

## Analytical Solution of Nonlinear Strain Hardening Preheated Pressure Tube

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### Abstract

The analytical solution of a nonlinear strain hardening preheated tube subjected to internal pressure is presented. A state of generalized plane strain, small deformations, and temperature gradients are assumed. The analytical plastic model is based on the incremental theory of plasticity, Tresca's yield criterion, its associated flow rule, and a Swift-type nonlinear hardening law. Solutions for linearly hardening and perfectly plastic materials are also presented.

**Key words:** Pressurized tube, Stress analysis, Elastoplasticity, Nonlinear hardening, Tresca's criterion.

### Introduction

Deformation analysis of tubes subjected to either internal or external pressure is an important topic in engineering because of rigorous applications in industry as well as in daily life. For this reason, the classical problem of a long pressurized tube has been the topic of a variety of theoretical investigations. It is treated in the purely elastic stress state by Timoshenko (1956), Timoshenko and Goodier (1970), Ugural and Fenster (1987), and Boreasi et al. (1993); in the fully plastic stress state by Boreasi et al. (1993), Mendelson (1986), and Nadai (1931); and in the partially plastic stress state by Bland (1956), Parker (2001), and Perry and Aboudi (2003). Recent studies on the subject by Horgan and Chan (1999), Tutuncu and Ozturk (2001), Jabbari et al. (2002), Ma et al. (2003), and Eraslan and Akis (2005a, 2006) include tubes made of functionally graded materials (FGM) under pressure. The results of stress and deformation analyses in 2-layer concentric pressure tubes in the elastic state by Eraslan and Akis (2005b) and Akis and Eraslan (2005) and in the partially plastic stress state by Eraslan and Akis (2004) have also

been reported.

The subject matter of this article is to derive consistent analytical solutions in order to predict the partially plastic stress responses of nonlinear strain hardening tubes under internal pressure. A sufficiently long tube of inner radius  $a$  and outer radius  $b$  with axially unconstrained ends is taken into account. Inner and outer surfaces are kept at different temperatures, which leads to a radial temperature gradient within the tube described by the logarithmic temperature distribution

$$T(r) = \frac{\ln(r/b)T_a - \ln(r/a)T_b}{\ln(a/b)}, \quad (1)$$

where  $T_a = T(a)$ ,  $T_b = T(b)$  are the temperatures of the inner and outer surfaces of the tube, respectively. The tube is then subjected to internal pressure. Since this is intended to be a pressurized tube problem rather than a thermal stress problem, small and negative temperature gradients of the order of 20 °C are considered. Under these circumstances and in the framework of generalized plane strain and small deformation theory, an elastic as well as a plastic region is formulated in terms of dimensionless vari-

ables. The formulation of the plastically deformed region is based on incremental theory, Tresca's yield criterion, its associated flow rule, and a Swift-type nonlinear hardening law. Exact solutions of the resulting field equations of Cauchy-Euler linear nonhomogeneous type for the elastic, and Abel nonlinear type for the plastic are obtained. Elastic, partially plastic and fully plastic stress states are investigated. The results imply that elastic and plastic limit pressures and the magnitudes of the response variables are affected notably by the existence of a small temperature gradient within the tube.

### Basic Equations

The notation given by Timoshenko and Goodier (1970) and the basic equations provided therein are employed. Hence, in the formulation,  $\sigma_j$  and  $\epsilon_j$  denote a normal stress and a normal strain component, respectively, and  $u$  is the radial component of displacement vector. Furthermore, we introduce and use the following nondimensional and normalized variables. They are radial coordinate:  $\bar{r} = r/b$ , bore radius:  $\bar{a} = a/b$ , normal stress:  $\bar{\sigma}_j = \sigma_j/\sigma_0$ , normal strain:  $\bar{\epsilon}_j = \epsilon_j E/\sigma_0$ , radial displacement:  $\bar{u} = uE/(b\sigma_0)$ , pressure:  $\bar{P} = P/\sigma_0$ , and the coefficient of thermal expansion:  $\bar{\alpha} = \alpha E/\sigma_0$ , with  $E$  being the modulus of elasticity and  $\sigma_0$  the uniaxial yield limit. The equations given below are written in terms of these variables, but to simplify the notation overbars are dropped.

A sufficiently long tube, a state of generalized plane strain, i.e.  $\epsilon_z = \text{constant}$ , and small strains are presumed. The strain displacement relations:  $\epsilon_r = u'$ ,  $\epsilon_\theta = u/r$ , the equation of equilibrium in the radial direction:

$$(r\sigma_r)' - \sigma_\theta = 0, \quad (2)$$

the compatibility relation:

$$(r\epsilon_\theta)' - \epsilon_r = 0, \quad (3)$$

and the equations of generalized Hooke's law of the form (Timoshenko and Goodier, 1970; Mendelson, 1986)

$$\epsilon_i = [\sigma_i - \nu(\sigma_j + \sigma_k)] + \epsilon_i^p + \alpha T, \quad (4)$$

constitute the basis for the entire analysis. In the above,  $\epsilon_i^p$  denotes the plastic counterpart of the normal strain  $\epsilon_i$ , and a prime implies differentiation with respect to nondimensional radial coordinate  $r$ .

