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Probabilistic Transformation Method in Reliability Analysis

Seifedine KADRY, Alaa CHATEAUNEUF
LaMI – UBP & IFMA, University Campus of Clermont-Fd, BP 265, 63175 Aubière cedex-FRANCE
e-mail: skadry@gmail.com
Khaled EL-TAWIL
Lebanese University, Faculty of Engineering, University Campus of Hadath-LEBANON

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Abstract

A technique is presented in order to evaluate the probability of failure in analytical form instead of approximation methods like FORM/SORM and no series expansion is involved in this expression. This technique is based on the Finite Element Method to obtain the expression of the response of stochastic systems, and the transformation of random variables to obtain the probability density function of the response. The transformation technique evaluates the probability density function (*pdf*) of the system output by multiplying the input *pdf* by the Jacobian of the inverse mechanical function. This approach has the advantage of giving directly the whole density function of the response in closed form, which is very helpful for reliability analysis.

Key words: Finite element method, Probabilistic methods, Reliability analysis, Sensitivity, Transformation method.

Introduction

Mechanical modeling of physical systems is often complicated by the presence of uncertainties. The implications of these uncertainties are particularly important in the assessment of reliability analysis. Even though significant effort may be needed to incorporate uncertainties into the modeling process, this could potentially result in the provision of useful information that can help in decision-making.

For several decades, the theory of probability has been used in mechanics to model the random structural properties (materials, geometry, boundary conditions etc.) and phenomena (turbulence, seismic wave, loads etc.) acting on mechanical systems. The probabilistic approach takes into account the uncertainties in the model data in order to improve the robustness of the forecasts and optimized configuration. The structural reliability has become a discipline of international interest, addressing issues such as performance-based cost-safety balancing (Procaccia and Morilhat, 1996).

In this work, a proposed technique is presented in order to evaluate the stochastic mechanical response, e.g., the probabilistic and statistical characteristics of the response of stochastic mechanical system (a mechanical system with an uncertain parameter like Young's modulus). The most important probabilistic characteristic of a stochastic system is the Probability Density Function (*pdf*) because on one hand the evaluation of other characteristics (mean, standard deviation etc.) is based on it, and on the other hand, it is very helpful for optimization of structure, reliability and fatigue analysis. Unfortunately, we do not have in the literature a deterministic method that gives us the probability density function of a stochastic mechanical system. Our method is based on the combination of the probabilistic transformation methods (PTM) for a random variable (e.g., Young's modulus or load) and the deterministic finite element method (FEM). The probabilistic transformation technique evaluates the Probability Density Function (*pdf*) in closed form of the system output by multiplying the input *pdf* by the Jacobian of the inverse mechanical function.

Reliability analysis

The problem of reliability analysis of stochastic mechanical systems is of central importance in the safety assessment of structures. In a stochastic system, a large number of random variables influence the performance of the system, e.g., Young's modulus and external loads. The performance of the system is evaluated by a *best-estimate* code. Consider a performance criterion Y of the system depending on the input variables $X_1, X_2, \ldots, X_n$ the function $Y = g(X_1, X_2, \ldots, X_n)$ is a random variable to be determined.

In order to get information about the uncertainty of Y, a number of *FE* runs have to be performed. For each of these runs, all identified uncertain parameters vary simultaneously.

According to the analysis of the FE results, the uncertainty in the response can be evaluated either in the form of an uncertainty range or in the form of a probability density function (*pdf*).

Uncertainty range

A 2-sided confidence interval $[m, M]$ of a response Y for a fractile α and a confidence level β is given by

$$P\left\{P(m \leq Y \leq M) \geq \alpha\right\} \geq \beta,$$

Such a relationship means that one can affirm, with at the most (1-β), percent of possible error that at least α percent of the response Y lie between the values m and M (Glaeser, 2000). To calculate the limits m and M, the technique usually used is a method of random simulation combined with the formula of Wilks (1941).

The advantage of this technique is that the number of code runs needed is independent of the number of uncertain parameters. However, for reliability evaluation, this method is not very useful because it is impossible to interpret the 2 levels of probability (α and β) in terms of reliability value for the system.

Probability density function

The uncertainty evaluation in the form of a *pdf* gives richer information than a confidence interval. Once the *pdf* of the system response is determined, the reliability can be directly obtained for a given failure criterion. However, the determination of this distribution can be expensive in computing time. The following paragraphs describe the various methods available for this evaluation.

Monte-carlo simulation The Monte-Carlo method (Rubenstein, 1981; Devictor, 1996) is used to build the *pdf*, as well as to assess the reliability of components or structures or to evaluate the sensitivity of parameters. Monte-Carlo simulation consists of drawing samples of the basic variables according to their probabilistic characteristics and then feeding them into the performance function. In this way, a sample of response $\{Y_j,\ j = 1,..,N\}$ is obtained.

The main advantage of the Monte-Carlo method is that it is valid for static, dynamic, and probabilistic models with continuous or discrete variables. The main drawback is that it often requires a large number of calculations and can be prohibitive when each calculation involves a long and onerous computer time.

Response surface method To avoid the problem of long computer time in the Monte-Carlo method, it can be interesting to build an approximate mathematical model called response surface.

Experiments are conducted with design variables $X_1, X_2, \ldots, X_n$ a sufficient number of times to define the response surface to the level of accuracy desired. Each experiment can be represented by a point with coordinates $x_{1j}, x_{2j}, \ldots, x_{nj}$ in an n-dimensional space. At each point, a value of y_i is calculated. The basic response procedure is to approximate by a simple mathematical model such as an n^{th} order polynomial with undetermined coefficients.

When a response surface has been determined, the system reliability can be easily assessed with Monte-Carlo simulation using the approximate mathematical model, yet this response surface must be qualified. The practical problems encountered by the use of the response surface method are in the analysis of strongly nonlinear phenomena where it is not obvious to find a family of adequate functions and in the analysis of discontinuous phenomena. El-Tawil et al. (1990, 1991) described the adaptive nature of these methods.

FORM/SORM

We present now specific usable methods for a direct evaluation of reliability without the need to define the *pdf* of the system performance.

The performance function of a stochastic system, according to a specified mission, is given by

M = performance limit – response indicator = $g(X_1, X_2, \dots, X_n)$ in which the X_i *(i=1, ... ,n)* are the n basic random variables (input parameters) with $g(.)$ being now the functional relationship between the random variables and the failure of the system. The performance function can be defined such that the limit state or failure surface is given by $M = 0$. The failure event is defined as the space where $M < 0$, and the success event is defined as the space where $M > 0$. Therefore, a probability of failure can be evaluated by the following integral:

$$P_f = \int\int ... \int f_X(x_1, ..., x_n) dx_1 ... dx_n, \qquad (1)$$

where f_X is the joint density function of $x_1 x_2, \dots, x_n$, and the integration is performed over the region where $M < 0$. As each of the basic random variables has a unique distribution that interacts with the others, the integral (1) cannot be easily evaluated. Two types of methods can be used to estimate the probability of failure: Monte-Carlo simulation and the approximate methods (FORM/SORM).

