of the eigenvalue problem for 1-dimensional structures with nonuniform mechanical properties. The cracked section is replaced with a massless rotational spring whose flexibility is a function of the crack depth and the height of the cross section of the column. Utilizing this procedure, the eigenvalue equations (buckling conditions) for the buckling of such columns with any 2 end conditions can be obtained from a system of 2 linear equations. For the common end conditions, namely fixed-free, pinned-pinned, fixed-pinned and fixed-fixed, eigenvalue equations are obtained explicitly. Taking 2 example columns, these equations are solved and their smallest roots, which are the buckling loads of the columns, are determined. Moreover, the effects of the crack depth and location are investigated and the results are given in figures.

Problem Formulation and Governing Equations

A slender prismatic column with a rectangular cross section and having a nonpropagating edge crack is shown in Figure 1(a). The mathematical model of the column is shown in Figure 1(b), in which after the local flexibility caused by the crack is considered, the cracked section is represented by a massless rotational spring with flexibility C. This quantity is a function of the crack depth and height of the cross section of the column and can be written as (Shifrin and Ruotolo, 1999)

$$C = 5.346\ hf(\xi) \tag{1}$$

where h is the height of the cross section of the column and $\xi = a/h$, where a is the depth of the crack, as seen in Figure 1(a). $f(\xi)$ is called the local flexibility function and is given by Shifrin and Ruotolo (1999)

$$\begin{aligned} f(\xi) &= 1.8624\xi^2 - 3.95\xi^3 + 16.375\xi^4 \\ &-37.226\xi^5 + 76.81\xi^6 - 126.9\xi^7 + 172\xi^8 \\ &-143.97\xi^9 + 66.56\xi^{10} \end{aligned} \tag{2}$$

It can be seen from Figure 1(b) that the column is divided into 2 segments, segment 1 ($0 \le x \le x_c$) and segment 2 ($x_c \le x \le L$), by the rotational spring.

The differential equation for buckling of segment 1 can be written as (Timoshenko and Gere, 1961)

$$\frac{d^4y_1}{dx^4} + k^2\frac{d^2y_1}{dx^2} = 0 \tag{3}$$

where $k^2 = P/EI$, and P and EI are the axial compressive force and the flexural rigidity, respectively. In this case, the relationships among the displacement, slope, bending moment and shear force are

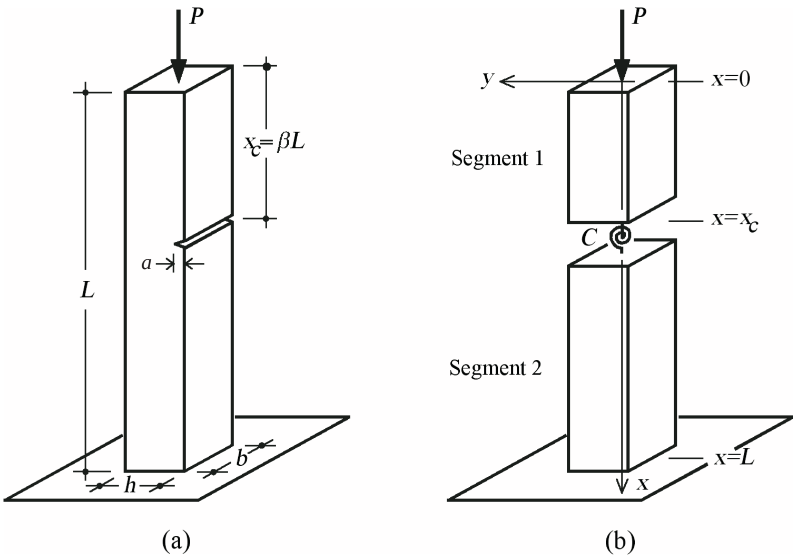

Figure 1. (a) A slender column with a nonpropagating edge crack, (b) Its mathematical model.

$$\left.\begin{aligned} \theta_1(x) &= \frac{dy_1}{dx} \\ M_1(x) &= -EI\frac{d^2y_1}{dx^2} \\ V_1(x) &= \frac{dM_1}{dx} - P\frac{dy_1}{dx} \end{aligned}\right\} \tag{4}$$