### Elastic Stresses and Onset of Yield

For purely elastic deformations  $\epsilon_i^p = 0$ . Moreover, in the state of generalized plane strain the axial stress reads

$$\sigma_z = \epsilon_z + \nu(\sigma_r + \sigma_\theta) - \alpha T. \quad (5)$$

The elastic equation may be obtained in a variety of ways. One way is to express  $\sigma_\theta$  and  $\sigma_z$  in terms of  $\sigma_r$  by the use of Eqs. (2) and (5) and then use the compatibility relation, Eq. (3). The result is

$$\frac{d^2 Y}{dr^2} + \frac{1}{r} \frac{dY}{dr} - \frac{Y}{r^2} = -\frac{\alpha}{(1-\nu)} \frac{dT}{dr}, \quad (6)$$

where  $Y = r\sigma_r$ . The general solution of this equation imposing

$$T(r) = \frac{T_a \ln r - T_b \ln(r/a)}{\ln a} \quad (7)$$

enables one to assemble

$$\sigma_r = \frac{C_1}{r^2} + C_2 + \frac{\alpha(T_b - T_a) \ln r}{2(1-\nu) \ln a}, \quad (8)$$

$$\sigma_\theta = -\frac{C_1}{r^2} + C_2 + \frac{\alpha(T_b - T_a)(1 + \ln r)}{2(1-\nu) \ln a}, \quad (9)$$

$$\sigma_z = -\frac{\alpha}{2(1-\nu) \ln a} \{ (2 \ln r + \nu) T_a + [2 \ln(a/r) - \nu(1 + 2 \ln a + 2 \ln(a/r))] T_b \} + 2C_2 \nu + \epsilon_z, \quad (10)$$

$$u = (1 + \nu) \left[ -\frac{C_1}{r} + (1 - 2\nu)rC_2 \right] + \frac{\alpha(1 + \nu)r}{2(1-\nu) \ln a} (D_1 - D_2) - \epsilon_z \nu r, \quad (11)$$

where  $C_1$  and  $C_2$  are arbitrary constants to be determined, and the dummy variables  $D_1$ ,  $D_2$  have been defined as

$$D_1 = [(1-\nu)(1 + 2 \ln a) - \ln r] T_b, \quad (12)$$

$$D_2 = (1 - \nu - \ln r) T_a. \quad (13)$$

Furthermore, the use of the conditions

$$\sigma_r(a) = -P, \quad \sigma_r(1) = 0, \quad \text{and}$$

$$\int \sigma_z dA = 2\pi \int_a^1 \sigma_z r dr = 0, \quad (14)$$

leads to

$$C_1 = \frac{a^2 P}{a^2 - 1} + \frac{a^2 \alpha (T_b - T_a)}{2(a^2 - 1)(1 - \nu)}, \quad (15)$$

$$C_2 = -\frac{a^2 P}{a^2 - 1} - \frac{a^2 \alpha (T_b - T_a)}{2(a^2 - 1)(1 - \nu)}, \quad (16)$$

$$\begin{aligned} \epsilon_z = & \frac{2a^2 \nu P}{a^2 - 1} - \frac{\alpha}{2(a^2 - 1) \ln a} [(1 - a^2 + 2 \ln a) T_b \\ & - (1 - a^2 + 2a^2 \ln a) T_a]. \end{aligned} \quad (17)$$

The inner surface of the tube is critical; consequently, the pressure tube fails with respect to plastic flow at this location as soon as the internal pressure reaches a limiting value  $P_E$ , called the elastic limit pressure. According to Tresca's yield criterion under the internal pressure  $P = P_E$ ;  $\sigma_\theta(a) - \sigma_r(a) = 1$ , and hence after some algebraic manipulations

$$P_E = \frac{1 - a^2}{2} - \frac{\alpha(1 - a^2 + 2 \ln a) \Delta T}{4(1 - \nu) \ln a} \quad (18)$$

is derived. Here,  $\Delta T = T_b - T_a$  is a measure of the radial temperature gradient. For an isothermal pressure tube,  $\Delta T = 0$ , the elastic limit simplifies to  $P_E = (1 - a^2)/2$ . In the case of negative temperature gradients, i.e.  $T_a > T_b$ ,  $P_E$  increases linearly with increasing magnitudes of  $\Delta T$  and vice versa for positive values of  $\Delta T$ .

Finally, we calculate a force integral  $F_E$  to accompany further elastoplastic calculations using

$$F_E(\beta, \gamma) = \int_\beta^\gamma \sigma_z r dr. \quad (19)$$

The result is

$$\begin{aligned} F_E(\beta, \gamma) = & -\frac{D_3 (\beta^2 \ln \beta - \gamma^2 \ln \gamma)}{2(1 - \nu^2)} \\ & + (\beta^2 - \gamma^2) \left[ \frac{D_3}{4(1 + \nu)} + \frac{\alpha T_b}{2} - \frac{\epsilon_z}{2} - \nu C_2 \right], \end{aligned} \quad (20)$$

in which

$$D_3 = \frac{\alpha(1 + \nu)(T_b - T_a)}{\ln a}. \quad (21)$$

## Plastic Formulation and Solution

In the case  $\Delta T < 0$ , the stress state satisfies  $\sigma_\theta > \sigma_z > \sigma_r$  throughout. Therefore, Tresca's yield criterion takes the form