Direct Monte Carlo: simulation techniques can be used to estimate the probability of failure defined in Eq. (1). Monte-Carlo simulation (Figure 1) consists of drawing samples of the basic variables according to their probabilistic characteristics and then feeding them into the performance function. An estimate of the probability of failure P_f (Sundararajan, 1995) can be found by

$$\overline{P_f} = \frac{N_f}{N},$$

where N_f is the number of simulation samples in which $g(.) < 0$, and N is the total number of simulation samples. As N approaches infinity, P_f approaches the true probability of failure.

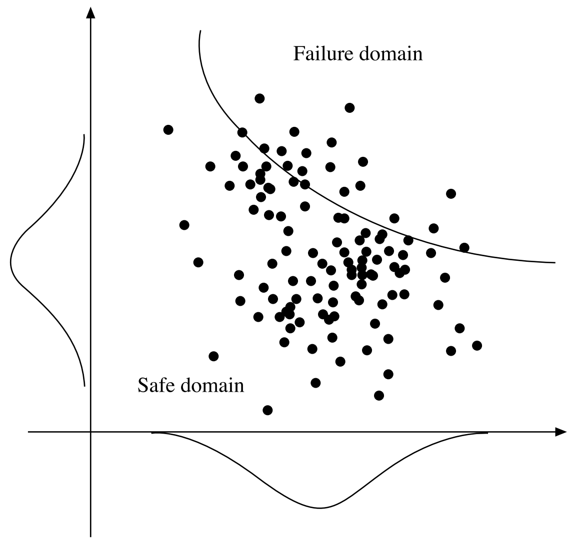


Figure 1. Reliability assessment by Monte-Carlo simulation.

The first and second order reliability methods (FORM/SORM) consist of 3 steps (Figure 2):

a. The transformation of the space of the basic random variables $X_1, X_2, \dots, X_n$ into a space of standard normal variables.

b. The search (in this transformed space) of the point of minimum distance from the origin on the limit state surface (this point is called the design point).

c. an approximation of the failure surface near the design point:

FORM (First Order Reliability Method) consists of approaching the surface of failure by a hyper plane tangent to the failure surface at the design point (Madsen and Ditlevsen, 1986). Then an estimate of the failure probability is obtained by

$$P_f = \Phi(-\beta),$$

where Φ is the cumulative Gaussian distribution of the standard normal law and β is the reliability index according to Hasofer and Lind. The precision of this approximation depends on the non-linearity of the failure surface.

If the linear approximation is not satisfactory, more precise evaluations can be obtained from approximations to higher orders of the failure surface at the design point. The approximation by a quadratic surface at the design point is called SORM (Second Order Reliability Method) (Melchers, 1999). The corresponding formula uses the knowledge of the (q-1) principal curves κ_i of the failure surface at the design point:

$$P_f \approx \Phi(-\beta_{HL}) \prod_{i=1}^{N-1} \frac{1}{\sqrt{(1+\beta_{Hl}\kappa_i)}},$$

This result is known as asymptotically exact in the sense that the approximation of the failure probability obtained is better for large reliability indexes. The computing time is influenced by the calculation of the matrix of the second-order derivatives.

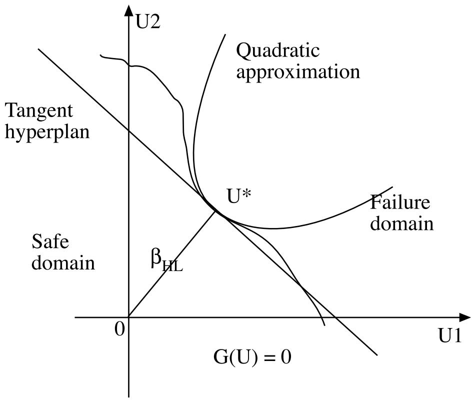


Figure 2. Reliability assessment with FORM/SORM methods.

The main drawbacks of FORM and SORM are that the mapping of the failure function onto a standardized set and the subsequent minimization of the function involve a significant computational effort for nonlinear black box numerical models. In addition, simultaneous evaluation of probabilities corresponding to multiple failure criteria would involve significant additional effort. Furthermore, these methods impose some conditions on the joint distributions of the random parameters that limit their applicability.

Proposed Technique FEM-PTM

The theory of the Probabilistic Transformation Method (PTM) is based on the following theorem (Soong, 1973; Hogg, 1989; Papoulis, 2002):

Theorem: Suppose that X is a continuous random variable with *pdf* $f_X(x)$ and $A \subset \Re$ is the one-dimensional space where $f_X(x) > 0$ (differentiable and monotonic). Consider the random variable $Y = u(X)$, where $y = u(x)$ defines a one-to-one transformation that maps the set A onto a set $B \subset \Re$ so that the equation $y = u(x)$ can be uniquely solved for x in terms of y, say $x = u^{-1}(y)$. Then, the *pdf* of Y is

$$f_Y(y) = f_X\left[u^{-1}(y)\right] \cdot |J| \ , \qquad y \in B$$

where $J = \frac{dx}{dy} = \frac{du^{-1}(y)}{dy}$ is the transformation Jacobian, which must be continuous for all points $y \in B$.

The proposed technique is a combination of the deterministic finite element method and the random variable transformation technique. In this technique, the differential equation is solved firstly using the deterministic theory of finite element, which yields accurate nodal solutions. These solutions are then used to obtain the approximate *pdf* using the random variable transformation between the input random variables and the output variable. The algorithm of this method is shown in Figure 3.

Algorithm of FEM-PTM:

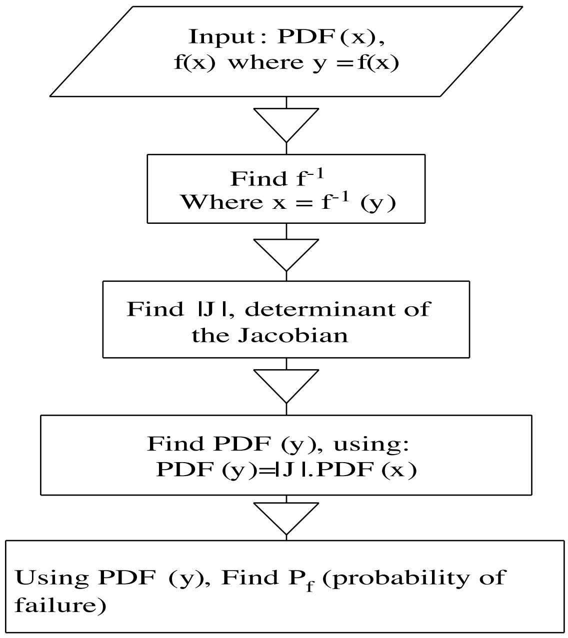


Figure 3. Algorithm of the proposed method.

Advantages and Disadvantages of FEM-PTM

The advantage of the FEM-PTM technique in the context of reliability analysis is clear. It gives the *pdf*, which is the most important characteristic in the probabilistic analysis of the response in closed form, contrary to other numerical methods like perturbation and Stochastic Finite Element Method (SFEM), when giving only the first and second moments of the response under some conditions (Sudret and Der Kiureghian, 2000).