The general solution of Eq. (3) is given by

$$y_1(x) = A_1 + A_2x + A_3\sin(kx) + A_4\cos(kx) \tag{5}$$

Using Eqs. (4) and (5), the following relationship can be written:

$$\begin{Bmatrix} y_1(x) \\ \theta_1(x) \\ M_1(x) \\ V_1(x) \end{Bmatrix} = [B(x)] \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} \tag{6}$$

where

$$[B(x)] = \begin{bmatrix} 1 & x & \sin(kx) & \cos(kx) \\ 0 & 1 & k\cos(kx) & -k\sin(kx) \\ 0 & 0 & P\sin(kx) & P\cos(kx) \\ 0 & -P & 0 & 0 \end{bmatrix} \tag{7}$$

The relationship between the parameters written above at the 2 ends of segment 1 can be expressed as

$$\begin{Bmatrix} y_1(x_c) \\ \theta_1(x_c) \\ M_1(x_c) \\ V_1(x_c) \end{Bmatrix} = [T_1] \begin{Bmatrix} y_1(0) \\ \theta_1(0) \\ M_1(0) \\ V_1(0) \end{Bmatrix} \tag{8}$$

in which

$$[T_1] = [B(x_c)][B(0)]^{-1} \tag{9}$$

$[T_1]$ is called the transfer matrix for segment 1, because this matrix transfers the parameters at the upper end ($x = 0$) to those at the lower end ($x = x_c$) of segment 1.

There is continuity among the displacements, bending moments and shear forces, at the boundary of segment 1 and segment 2, but there is a discontinuity between slopes at this point, caused by the bending moment and rotation of the spring representing the cracked section (Figure 2),

$$\left.\begin{aligned} y_1(x_c) &= y_2(x_c) \\ y_1''(x_c) &= y_2''(x_c) \\ y_1'''(x_c) &= y_2'''(x_c) \end{aligned}\right\} \tag{10a}$$

$$\theta_2(x_c) - \theta_1(x_c) = y_2'(x_c) - y_1'(x_c)$$
$$= \Delta\theta(x_c) = C y_1''(x_c) = -C\frac{M_1(x_c)}{EI} \tag{10b}$$

Equation (10b) is written by imposing equilibrium between the transmitted bending moment and the rotation of the spring.

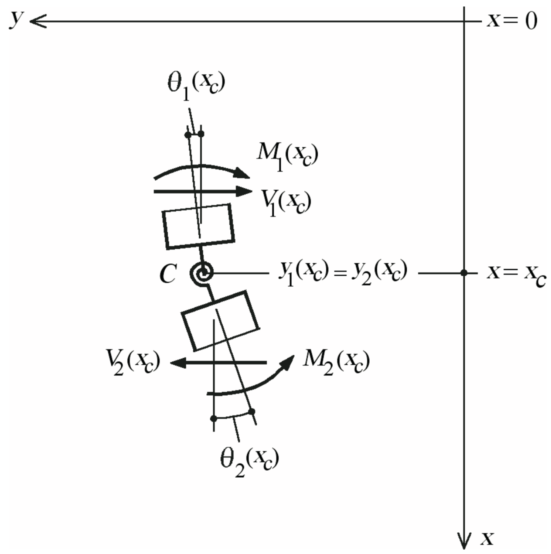

Figure 2. Continuity of displacements, bending moments and shear forces, discontinuity of slopes at the rotational spring.

Equations (10a) and (10b) can be written in matrix form as

$$\begin{Bmatrix} y_2(x_c) \\ \theta_2(x_c) \\ M_2(x_c) \\ V_2(x_c) \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{C}{EI} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} y_1(x_c) \\ \theta_1(x_c) \\ M_1(x_c) \\ V_1(x_c) \end{Bmatrix} \tag{11}$$

Substitution of Eq. (8) into Eq. (11) yields

$$\begin{Bmatrix} y_2(x_c) \\ \theta_2(x_c) \\ M_2(x_c) \\ V_2(x_c) \end{Bmatrix} = [T_{1C}] \begin{Bmatrix} y_1(0) \\ \theta_1(0) \\ M_1(0) \\ V_1(0) \end{Bmatrix} \tag{12}$$

in which

$$[T_{1C}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{C}{EI} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} [T_1] \tag{13}$$