$$\sigma_Y = \sigma_\theta - \sigma_r. \quad (22)$$

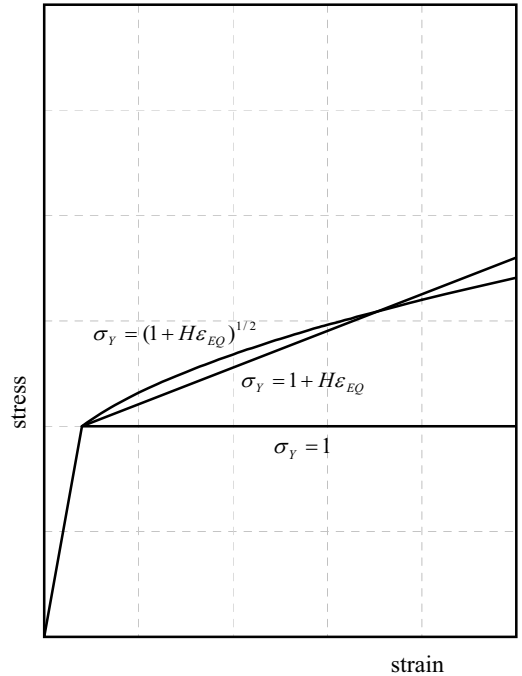
The associated flow rule states  $\epsilon_\theta^p = -\epsilon_r^p$  and  $\epsilon_z^p = 0$  (Mendelson, 1986). Furthermore, with  $\epsilon_{EQ}$  being the equivalent plastic strain, we deduce from the increment of plastic work that  $\epsilon_\theta^p = \epsilon_{EQ}$ . On the other hand, a Swift-type nonlinear strain hardening law relates the yield stress  $\sigma_Y$  and the equivalent plastic strain  $\epsilon_{EQ}$  as

$$\sigma_Y = (1 + H \epsilon_{EQ})^{1/2}, \quad (23)$$

where  $H$  is the nondimensional hardening parameter. The inverse relation is

$$\epsilon_{EQ} = (\sigma_Y^2 - 1)/H. \quad (24)$$

The stress-strain curve anticipated by relation (23) can be examined in Figure 1 in comparison to linear strain hardening and perfectly plastic models. It is noted that closed form solutions of the present problem for perfectly plastic and linear strain hardening materials are also obtained. These solutions are presented in the Appendix.



**Figure 1.** Graphical representation of perfectly plastic, linear strain hardening, and nonlinear strain hardening plastic models.

Making use of the flow rule and the relation  $\epsilon_\theta^p = \epsilon_{EQ}$  the total strains can be expressed as

$$\epsilon_r = \sigma_r - \nu(\sigma_\theta + \sigma_z) - (\sigma_Y^2 - 1)/H + \alpha T, \quad (25)$$

$$\epsilon_\theta = \sigma_\theta - \nu(\sigma_r + \sigma_z) + (\sigma_Y^2 - 1)/H + \alpha T. \quad (26)$$

Like in the elastic treatment, the stresses here may be expressed in terms of  $\sigma_r$ . Knowing that  $\sigma_Y =$

$\sigma_\theta - \sigma_r$ , we find from the equation of equilibrium  $\sigma_Y = r \sigma_r'$  and  $\sigma_\theta = (r \sigma_r)'$ . Moreover, since  $\epsilon_z^p = 0$ , the axial stress  $\sigma_z$  is no more than the one given in Eq. (5). If the total strains all expressed in terms of  $\sigma_r$  are substituted in the compatibility relation (3) we end up with the governing equation for the plastic region. The result turns out to be

$$-2 - HD_3 + 3Hr(1 - \nu^2) \frac{d\sigma_r}{dr} + 4r^2 \left( \frac{d\sigma_r}{dr} \right)^2 + r^2 \left[ H(1 - \nu^2) + 2r \frac{d\sigma_r}{dr} \right] \frac{d^2\sigma_r}{dr^2} = 0. \quad (27)$$

For the solution, we introduce a new variable  $V$  as  $V = \sigma_r'$ , so that Eq. (27) becomes

$$-2 - HD_3 + 3Hr(1 - \nu^2)V + 4r^2V^2 + r^2 [H(1 - \nu^2) + 2rV] \frac{dV}{dr} = 0. \quad (28)$$

This is Abel's B class nonlinear differential equation (see for example, ode advisor in Maple (Garvan, 2002)), which assumes 1 of the 2 exact solutions:

$$V(r) = -\frac{H(1 - \nu^2)}{2r} \pm \frac{\sqrt{r^2 D_4 + 2HC_3}}{2r^2}, \quad (29)$$

in which  $C_3$  is an arbitrary integration constant and

$$D_4 = 4 + 2HD_3 + [H(1 - \nu^2)]^2. \quad (30)$$

Among the 2 solutions, the one that satisfies

$$\lim_{H \rightarrow 0} \sigma_Y = \lim_{H \rightarrow 0} (r \sigma_r') = \lim_{H \rightarrow 0} (rV) = 1 \quad (31)$$

is

$$V(r) = -\frac{H(1 - \nu^2)}{2r} + \frac{\sqrt{r^2 D_4 + 2HC_3}}{2r^2}. \quad (32)$$