However, the main disadvantage of this technique is that the transformation function must be one-to-one (e.g., bijective). Kadry et al. (2006) have proposed a technique to solve this limitation.

Applications

I) In the first application, we are going to analyze the reliability of a cantilever beam (Figure 4) with random parameters (Young's modulus E and distributed load W).

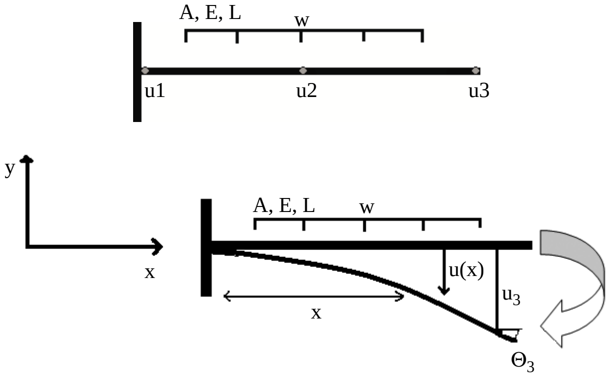


Figure 4. Cantilever beam.

FEM modeling of the beam with 2 elements

The deformation and the bending stresses are given by (Chateauneuf, 2005):

$$\varepsilon(x) = -u\frac{d^2v(x)}{dx^2}$$
$$\sigma(x) = E.\varepsilon(x)$$

Let $l = \frac{L}{2}$, $u_1 = 0$ and $\theta_1 = 0$, the assembly of 2 elements leads to the following system:

$$\frac{8EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & -12L & -12 & 6L \\ 6L & 2L & -12L & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & 6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \\ u_3 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} T + \frac{wL}{3} \\ M + \frac{wL^2}{48} \\ \frac{wL}{3} \\ \frac{wL^2}{48} \\ \frac{wL}{3} \\ \frac{wL^2}{48} \end{bmatrix}$$

After simplification, the displacement of third node is

$$u_3 = \frac{WL^4}{8EI},$$

Probabilistic study of u_3

Case 1: E is uniformly distributed in the range $[10^8(N/m^2),\ 3{\times}10^8(N/m^2)]$ with $\overline{E} = 2 \times 10^8(N/m^2)$.

Using the proposed technique,

$$PDF(u_3) = |J|\,PDF(E) = \frac{wL^4}{8Iu_3^2}PDF(E)$$
$$= \begin{cases} \frac{WL^4}{16.10^8Iu_3^2} & \text{if } \frac{wL^4}{24.10^8I} \le u_3 \le \frac{wL^4}{8.10^8I} \\ 0 & if\ not \end{cases}$$

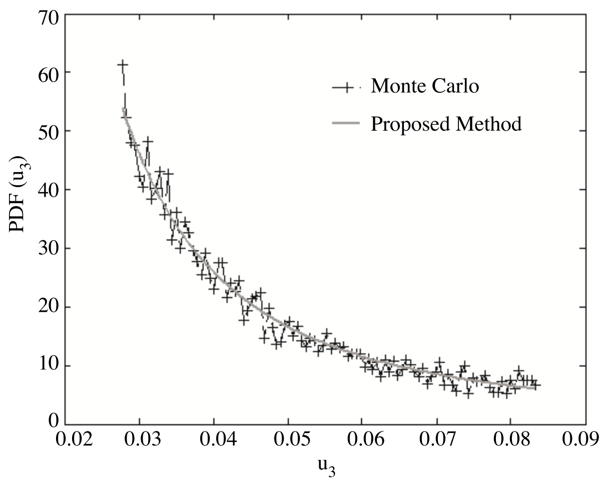


Figure 5. *pdf* (u_3) when E is uniformly distributed.

Reliability analysis

Let us suppose the limit displacement is $u_{3l} = \frac{L}{180} = 0.0556mm$. It is required to find the failure probability $P_f = P(u \ge u_{3l})$, which is as follows:

Numerical values:

$$w = 12N/m$$
$$L = 10m$$
$$I = 1,8.10^{-3}m^4$$

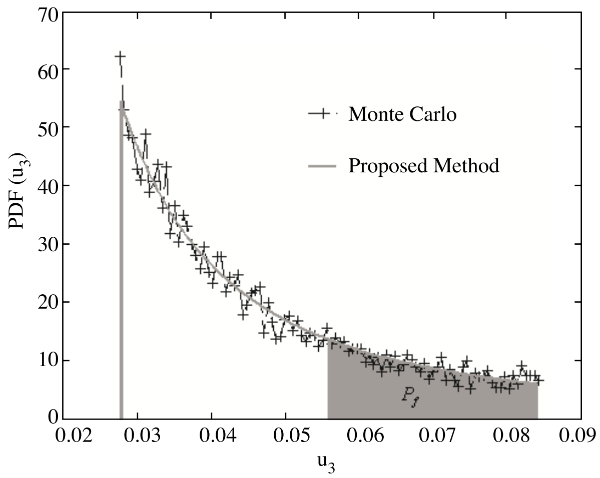


Figure 6. Probability of failure.

$$\begin{aligned} P_f &= \int_{0.0556}^{\infty} PDF(u_3) = \int_{0.0556}^{\infty} \frac{WL^4}{16.10^8 I u_3^2} du_3 \\ &= \int_{0.0556}^{0.0833} \frac{12.10^4}{16.10^8.1,8.10^{-3}.u_3^2} du_3 = \frac{1}{4} = 0.25 \end{aligned}$$

Table 1. Comparison with Monte Carlo.

	Proposed Method	Monte-Carlo simulation (10,000)
P_f	0.25	0.2458

Case 2: W is normally distributed with mean equal to 12 and standard deviation equal to 1.

Using the proposed technique, the displacement pdf is written as $PDF(u_3) = \left\{ \frac{8EI}{L^4\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{8EIu_3}{L^4}-12\right)^2} \right.$

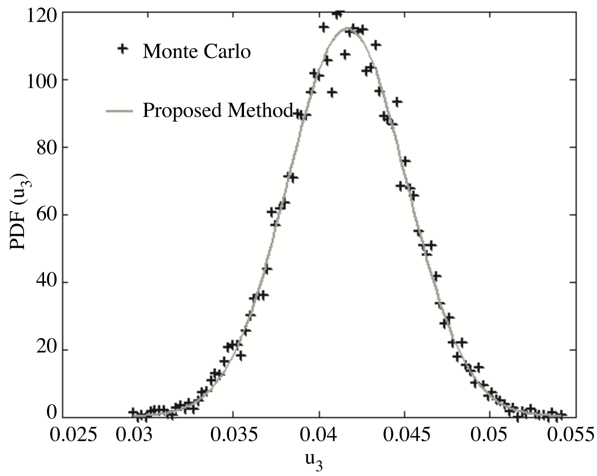


Figure 7. *pdf* (u_3) when W is normally distributed.