The equation for segment 2 can be obtained by using Eqs. (12) and (8)

$$\begin{Bmatrix} y_2(L) \\ \theta_2(L) \\ M_2(L) \\ V_2(L) \end{Bmatrix} = [T_2] \begin{Bmatrix} y_2(x_c) \\ \theta_2(x_c) \\ M_2(x_c) \\ V_2(x_c) \end{Bmatrix}$$

$$= [T_2]\,[T_{1C}] \begin{Bmatrix} y_1(0) \\ \theta_1(0) \\ M_1(0) \\ V_1(0) \end{Bmatrix} = [T] \begin{Bmatrix} y_1(0) \\ \theta_1(0) \\ M_1(0) \\ V_1(0) \end{Bmatrix} \tag{14}$$

in which

$$[T] = [T_2]\,[T_{1C}] \tag{15}$$

The matrix $[T]$ has the following form:

$$[T] = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{bmatrix} \tag{16}$$

Eigenvalue Equations and Eigenvalues

The eigenvalue equations can be established by using Eq. (14) and the end conditions as follows:

(a) Fixed-free ended column: for this case Eq. (14) becomes

$$\begin{Bmatrix} 0 \\ 0 \\ M_2(L) \\ V_2(L) \end{Bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{bmatrix} \begin{Bmatrix} y_1(0) \\ \theta_1(0) \\ 0 \\ 0 \end{Bmatrix} \quad (17a)$$

Equation (17a) reduces to the following form:

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{Bmatrix} y_1(0) \\ \theta_1(0) \end{Bmatrix} \quad (17b)$$

For a nontrivial solution, by setting the determinant of the matrix in Eq. (17b) equal to zero, one obtains

$$T_{11}T_{22} - T_{12}T_{21} = 0 \quad (18)$$

It can be seen that the eigenvalue equation is obtained from a second order determinant.

Following the same procedure, the eigenvalue equations for other boundary conditions are obtained as

(b) Pinned-pinned column,

$$T_{12}T_{34} - T_{14}T_{32} = 0 \quad (19)$$

(c) Fixed-pinned column,

$$T_{12}T_{24} - T_{14}T_{22} = 0 \quad (20)$$

(d) Fixed-fixed column,

$$T_{13}T_{24} - T_{14}T_{23} = 0 \quad (21)$$

After determining the elements T_{ij} of the matrix $[T]$ and then using Eqs. (18), (19), (20) and (21), the eigenvalue equations are obtained in explicit form as

(a) Fixed-free ended column,

$$\cos(kL) - Ck\sin(\beta kL)\cos[(1-\beta)kL] = 0 \quad (22)$$

(b) Pinned-pinned column,

$$\sin(kL) - Ck\sin(\beta kL)\sin[(1-\beta)kL] = 0 \quad (23)$$

(c) Fixed-pinned column,

$$\begin{aligned}&[(kL)\cos(kL) - \sin(kL)] + Ck\sin(\beta kL)\\ &\{\sin[(1-\beta)kL] - (kL)\cos[(1-\beta)kL]\} = 0\end{aligned} \quad (24)$$

(d) Fixed-fixed column,

$$\begin{aligned}&4\sin(kL/2)[\sin(kL/2) - (kL/2)\cos(kL/2)] +\\ &Ck\{\sin(kL) - (kL)\cos(\beta kL)\cos[(1-\beta)kL]\} = 0\end{aligned} \quad (25)$$

In Eqs. (22) to (25) $\beta = x_c/L$ (Figure 1(a)) and $k^2 = P/EI$ as stated earlier. By using any of a number of root-finder algorithms, the roots (eigenvalues) of the above transcendental equations can be obtained.

It must be noted that, by setting $C = 0$ in the above equations, one obtains the buckling conditions, i.e. eigenvalue equations, of the corresponding intact columns.

Numerical Examples and Discussion

In the first part of this section, 2 specimen columns are considered to analyze the effects of cracks on the critical buckling loads of the columns.

Example 1. As a first example, a fixed-free ended column shown in Figure 3(a) is considered. For the column, the following data are taken: $h = b = 20$ cm, $L = 3$ m, $E = 2 \times 10^6$ N/cm^2, $a = 0.3\ h = 6$ cm and $x_c = 0.7\ L = 2.10$ m. Consequently ξ and β become $\xi = a/h = 0.3$ and $\beta = x_c/L = 0.7$, respectively. Using Eq. (2) $f(\xi) = f(0.30) = 0.14023$ is obtained and then Eq. (1) provides a $C = 0.149934$ m value. Substituting the values of L, β and C into Eq. (22), the following equation is obtained:

$$\cos(3k) - (0.149934k)\sin(2.1k)\cos(0.9k) = 0 \quad (26)$$

Solving this eigenvalue equation gives $k = 0.503861$. Remembering that $k^2 = P_{cr}/EI$, then the critical buckling load of this column is calculated as $P_{cr,(1)} = k^2 EI = 0.253876EI$. The Euler buckling load for the intact column is $P_E = \pi^2 EI/(2 \times 3)^2 = 0.274155EI$. It is calculated that $P_{cr,(1)}$ is 7.4% smaller than P_E, i.e. this crack, which has specified properties in the above, causes a reduction of 7.4% in the buckling load of the column.