Since  $\sigma_r = \int V dr$  we finally obtain

$$\sigma_r = \frac{\sqrt{D_4}}{2} \ln \left( 2r\sqrt{D_4} + 2\sqrt{2HC_3 + r^2 D_4} \right) - \frac{\sqrt{2HC_3 + r^2 D_4}}{2r} + C_4 - \frac{H(1 - \nu^2) \ln r}{2}, \quad (33)$$

and hence

$$\sigma_\theta = \frac{\sqrt{D_4}}{2} \ln \left( 2r\sqrt{D_4} + 2\sqrt{2HC_3 + r^2 D_4} \right) + C_4 - \frac{H(1 - \nu^2)}{2} (1 + \ln r), \quad (34)$$

$$\begin{aligned} \sigma_z &= \epsilon_z + 2\nu C_4 + \nu\sqrt{D_4} \ln \left( 2r\sqrt{D_4} + 2\sqrt{2HC_3 + r^2 D_4} \right) - \frac{H\nu(1 - \nu^2)}{2} (1 + 2 \ln r) \\ &\quad - \frac{\nu\sqrt{2HC_3 + r^2 D_4}}{2r} - \alpha T_b + \frac{D_3 \ln r}{1 + \nu}, \end{aligned} \quad (35)$$

$$\begin{aligned} u &= \frac{C_3}{2r} - \frac{r[1 - HD_5 C_4]}{H} - \frac{Hr(1 + \nu)(1 - \nu^2)}{2} [1 - \nu + (1 - 2\nu) \ln r] + \frac{Hr(1 - \nu^2)^2}{4} \\ &\quad + \frac{r\sqrt{D_4}}{2} D_5 \ln \left( 2r\sqrt{D_4} + 2\sqrt{2HC_3 + r^2 D_4} \right) + \frac{rD_4}{4H} + r\alpha(1 + \nu)T_b - r\nu\epsilon_z \\ &\quad - \frac{1}{2} D_5 \sqrt{2HC_3 + r^2 D_4} - rD_3 \ln r, \end{aligned} \quad (36)$$

$$\epsilon_{\theta}^p = -\epsilon_r^p = \frac{C_3 - r(1 - \nu^2)\sqrt{2HC_3 + r^2D_4}}{2r^2} + \frac{1}{H} \left( \frac{D_4}{4} - 1 \right) + \frac{H(1 - \nu^2)^2}{4}, \quad (37)$$

where

$$D_5 = (1 + \nu)(1 - 2\nu). \quad (38)$$

Having found the analytical expression for  $\sigma_z$  we evaluate the force integral as

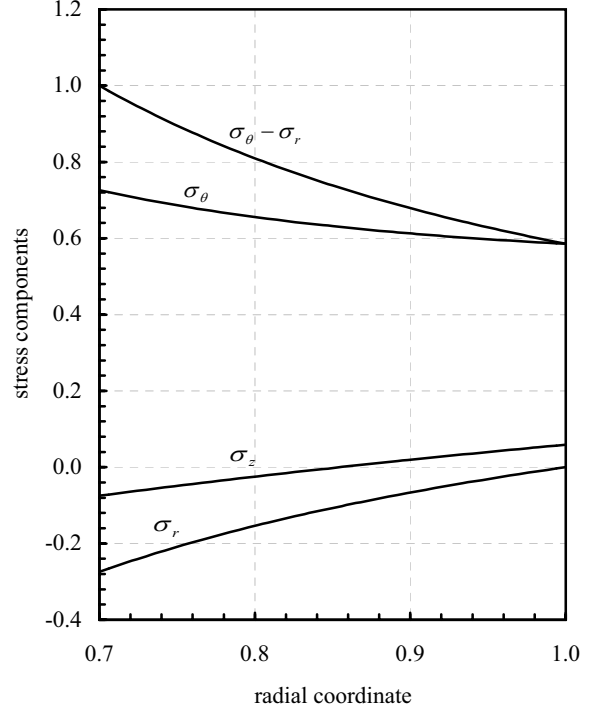
$$\begin{aligned} F_P(\beta, \gamma) &= \int_{\beta}^{\gamma} \sigma_z r dr = -(\beta^2 - \gamma^2) \left[ \frac{\epsilon_z}{2} - \frac{\alpha(T_b - T_a)}{4 \ln a} + \nu C_4 \right] \\ &+ \frac{\nu}{2} \beta^2 \left[ H(1 - \nu^2) \ln \beta - \sqrt{D_4} \ln \left( 2\beta\sqrt{D_4} + 2\sqrt{2HC_3 + \beta^2 D_4} \right) \right] \\ &+ \frac{\nu}{2} \gamma^2 \left[ -H(1 - \nu^2) \ln \gamma + \sqrt{D_4} \ln \left( 2\gamma\sqrt{D_4} + 2\sqrt{2HC_3 + \gamma^2 D_4} \right) \right] \\ &+ \beta\sqrt{2HC_3 + \beta^2 D_4} - \gamma\sqrt{2HC_3 + \gamma^2 D_4} + \frac{\alpha(\beta^2 \ln \beta - \gamma^2 \ln \gamma) T_a}{2 \ln a} \\ &+ \frac{\alpha(\gamma^2 \ln \gamma/a - \beta^2 \ln \beta/a) T_b}{2 \ln a}. \end{aligned} \quad (39)$$