Reliability analysis

Let us suppose now the limit displacement is $u_{3l} = \frac{L}{220} = 0.0455mm$. The failure probability $P_f = P(u \geq u_{3l})$is:

Numerical values:

$$\begin{aligned} E &= 2.10^8 N/m \\ L &= 10m \\ I &= 1,8.10^{-3} m^4 \end{aligned}$$

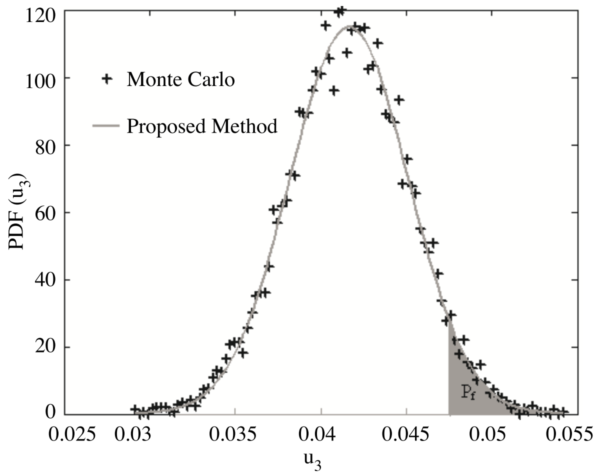


Figure 8. Probability of failure.

$$P_f = \int_{0.0455}^{\infty} PDF(u_3)$$

$$= \int_{0.0455}^{\infty} \frac{8EI}{L^4\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{8EIu_3}{L^4}-12\right)^2} du_3$$

$$= \int_{0.0455}^{0.0546} \frac{16.1,8.10^5}{10^4\sqrt{2.3,14}} e^{-\frac{1}{2}\left(\frac{16.1,8.10^5 u_3}{10^4}-12\right)^2} du_3$$

$$= 0.1347$$

Table 2. Comparison with Monte Carlo.

	Proposed Method	Monte-Carlo simulation (10,000)
P_f	0.1347	0.1328

II) In the second application, we are going to analyze the reliability of a 3-bar truss structure (Figure 9) with random parameters (Young's modulus E or concentrated load P).

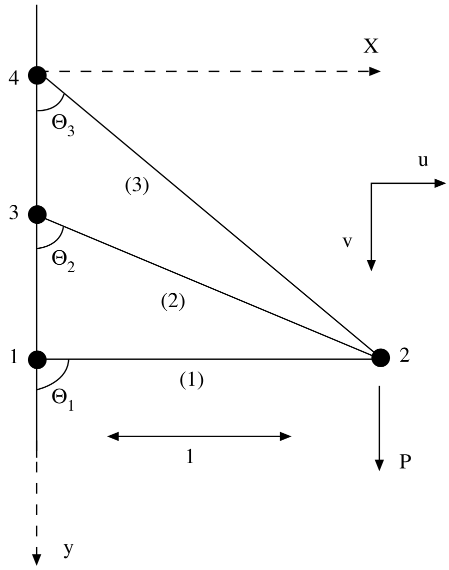


Figure 9. Three-bar truss structure.

FEM modeling the 3-bar truss:

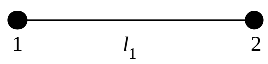
The stiffness matrix in the global coordinate system is given by

$$[K^e]^{(i)} = \frac{A^i E^i}{l_i}\begin{bmatrix} \lambda^2 & \lambda\mu & -\lambda^2 & -\lambda\mu \\ \lambda\mu & \mu^2 & -\lambda\mu & -\mu^2 \\ -\lambda^2 & -\lambda\mu & \lambda^2 & \lambda\mu \\ -\lambda\mu & -\mu^2 & \lambda\mu & \mu^2 \end{bmatrix}$$

where

- $-i\ :\ number\ of\ element$
- $-A : cross\ \sec tion$
- $-E : Young's\ modulus$
- $-l : length\ of\ bar$
- $-\lambda : \cos\alpha$
- $-\mu : \sin\alpha$
- (α : *angle between the element and the horizontal*)

Element (1): 1-2



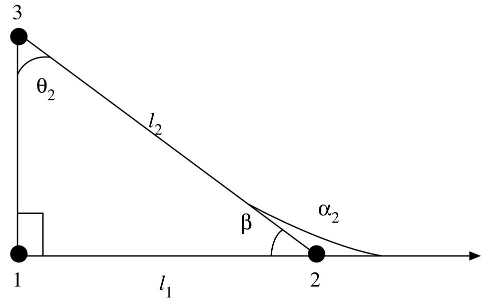
$$\begin{cases} \lambda = \cos\alpha_1 = 1 \\ \mu = \sin\alpha_1 = 0 \end{cases}$$

The stiffness matrix of element (1) is given by

$$[K^e]^{(1)} = \frac{AE}{l_1}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

To simplify, we suppose$A^i = A,\ E^i = E.$

Element (2): 2-3



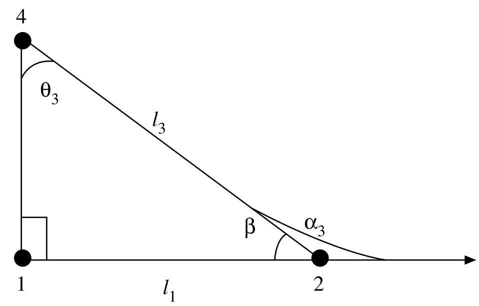
$\alpha_2 = \pi - \beta = \frac{\pi}{2} - \theta_2$
$\lambda = \cos\alpha_2 = \cos(\frac{\pi}{2} - \theta_2) = \sin\theta_2 = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$
$\mu = \sin\alpha_2 = \sin(\frac{\pi}{2} - \theta_2) = \cos\theta_2 = \cos\frac{\pi}{3} = \frac{1}{2}$
$l_2 = \frac{l_1}{\cos\beta} = \frac{2l_1}{\sqrt{3}}$

The stiffness matrix of element (2) is

$$[K^e]^{(2)} = \frac{AE}{l_2}\begin{bmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{3}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}$$

$$\text{i.e. } [K^e]^{(2)} = \frac{AE}{l_1}\begin{bmatrix} \frac{3\sqrt{3}}{8} & \frac{3}{8} & -\frac{3\sqrt{3}}{8} & -\frac{3\sqrt{3}}{8} \\ \frac{3}{8} & \frac{\sqrt{3}}{8} & -\frac{3}{8} & -\frac{\sqrt{3}}{8} \\ -\frac{3\sqrt{3}}{8} & -\frac{3}{8} & \frac{3\sqrt{3}}{8} & \frac{3}{8} \\ -\frac{3}{8} & -\frac{\sqrt{3}}{8} & \frac{3}{8} & \frac{\sqrt{3}}{8} \end{bmatrix}$$

Element (3): 2-4



$\alpha_3 = \pi - \gamma = \frac{\pi}{2} - \theta_3$
$\lambda = \cos\alpha_3 = \cos(\frac{\pi}{2} - \theta_3) = \sin\theta_3 = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$
$\mu = \sin\alpha_3 = \sin(\frac{\pi}{2} - \theta_3) = \cos\theta_3 = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$
$l_3 = \frac{l_1}{\cos\gamma} = \frac{2l_1}{\sqrt{2}}$