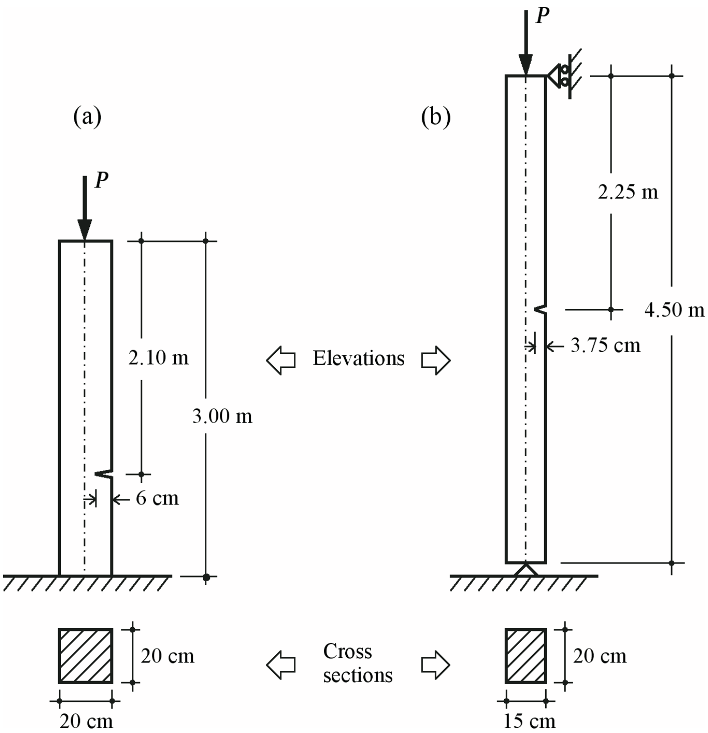

Figure 3. Columns considered as numerical examples; (a) Fixed-free ended column, (b) Pinned-pinned column.

If the depth of the crack is increased to $a = 0.45$ $h = 9$ cm from 0.3 $h = 6$ cm, then following the same steps, the critical buckling load is calculated as $P_{cr,(2)} = 0.22625EI$. This value is 10.88% and 17.47% smaller than $P_{cr,(1)}$ and P_E, respectively. It is seen that a greater crack depth causes more reduction in the critical buckling load, as expected.

If the location of the crack is shifted to $x_c = 0.95$ $L = 2.85$ m from $x_c = 0.7$ $L = 2.10$ m, then the buckling load is obtained as $P_{cr,(3)} = 0.24898EI$, which is 1.93% and 9.18% smaller than $P_{cr,(1)}$ and P_E, respectively. This result shows that, for a fixed-free ended column, a crack nearer to the fixed end causes a greater reduction in the critical buckling load, and is therefore more critical in the stability behavior of the column.

Example 2. As a second example, a pinned-pinned column depicted in Figure 3(b) is considered. For this column, the following data are taken: $h = 15$ cm, $b = 20$ cm, $L = 4.5$ m, $E = 2 \times 10^6$ N/cm^2, $a = 0.25$ $h = 3.75$ cm and $x_c = 0.5$ $L = 2.25$ m. Therefore, ξ and β become 0.25 and 0.5, respectively. Equation (1) gives $f(0.25) = 0.095438$ and then Eq. (1) yields a $C = 0.076532$ m value. Substitution of the values of L, β and C into Eq. (23) gives the following equation:

$$\sin(4.5k) - (0.076532k)\sin^2(2.25k) = 0 \qquad (27)$$

The first root of this eigenvalue equation is $k = 0.686459$. Therefore, the critical buckling load is $P_{cr,(1)} = k^2EI = 0.471226EI$. The Euler buckling load for the intact column is $P_E = \pi^2EI/(4.5)^2 = 0.487388EI$. $P_{cr,(1)}$ is 3.32% smaller than P_E, i.e. the crack causes a 3.32% reduction in the buckling load of the column.

If the depth of the crack is increased to $a = 0.5$ $h = 7.5$ cm from 0.25 $h = 3.75$ cm, then the critical buckling load becomes $P_{cr,(2)} = 0.411745EI$, which is 12.62% and 15.52% smaller than $P_{cr,(1)}$ and P_E, respectively. It is again seen that a deeper crack causes more reduction in the buckling load.