## Numerical Results

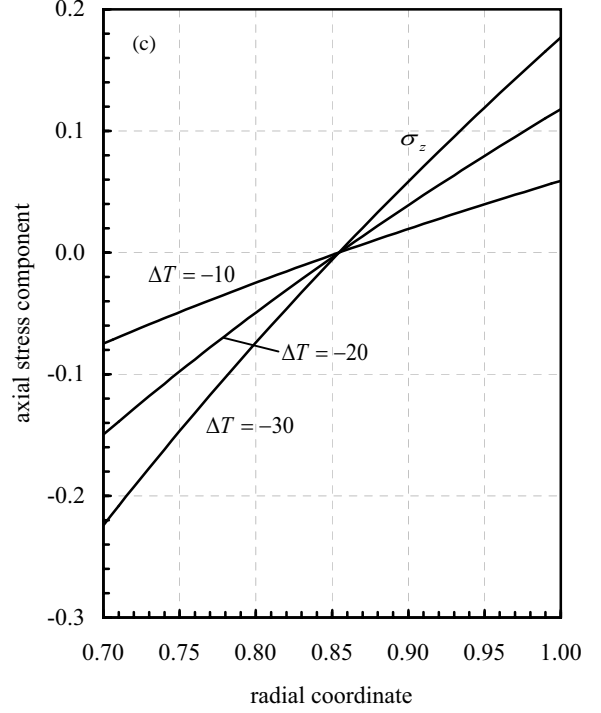
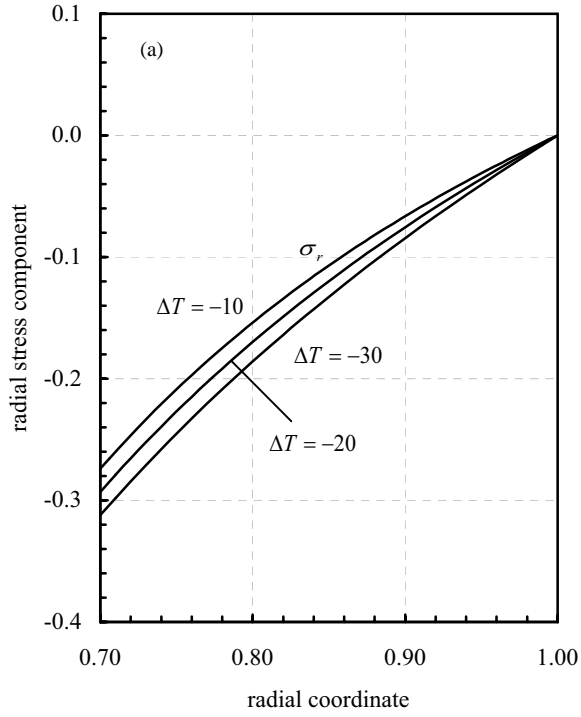
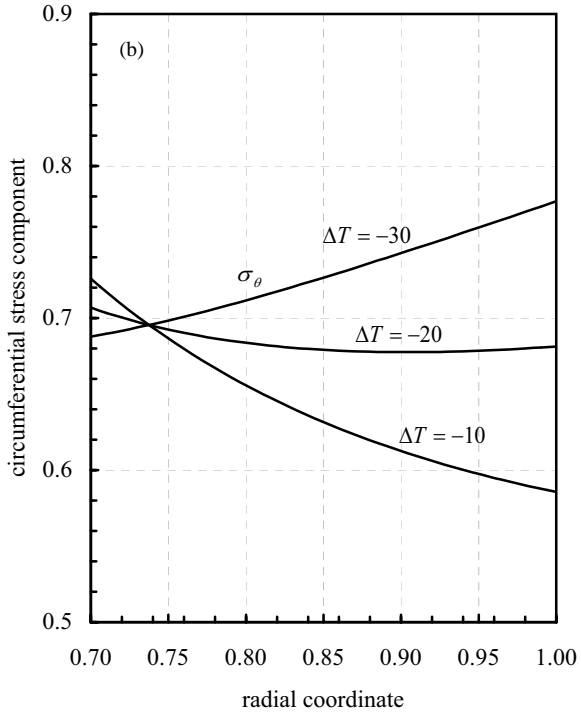
In the following calculations the material properties of steel are used. Hence,  $\nu = 0.3$  and  $\bar{\alpha} = \alpha E / \sigma_0 = 11.7 \times 10^{-6} \times 200 \times 10^9 / 250 \times 10^6 = 9.36 \times 10^{-3} \text{ 1/}^\circ\text{C}$ . First, the elastic response of a tube of  $a = 0.7$  is studied. Assigning  $T_a = 15 \text{ }^\circ\text{C}$  and  $T_b = 5 \text{ }^\circ\text{C}$  ( $\Delta T = -10 \text{ }^\circ\text{C}$ ) and using Eq. (18) we calculate  $P_E = 0.274059$ . Moreover, from Eqs. (15)-(17) we obtain  $C_1 = -0.199076$ ,  $C_2 = 0.199076$ ,  $\epsilon_z = -0.699042 \times 10^{-3}$ . The consequent distributions of the stresses are plotted in Figure 2. As seen in this figure, the plastic flow has just begun at  $r = a$  since  $\sigma_{\theta}(a) - \sigma_r(a) = 1$ , and also the stress state satisfies  $\sigma_{\theta} > \sigma_z > \sigma_r$  throughout. The effect of  $\Delta T$  on the maximum elastic stresses can be examined in Figure 3(a), (b), and (c). The stresses corresponding to  $\Delta T = -10 \text{ }^\circ\text{C}$  come from the above calculation, the others are drawn at their elastic limits keeping  $T_b$  at  $5 \text{ }^\circ\text{C}$  and changing  $T_a$  accordingly.