The stiffness matrix of element (3) is

$$[K^e]^{(3)} = \frac{AE}{l_3}\begin{bmatrix} \frac{2}{4} & \frac{2}{4} & -\frac{2}{4} & -\frac{2}{4} \\ \frac{2}{4} & \frac{2}{4} & -\frac{2}{4} & -\frac{2}{4} \\ -\frac{2}{4} & -\frac{2}{4} & \frac{2}{4} & \frac{2}{4} \\ -\frac{2}{4} & -\frac{2}{4} & \frac{2}{4} & \frac{2}{4} \end{bmatrix}$$

i.e. $[K^e]^{(3)} = \frac{AE}{l_1}\begin{bmatrix} \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \end{bmatrix}$

Using the Assembly technique (Chateauneuf, 2005) to form the global stiffness matrix for the entire structure using the stiffness matrix of each element:

$$[K]^g = \frac{AE}{l_1}\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & \frac{8+3\sqrt{3}+2\sqrt{2}}{8} & \frac{3+2\sqrt{2}}{8} & \frac{-3\sqrt{3}}{8} & \frac{-3\sqrt{3}}{8} & \frac{-\sqrt{2}}{4} & \frac{-\sqrt{2}}{4} \\ 0 & 0 & \frac{3+2\sqrt{2}}{8} & \frac{3+2\sqrt{2}}{8} & \frac{-3}{8} & \frac{-\sqrt{3}}{8} & \frac{-\sqrt{2}}{4} & \frac{-\sqrt{2}}{4} \\ 0 & 0 & \frac{-3\sqrt{3}}{8} & \frac{-3}{8} & \frac{3\sqrt{3}}{8} & \frac{3}{8} & 0 & 0 \\ 0 & 0 & \frac{-3}{8} & \frac{-\sqrt{3}}{8} & \frac{3}{8} & \frac{\sqrt{3}}{8} & 0 & 0 \\ 0 & 0 & \frac{-\sqrt{2}}{4} & \frac{-\sqrt{2}}{4} & 0 & 0 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ 0 & 0 & \frac{-\sqrt{2}}{4} & \frac{-\sqrt{2}}{4} & 0 & 0 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \end{bmatrix}$$

Therefore, the assembly of 3 elements leads to the following system:

$$\{F\} = [K]^g \cdot \{U\}$$

where

$$\{F\} = \begin{bmatrix} F_{1x} \\ F_{1y} \\ 0 \\ P \\ F_{3x} \\ F_{3y} \\ F_{4x} \\ F_{4y} \end{bmatrix}, \quad \{U\} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

After resolution, the vertical displacement of node 2 is

$$v_2 = 3.25\frac{Pl_1}{AE}$$

Probabilistic study of v_2

Case 1: E(N/m^2) is uniformly distributed in the range [5, 10]

Using our technique, the pdf of the displacement is thus

$$PDF(v_2) = |J|\, PDF(E) = \frac{3.25Pl_1}{Av_2^2}PDF(E)$$
$$= \begin{cases} \frac{3.25Pl_1}{Av_2^2} & if\ \frac{3.25Pl_1}{10A} \le v_2 \le \frac{3.25Pl_1}{5A} \\ 0 & if\,not \end{cases}$$

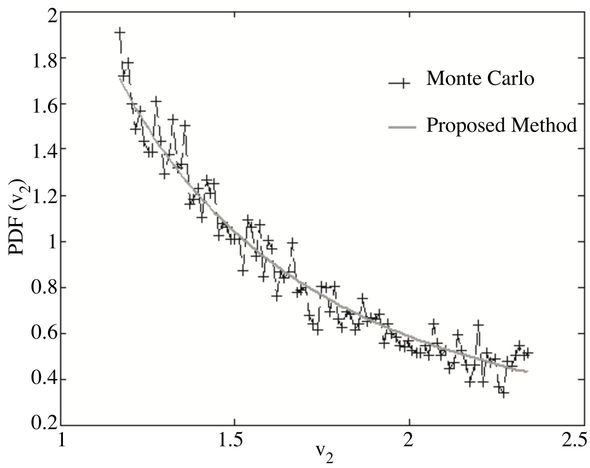


Figure 10. *pdf* (v_2) when E is uniformly distributed.

Reliability analysis

Let us suppose now the limit displacement is $v_{2l} = 2mm$. It is requested to find the failure probability $P_f = P(v \ge v_{2l})$, which is as follows:

Numerical values:

$$P = 1.2\ N$$
$$l_1 = 3\ m$$
$$A = 1\ m^2$$

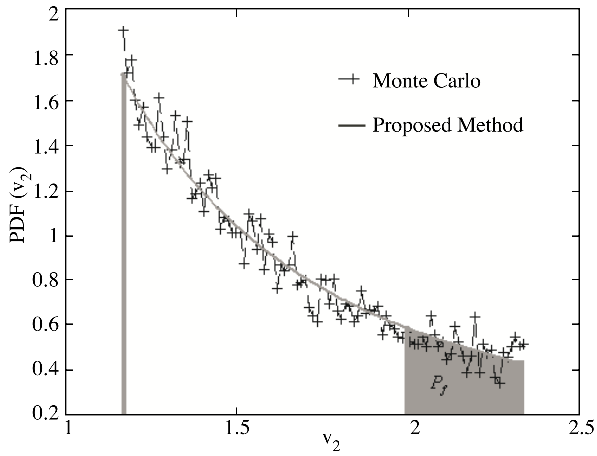


Figure 11. Probability of failure.

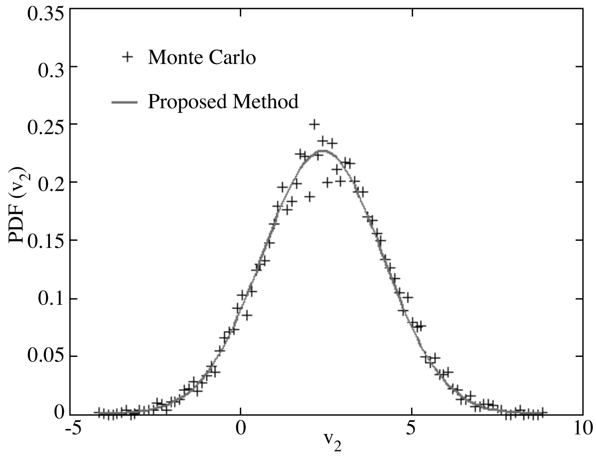


Figure 12. *pdf* (v_2) when P is normally distributed.

$$
\begin{aligned}
P_f &= \int_2^{\infty} PDF(v_2) = \int_2^{\infty} \frac{3.25 P l_1}{A v_2^2} dv_2 \\
&= \int_2^{2.34} \frac{3.25 \times 1.2 \times 3}{1.v_2^2} dv_2 = \frac{17}{100} = 0.17
\end{aligned}
$$

Table 3. Comparison with Monte Carlo.