When the location of the crack is changed to $x_c = 0.85$ $L = 3.825$ m from $x_c = 0.5$ $L = 2.25$ m, then the buckling load is obtained as $P_{cr,(3)} = 0.483937EI$. $P_{cr,(3)}$ is 2.63% greater than $P_{cr,(1)}$ and 0.71% smaller than P_E, respectively. This result shows that, for a pinned-pinned column, a crack nearer to the mid-length of the column causes a greater reduction in the critical buckling load, and is therefore more critical in the stability behavior of the column.

In the second part of this section, in order to see more clearly the effects of the crack depth ($a = \xi h$) and the location ($x_c = \beta L$), 4 compression rods having fixed-free, pinned-pinned, fixed-pinned and fixed-fixed support conditions are considered. The rods have the same cross-sectional dimensions of $h = b = 0.03$ m, but different lengths of 0.65 m, 1.30 m, 1.85 m and 2.60 m, respectively. With these geometric properties, all rods buckle in the elastic range. For these rods, the first critical buckling load to the Euler buckling load ratio (P_{cr}/P_E) versus crack location parameter ($\beta = x_c/L$) curves, corresponding to the 0.15, 0.35 and 0.50 values of the crack depth parameter ($\xi = a/h$), are drawn and shown in Figures 4(a) to (d).