A tube of inner radius  $a = 0.7$  having the surface temperatures held at  $T_a = 30 \text{ }^\circ\text{C}$  and  $T_b = 5 \text{ }^\circ\text{C}$  undergoes plastic deformation when  $P = P_E = 0.302646$ . For pressures  $P > P_E$  the tube behaves as a partially plastic material. The plastic region initiated at  $r = a$  propagates into the tube with increasing pressures. The solution of this elastoplastic problem requires the evaluation of 6 unknowns, namely  $C_3$ ,  $C_4$  (plastic constants),  $r_{EP}$  (plastic-elastic border radius),  $C_1$ ,  $C_2$  (elastic constants), and  $\epsilon_z$ . There exist 6 nonredundant conditions for the evaluation. Five of them are:  $\sigma_r^p(a) = -P$ ,  $u^p(r_{EP}) = u^e(r_{EP})$ ,

$\sigma_r^p(r_{EP}) = \sigma_r^e(r_{EP})$ ,  $\sigma_{\theta}^p(r_{EP}) - \sigma_r^e(r_{EP}) = 1$ ,  $\sigma_r^e(1) = 0$ , with the superscripts  $p$  and  $e$  implying the plastic and elastic regions, respectively. The remaining condition is



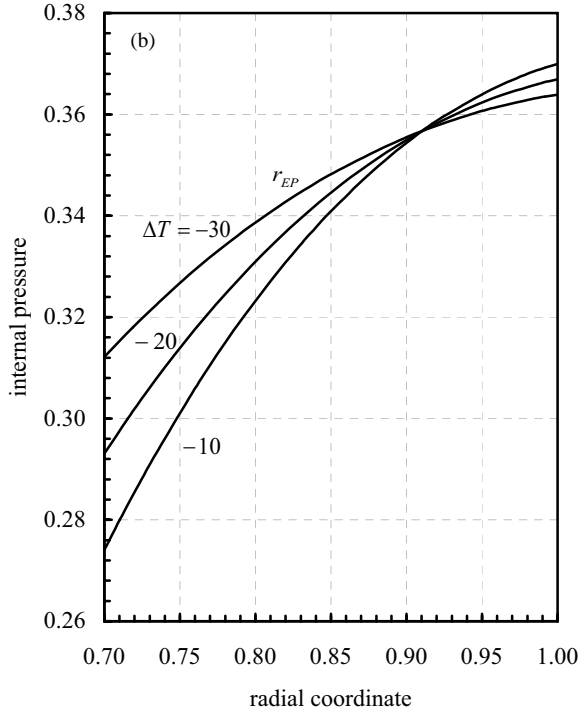
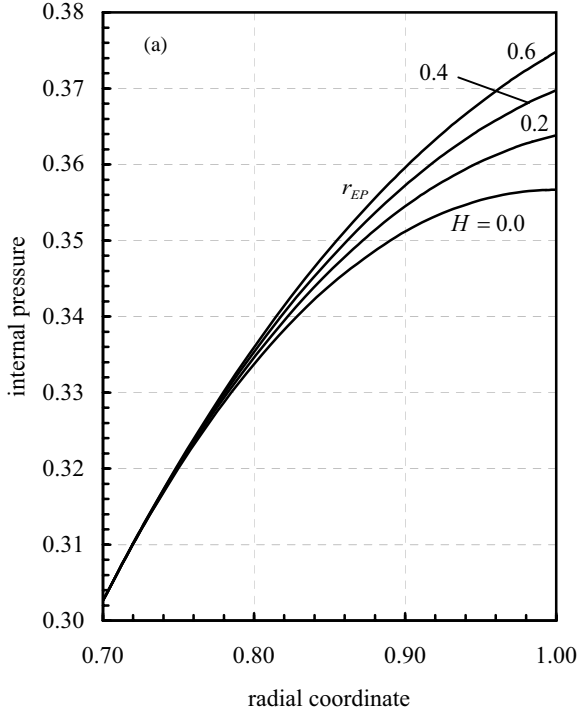
**Figure 2.** Elastic response of a tube of inner radius  $a = 0.7$  subjected to an internal pressure of  $P = 0.274059$  and  $\Delta T = -10 \text{ }^\circ\text{C}$ .