	Proposed Method	Monte-Carlo simulation (10,000)
P_f	0.17	0.1681

Case 2: P is normally distributed with mean equal to 1.2 and standard deviation equal to 0.9.

Using our technique, the pdf of the displacement is thus

$$PDF(v_2) = |J|\, PDF(E) = \tfrac{AEv_2}{3.25 l_1} PDF(P)$$

$$= \left\{ \frac{AE}{3.25 l_1} . \frac{1}{0.9\sqrt{2\pi}} .e^{\frac{-(\frac{AEv_2}{3.25 l_1} - 1.2)^2}{2 \times 0.9^2}} \right.$$

Reliability analysis

Let us suppose now the limit displacement is $v_{2l} = 5mm$. The failure probability $P_f = P(v \geq v_{2l})$ is

Numerical values:

$$
\begin{aligned}
E &= 5\ N/m^2 \\
l_1 &= 3\ m \\
A &= 1\ m^2
\end{aligned}
$$

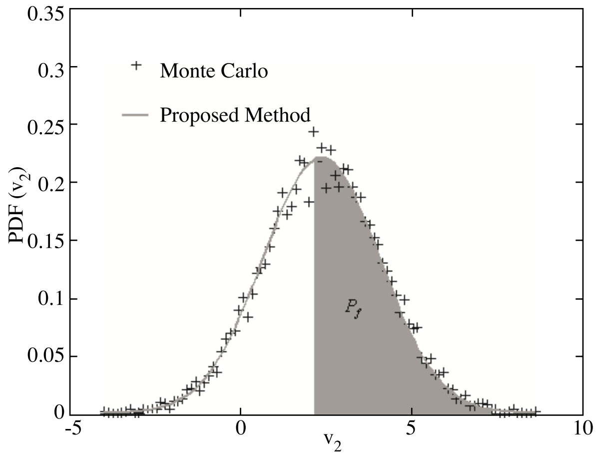


Figure 13. Probability of failure.

$$
\begin{aligned}
P_f &= \int_2^{\infty} PDF(v_2) \\
&= \int_2^{\infty} \frac{AE}{3.25 l_1} . \frac{1}{0.9\sqrt{2\pi}} .e^{\frac{-(\frac{AEv_2}{3.25 l_1} - 1.2)^2}{2 \times 0.9^2}} dv_2 \\
&= \int_2^{8.7562} \frac{5}{9.75} . \frac{1}{0.9\sqrt{2\pi}} .e^{\frac{-(\frac{5v_2}{9.75} - 1.2)^2}{2 \times 0.9^2}} dv_2 \\
&= 0.15
\end{aligned}
$$

Table 4. Comparison with Monte Carlo.

	Proposed Method	Monte-Carlo simulation (10,000)
P_f	0.5840	0.5823

III) In the third application, we are going to analyze the reliability of a 25-bar truss structure (Figure 14) by using the unit load method with random parameters (Young modulus E, section S or the horizontal load q).

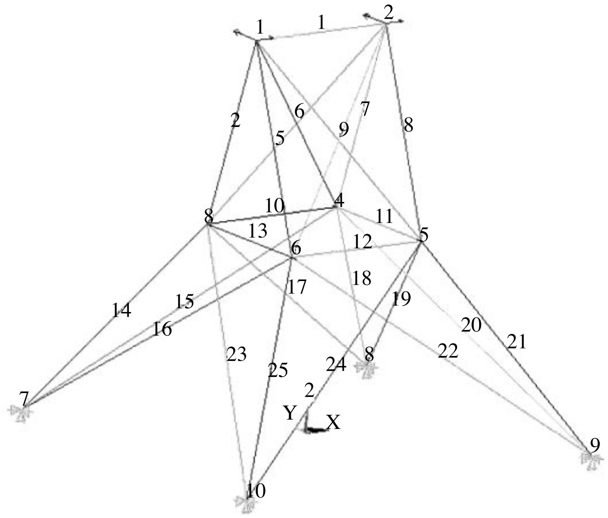


Figure 14. 25-Bar truss structure.

The method of the unit load permits the calculation of the displacement at a point using the following formula:

$$u = \sum_{i=1}^{n} \frac{N_i \overline{N_i}}{ES_i} L_i,$$

where N_i is the normal force due to the applied load, $\overline{N_i}$ is the normal force due to a unit load to the point and in the direction of searching displacement, E is the Young's modulus, and S_i and L_i are respectively the section and the length of the bar i.

By symmetry, the sections of some bars are identical. We adopt the following distribution:

Bar	Section
1	S_1
2,5,7,8	S_2
3,4,6,9	S_3
10,11,12,13	S_4
14,18,21,25	S_5
15,16,17,19,20,22,23,24	S_6

Normal forces:

Bar	Vertical Load (N)	Horizontal Load (N)	Fx1 =1N	Fy1 =1N	Fz1 =1N	Fx2 =1N	Fy2 =1N	Fz2 =1N	Length Li (mm)
1	118,496	0	-0.44	0	-0.11	0.44	0	-0.11	18,000
2	-182,632	-108,058	0.39	-0.88	0.45	0.31	-0.04	-0.08	25,632
3	-103,094	-181,168	-0.48	-0.65	0.10	-0.38	0.05	0.10	31,321
4	-103,094	24,564	-0.48	0.65	0.10	-0.38	-0.05	0.10	31,321
5	-182,632	236,220	0.39	0.88	0.45	0.31	0.04	-0.08	25,632
6	-103,094	-34,814	0.38	0.05	0.10	0.48	-0.65	0.10	31,321
7	-182,632	-227,820	-0.31	-0.04	-0.08	-0.39	-0.88	0.45	25,632
8	-182,632	99,670	-0.31	0.04	-0.08	-0.39	0.88	0.45	25,632
9	-103,094	191,418	0.38	-0.05	0.10	0.48	0.65	0.10	31,321
10	-2112.2	27,160	0.02	0.07	-0.01	-0.04	0.07	-0.01	18,000
11	25,438	25,416	0.14	0	-0.01	0.14	0	-0.07	18,000
12	-2112.2	-27,160	0.02	-0.07	-0.01	-0.02	-0.07	-0.01	18,000
13	25,438	-25,416	-0.14	0	-0.07	-0.14	0	-0.01	18,000
14	-261,700	-84,148	0.57	-0.52	0.35	0.57	-0.60	0.02	32,031
15	-142,550	-129,638	-0.14	-0.03	-0.04	-0.15	-0.54	0.24	43,474
16	-136,254	138,918	0.27	0.17	0.17	0.27	0.12	0.00	43,474
17	-142,550	-78,852	0.15	-0.54	0.24	0.14	-0.03	-0.04	43,474
18	-261,700	-322,780	-0.57	-0.60	0.02	-0.57	-0.52	0.35	32,031
19	-136,254	-30,232	-0.27	0.12	0.00	-0.27	0.17	0.17	43,474
20	-136,254	-70,148	-0.27	-0.12	0.00	-0.27	-0.17	0.17	43,474
21	-261,700	116,470	-0.57	0.6	0.02	-0.57	0.52	0.35	32,031
22	-142,550	133,196	0.15	0.54	0.24	0.14	0.03	-0.04	43,474
23	-136,254	-38,536	0.27	-0.17	0.17	0.27	-0.12	0.00	43,474
24	-142,550	75,296	-0.14	0.034	-0.04	-0.15	0.54	0.24	43,474
25	-261,700	290,460	0.57	0.52	0.35	0.57	0.60	0.02	32,031