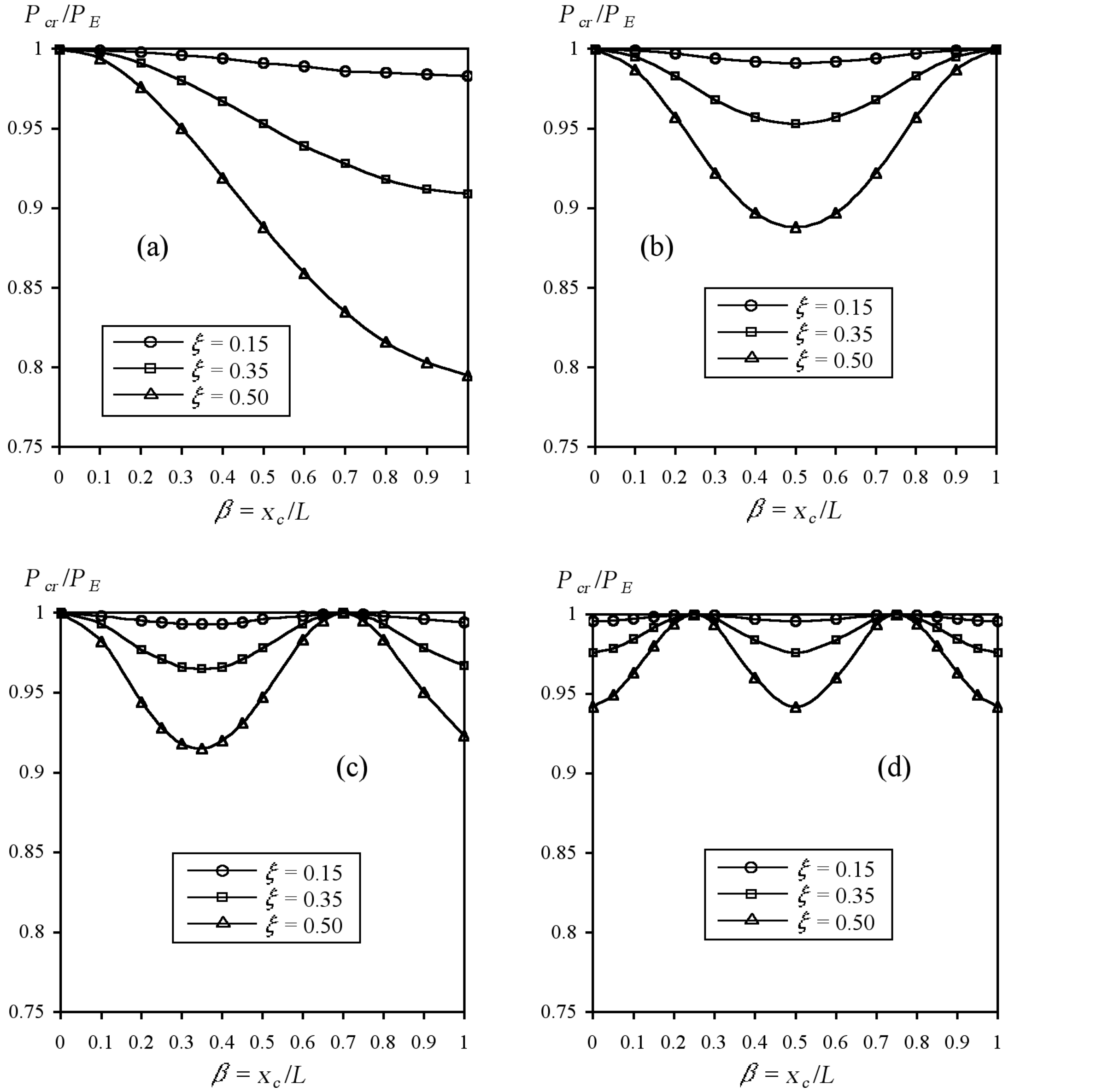

Figure 4. Variation of the first critical buckling load to the Euler buckling load ratio (P_{cr}/P_E) depending on the dimensionless crack depth (ξ) and the dimensionless crack location (β); (a) Fixed-free, (b) Pinned-pinned, (c) Fixed-pinned and (d) Fixed-fixed supported rods.

It is evident from the figures that for all rods, when the crack depth and thus crack depth parameter increases, the buckling load and thus the P_{cr}/P_E ratio decrease. This is an expected result. The largest decrease is in the fixed-free ended rod, with a decrease of 20.50% ($P_{cr}/P_E = 0.795$), and the smallest decrease occurs in the fixed-fixed rod, with a decrease of 5.84% ($P_{cr}/P_E = 0.9416$).

The crack location has different effects depending on the end conditions. For a constant crack depth, in a fixed-free ended rod, a crack at the fixed end causes the largest decrease in the buckling load, while in a pinned-pinned rod a crack located at mid-length has the largest effect. In a pinned-pinned rod, when the crack shifts towards any of the supports, its effect diminishes. For fixed-pinned and fixed-fixed rods a crack at $x_c = 0.35\ L$ and $x_c = 0.50\ L$, respectively, causes the largest reduction in the buckling load. As is well known from fracture mechanics and strength of materials, strain energy stored in an elastic body under a bending effect is directly related to the magnitude of the bending moments. Therefore, as the calculated results show, for all rod types, a crack located in the section of maximum bending moments of the corresponding intact rods causes maximum energy losses and consequently the largest decrease in the buckling loads. Naturally, a crack located in the inflexion points (moment zero points) of the corresponding intact rods has no effect on the critical buckling load.

Conclusions

Buckling analysis of slender prismatic columns with a single nonpropagating open edge crack subjected to axial loads has been presented and the following conclusions are drawn:

1) The transfer matrix method is a simple and efficient method with which to analyze the buckling of cracked columns with various support conditions. The eigenvalue equations of cracked columns can be easily established from a system of 2 linear equations.

2) Using the derived eigenvalue equations (buckling conditions) in explicit form, one can readily obtain the buckling loads of slender prismatic columns with a single nonpropagating open edge crack or notch.

3) The effects of a crack on the buckling load of a column depend on the depth and the location of the crack.

4) In columns under axial compression, the effect of a crack is to decrease the buckling load. As expected, the load carrying capacity decreases as the crack depth increases. On the other hand, the effect of crack location depends on the end conditions of the columns. Generally, a crack located in the section of maximum bending moments causes the largest decrease in the buckling loads. If a crack is located just in the inflexion points of the corresponding intact columns, it has no effect on the buckling load.

5) The analysis of the present study is mainly for columns having only 1 crack. However, extension to columns with multiple cracks can be carried out trivially by modeling each cracked section with a rotational spring. Other possible extensions of the present analysis are the inclusion of elastic support conditions and non-prismatic columns, as well as the propagation of cracks, which are left for future studies.

Nomenclature

A_i	constants
a	depth of the crack
b	width of the cross section
C	flexibility of rotational spring
E	modulus of elasticity
EI	flexural rigidity
$f(\xi)$	local flexibility function
h	height of the cross section
I	second moment of area of the cross section
L	length of the column
M	bending moment
P	concentrated axial compressive force
P_{cr}	critical buckling load
P_E	Euler buckling load
$[T_1]$	transfer matrix
V	shear force
x	axial coordinate
x_c	axial coordinate of the cracked section
y	displacement
β	dimensionless crack location parameter
θ	slope
ξ	dimensionless depth of the crack

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