**Figure 3.** Continued.


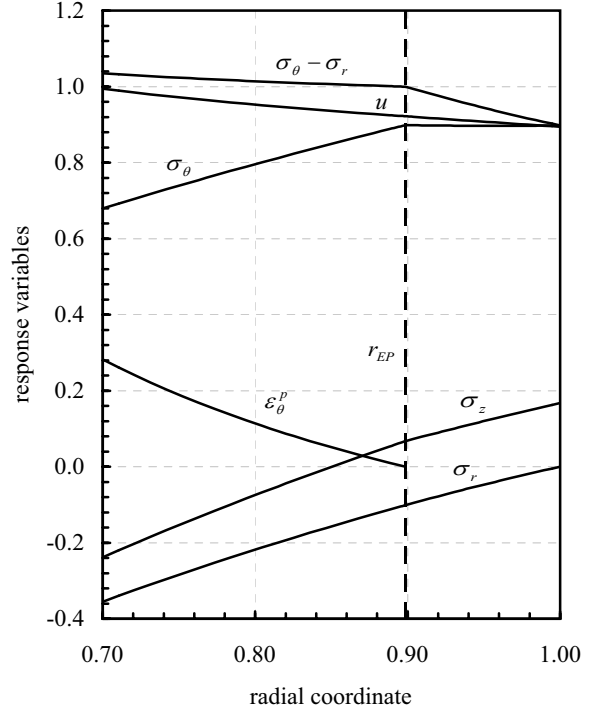
**Figure 3.** Effect of  $\Delta T$  on the maximum elastic stresses. Variation of (a) radial stress component, (b) circumferential stress component, (c) axial stress component with  $\Delta T$ .

$$\int \sigma_z dA = 2\pi \int_a^{r_{EP}} \sigma_z^p r dr + 2\pi \int_{r_{EP}}^1 \sigma_z^e r dr = 0 \quad (40)$$

Since the force integrals have already been evaluated in the preceding sections, this last condition takes simply the form  $F_P(a, r_{EP}) + F_E(r_{EP}, 1) = 0$ . The imposition of these conditions leads to a  $6 \times 6$  highly nonlinear system of equations, the simultaneous solution of which is achieved by Newton iterations. The spread of the plastic zone into the tube from  $a$  to  $r_{EP}$  with increasing pressures is calculated for different values of the hardening parameter  $H$  and the results are plotted in Figure 4(a). The curve for  $H = 0$  is obtained using the perfectly plastic solution presented in Appendix A. For all  $P_E = 0.302646$ , but the fully plastic limits  $P_{FP}$  differ notably. The calculated plastic limits are  $P_{FP} = 0.356675, 0.363838, 0.369783,$  and  $0.374821$  for  $H = 0, 0.2, 0.4,$  and  $0.6$ , respectively. The effect of  $\Delta T$  on the rate of propagation of the plastic region for  $H = 0.25$  can be examined in Figure 4(b). As stated earlier, for  $\Delta T < 0$  the increase in the magnitude of  $\Delta T$  increases the elastic limit pressure  $P_E$ . In contrast, the fully plastic limits decrease with increasing  $|\Delta T|$ . However, the effect of  $\Delta T$  on  $P_E$  is highly pronounced in comparison to that on  $P_{FP}$ , as seen in Figure 4(b).



**Figure 4.** Propagation of elastic-plastic border with increasing pressures for  $a = 0.7$  (a)  $\Delta T = -25$  °C using  $H$  as a parameter, (b)  $H = 0.25$  using  $\Delta T$  as a parameter.

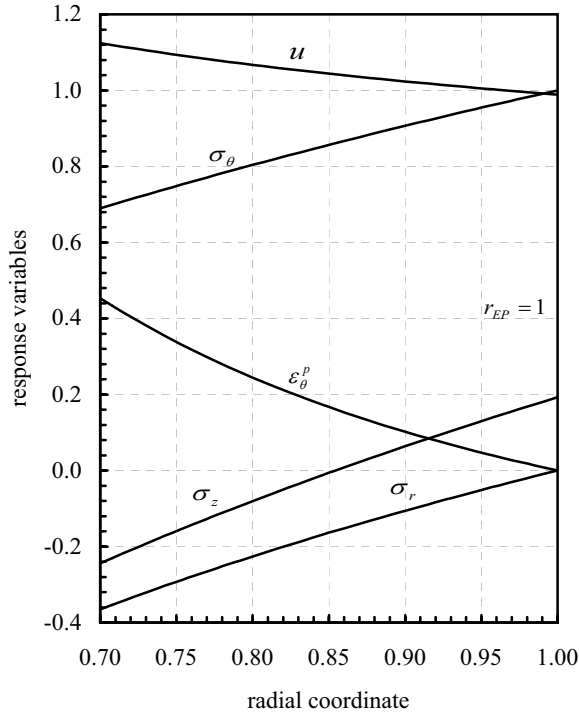


**Figure 5.** Partially plastic response of a tube of inner radius  $a = 0.7$  subjected to an internal pressure of  $P = 0.355$  and  $\Delta T = -25$  °C for  $H = 0.25$ .

For the pressure tube with parameters  $a = 0.7$ ,  $T_a = 30$  °C, and  $T_b = 5$  °C we have determined  $P_E = 0.302646$  above. Assigning  $P = 0.355 > P_E$  and taking  $H = 0.25$  an elastoplastic calculation is carried out. We find  $C_3 = 0.780949$ ,  $C_4 = -1.17398$ ,  $r_{EP} = 0.898609$ ,  $C_1 = -0.214546$ ,  $C_2 = 0.214546$ , and  $\epsilon_z = -0.546408 \times 10^{-3}$ . The corresponding distributions of the response variables are shown in Figure 5. The tube becomes fully plastic when the internal pressure reaches  $P = P_{FP} = 0.365424$  for  $H = 0.25$ . Under  $P = 0.365424$  we evaluate  $C_3 = 0.967123$ ,  $C_4 = -1.17369$ ,  $r_{EP} = 1.0$ , and  $\epsilon_z = -0.606502 \times 10^{-3}$ . The consequent fully plastic state of stress is depicted in Figure 6.