For the calculation of the horizontal displacement u_{y2} at the point 2, according to y direction, and due to the load q, we put:

$$u_{y2} = \frac{q}{180000E}\left[\begin{array}{l} [(1.57898e-10)(3.7646e-17)18000]\,\frac{1}{S_1}+ \\ \left[\begin{array}{l}\text{(-108,058) (-0.04387) 25,632 + (236,220) (0.04387) 25,632 +} \\ \text{(-227,820) (-0.88916) 25,632 + (99,670) (0.88916) 25,632}\end{array}\right]\frac{1}{S_2}+ \\ \left[\begin{array}{l}\text{(-181,168) (0.053606) 31,321 + (24,564) (-0.053606) 31,321 +} \\ \text{(-34,814) (-0.65356) 31,321 + (191,418) (0.65356) 31,321}\end{array}\right]\frac{1}{S_3}+ \\ \left[\begin{array}{l}\text{(27,160) (0.075444) 18,000 + (25,416) (0) 18,000 +} \\ \text{(-27,160) (-0.075444) 18,000 + (-25,416) (0) 18,000}\end{array}\right]\frac{1}{S_4}+ \\ \left[\begin{array}{l}\text{(-84,148) (-0.6071) 32,031 + (-322,780) (-0.52328) 32,031 +} \\ \text{(116,470) (0.52328) 32,031 + (290,460) (0.6071) 32,031}\end{array}\right]\frac{1}{S_5}+ \\ \left[\begin{array}{l}\text{(-129,638) (-0.54426) 43,474 + (138,918) (0.122694) 43,474 +} \\ \text{(-78,852) (-0.034886) 43,474 + (-30,232)(0.17921) 43,474+} \\ \text{(-70,148) (-0.17921) 43,474 + (133,196) (0.034886) 43,474 +} \\ \text{(-38,536) (-0.122694)43,474 + (75,296) (0.54426) 43,474}\end{array}\right]\frac{1}{S_6}\end{array}\right]$$

After simplification, the horizontal displacement of node 2 becomes

$$u_{y2} = \frac{q}{180000E}\left(\begin{array}{l}\frac{0}{S_1}+\frac{7,850,940,221}{S_2}+\frac{4,285,580,903}{S_3}+\frac{73,766,125.\ 44}{S_4}+ \\ \frac{1.\ 464\ 698\ 375\times 10^{10}}{S_5}+\frac{6,428,099,237}{S_6}\end{array}\right)$$

with:

E: Young's modulus.

q: Vertical load.

S_i: Section of bar i.

Probabilistic study of u_{y2}

For simplification, we suppose $S_i = S$ leading to $u_{y2} = \frac{q}{ES}(184,918.723528)$

Case 1: E(N/m^2) is uniformly distributed in the range [10^5, 310^5]

Using our technique, the pdf of the displacement is thus

$$PDF(u_{y2}) = |J|\,PDF(E)$$

$$= \left(\frac{q}{u_{y2}^2 S}(184,918.723528)\right) PDF(E)$$

$$= \begin{cases} \frac{q10^{-5}}{2u_{y2}^2 S}(184,918.723528) & \text{if } \frac{q(184,918.723528)}{3.10^5 S} \\ 0 \quad \text{if not} & \end{cases}$$

$$\leq u_{y2} \leq \frac{q(184,918.723528)}{10^5 S}$$

Numerical values:

q=180000N

S=2000mm^2

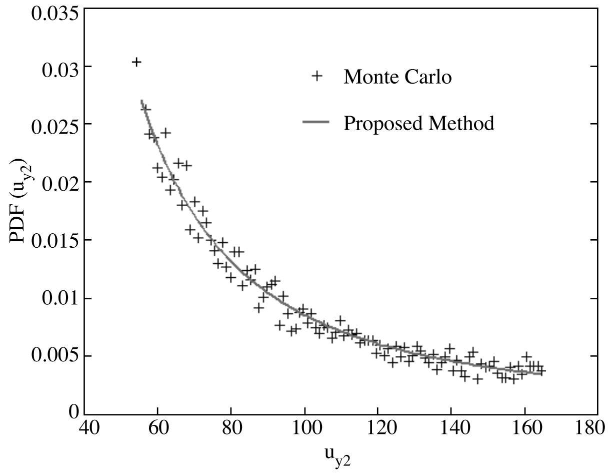


Figure 15. *pdf* (u_{y2}) when E is uniformly distributed.

Reliability analysis

Let us suppose now that the limit displacement is $u_{y2} = 1.20mm$. It is required to find the failure probability $P_f = P(u \geq u_{y2})$, which is as follows:

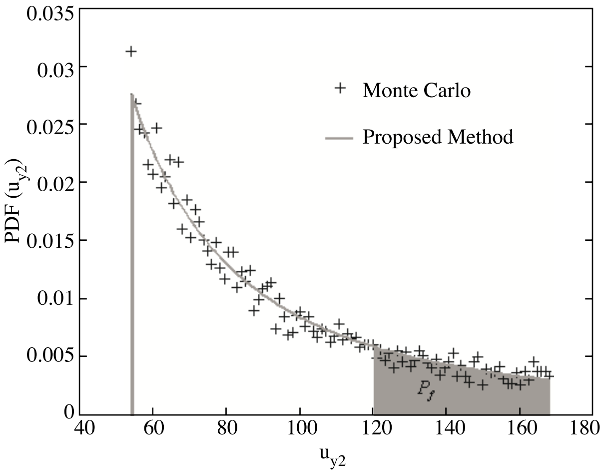


Figure 16. Probability of failure.

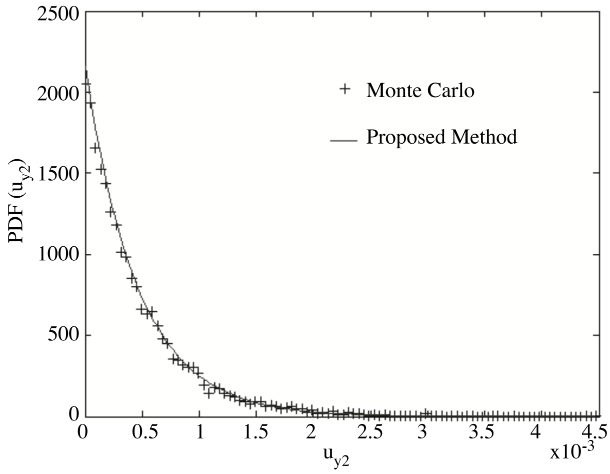


Figure 17. *pdf* (u_{y2}) when q is exponentially distributed.

$$
\begin{aligned}
P_f &= \int_{120}^{\infty} PDF(u_{y2}) du_{y2} \\
&= \int_{120}^{\infty} \frac{q10^{-5}}{2u_{y2}^2 S} (184,918.723528)\, du_{y2} \\
&= \int_{120}^{166} \frac{16,642,685.118 \times 10^{-5}}{2.u_{y2}^2} du_{y2} = 0.19
\end{aligned}
$$

Table 5. Comparison with Monte Carlo.