## Conclusion

It is apparent that closed form solutions to simplified versions of real engineering problems are important as they not only facilitate the analysis of limiting cases but also provide benchmark results for the sophisticated FEM computer programs. In this short article, the closed form solution to a nonlinear elastoplastic problem has been presented. A sufficiently



**Figure 6.** Fully plastic response of a tube of inner radius  $a = 0.7$  subjected to an internal pressure of  $P = 0.365424$  and  $\Delta T = -25$  °C for  $H = 0.25$ .

long, thick-walled pressure tube subjected to a small negative temperature gradient has been considered. It has been shown under these conditions that the stress state in the tube satisfies  $\sigma_\theta > \sigma_z > \sigma_r$  throughout and the analytical solution to a nonlinear hardening law could be found by using incremental theory of plasticity, Tresca’s yield criterion and the associated flow rule. With this solution, the elastic, partially plastic, and fully plastic stress states have been computed and presented graphically.

**Nomenclature**

$a, b$	inner and outer radii of the tube
$C_i$	integration constant
$E$	modulus of elasticity
$F$	axial force
$H$	nondimensional hardening parameter
$P$	pressure
$r, \theta, z$	cylindrical coordinates
$T$	temperature
$u$	radial displacement
$\alpha$	coefficient of thermal expansion
$\nu$	Poisson’s ratio
$\epsilon_{EQ}$	equivalent plastic strain
$\epsilon_i$	normal strain component in $i$ -direction
$\sigma_i$	normal stress component in $i$ -direction
$\sigma_Y$	yield stress

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## APPENDIX A

### Solution for a Perfectly Plastic Material

In the expressions below, the boundary condition  $\sigma_r(a) = -P$  has already been used to eliminate one of the integration constants and  $D_3 = \alpha(1 + \nu)(T_b - T_a)/\ln a$ .

$$\sigma_r = -P + \ln(r/a), \quad (41)$$

$$\sigma_\theta = 1 - P + \ln(r/a), \quad (42)$$

$$\sigma_z = \epsilon_z + \nu \{1 + 2[-P + \ln(r/a)]\} + \frac{D_3 \ln r}{1 + \nu} - \alpha T_b, \quad (43)$$

$$u = \frac{C_3}{r} + rD_5[-P + \ln(r/a)] + \frac{D_3 r}{2}(1 - 2 \ln r) + rT_b \alpha(1 + \nu) - r\nu\epsilon_z, \quad (44)$$

$$\epsilon_\theta^p = \frac{C_3}{r^2} - 1 + \nu^2 + \frac{D_3}{2}, \quad (45)$$

$$F_P(\beta, \gamma) = -\nu(\beta^2 \ln \beta - \gamma^2 \ln \gamma) + (\beta^2 - \gamma^2) \left[ \nu(P + \ln a) + \frac{\alpha T_b - \epsilon_z}{2} \right] + \frac{D_3 [\beta^2(1 - 2 \ln \beta) - \gamma^2(1 - 2 \ln \gamma)]}{4(1 + \nu)}. \quad (46)$$

**APPENDIX B**

## Solution for a Linear Strain Hardening Material

In the expressions below  $D_3 = \alpha(1 + \nu)(T_b - T_a)/\ln a$  and  $D_5 = (1 + \nu)(1 - 2\nu)$  as in the text.

$$\sigma_r = -\frac{C_3}{2r^2} + C_4 + \frac{\ln r}{1 + H(1 - \nu^2)} + \frac{HD_3 \ln r}{2[1 + H(1 - \nu^2)]}, \quad (47)$$

$$\sigma_\theta = \frac{C_3}{2r^2} + 2C_4 + \frac{1 + \ln r}{1 + H(1 - \nu^2)} + \frac{HD_3(1 + \ln r)}{2[1 + H(1 - \nu^2)]}, \quad (48)$$

$$\sigma_z = 2\nu C_4 + \epsilon_z + \frac{\nu(1 + 2 \ln r)(1 + HD_3/2)}{1 + H(1 - \nu^2)} - \alpha T(r), \quad (49)$$

$$u = \frac{[2 + H(1 + \nu)]C_3}{2Hr} + rD_5C_4 - r\nu\epsilon_z + \frac{rD_5 \ln r(2 + HD_3)}{2[1 + H(1 - \nu^2)]} + \frac{rD_3(1 - 2 \ln r)}{2} \quad (50)$$

$$+ T_b r \alpha (1 + \nu), \quad (51)$$

$$\epsilon_\theta^p = -\epsilon_r^p = \frac{C_3}{Hr^2} - \frac{1}{H} + \frac{1}{H[1 + H(1 - \nu^2)]} + \frac{D_3}{2[1 + H(1 - \nu^2)]}, \quad (52)$$

$$F_P(\beta, \gamma) = -\frac{\beta^2 - \gamma^2}{2} \left[ 2\nu C_4 + \epsilon_z - \alpha T_b - \frac{D_3}{2(1 + \nu)} \right] - \frac{(\beta^2 \ln \beta - \gamma^2 \ln \gamma) D_3}{2(1 + \nu)} - \frac{\nu(\beta^2 \ln \beta - \gamma^2 \ln \gamma)(2 + HD_3)}{2[1 + H(1 - \nu^2)]}. \quad (53)$$