	Proposed Method	Monte-Carlo simulation (10,000)
P_f	0.19	0.1890

Case 2: q is exponentially distributed with unit mean.

Using our technique, we can write

$$PDF(u_{y2}) = |J|\, PDF(q)$$

$$= \left(\frac{ES}{(184,918.723528)} \right) PDF(q)$$

$$= \left\{ \frac{ES}{(184,918.723528)} .e(- \frac{ESu_{y2}}{184,918.723528}) \right.$$

Numerical values:

E=200,000 N/m^2

S=2000mm^2

Reliability analysis

Let us suppose now that the limit displacement is $u_{y2} = 0.0005mm$. It is requested to find the failure probability $P_f = P(u \geq u_{y2})$, which is as follows:

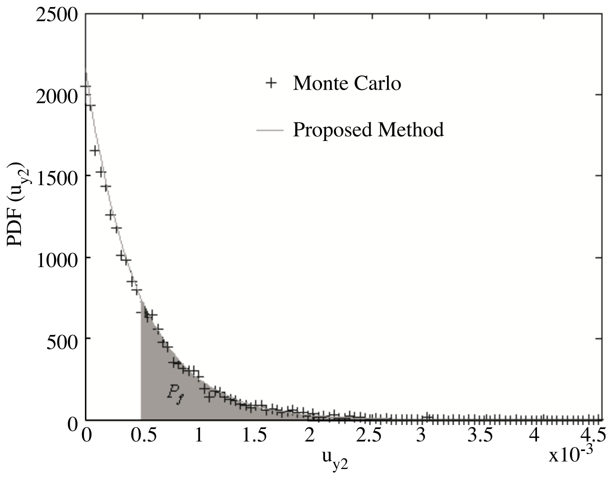


Figure 18. Probability of failure.

$$
\begin{aligned}
P_f &= \int_{0.0005}^{\infty} PDF(u_{y2}) du_{y2} \\
&= \int_{0.0005}^{\infty} \frac{ES}{(184,918.723528)} .e(- \frac{ESu_{y2}}{184,918.723528}) du_{y2} \\
&= \int_{0.0005}^{0.004} 4.610^{-4} e(-4.610^{-4} u_{y2}) du_{y2} = 0.34
\end{aligned}
$$

Table 6. Comparison with Monte Carlo.

	Proposed Method	Monte-Carlo simulation (10,000)
P_f	0.34	0.3382

Case 3: S(mm^2) normally distributed with mean equal to 2000 and standard deviation equal to 1.

Using our technique, the displacement pdf is written:

$$PDF(u_{y2}) = |J|\, PDF(S)$$

$$= \left(\frac{q}{u_{y2}^2 E} (184, 918.723528) \right) PDF(S)$$

$$= \left\{ \frac{q}{u_{y2}^2 E} (184, 918.723528) \, . \right.$$

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{(\frac{q}{u_{y2} E}(184,918.723528)-20)^2}{2}}$$

Numerical values:
q=180,000N
E=200,000N/m^2

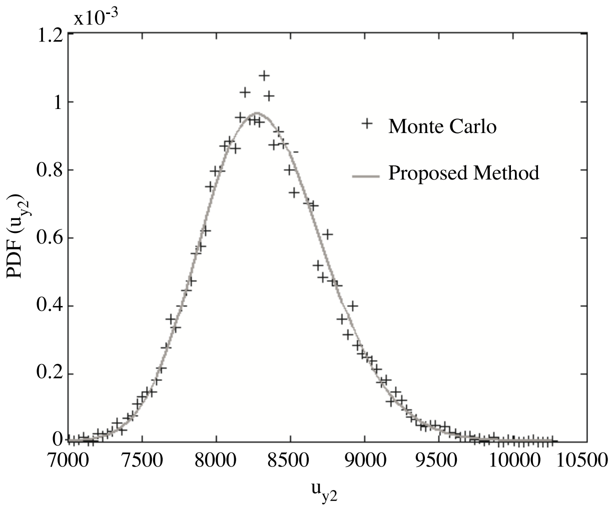


Figure 19. *pdf* (u_{y2}) when S is normally distributed.

Reliability analysis

Let us suppose now the limit displacement is $u_{y2} =$ 8.500 mm. It is requested to find the failure probability $P_f = P(u \geq u_{y2})$, which is as follows:

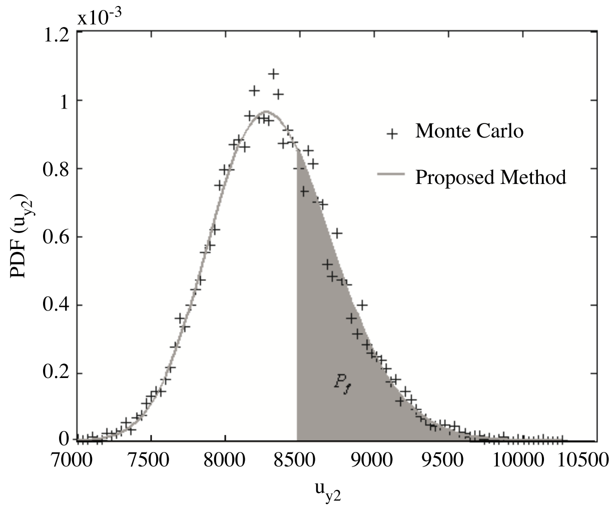


Figure 20. Probability of failure.

$$P_f = \int_{8500}^{\infty} PDF(u_{y2}) du_{y2}$$

$$\int_{8500}^{10272} \frac{q}{u_{y2}^2 E} (184, 918.723528) \, .$$

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{(\frac{q}{u_{y2} E}(184,918.723528)-20)^2}{2}} du_{y2}$$

$$= \int_{8500}^{10272} \frac{1.66426}{u_{y2}^2} . \frac{1}{\sqrt{2\pi}} e^{-\frac{(\frac{1.66426}{u_{y2}}-20)^2}{2}} du_{y2} = 0.33$$

Table 7. Comparison with Monte Carlo.

	Proposed Method	Monte-Carlo simulation (10,000)
P_f	0.33	0.3334

Conclusion

In this paper, the reliability analysis of a mechanical system with parameter uncertainties has been considered. The uncertainty has been considered in the material properties, e.g., Young's modulus, cross section and load. A proposed technique for the evaluation in "exact" form of the probability of failure is developed. Compared to other numerical methods like FORM/SORM, Monte-Carlo and Response surface method, no-series expansion is involved in this expression. This technique is based on the combination of the probabilistic transformation method (PTM) and the deterministic finite element method (FEM). To prove the accuracy of the FEM-PTM technique, the result is compared with 10,000 sampling of direct Monte-Carlo simulation.

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