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An Approximate Torsion Analysis of Closed Moderately Thick-Walled, Thick-Walled, and Solid Cross-Sections

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Abstract

An approximated model and mathematical formulation are presented for the uniform torsion analysis of closed moderately thick-walled, thick-walled, and solid cross-sections. For this purpose, the considered section is ideally divided into thin-walled closed strips one inside the other. Using this model and Bredt's theory, the torsional moments carried by each strip are obtained. Then, considering that the number of the imaginary strips is sufficiently high, formulas are derived for the maximum shearing stress and the angle of twist of the considered cross-section. To demonstrate the accuracy and efficiency of the formulation, results for several cross-sections having exact or numerical solutions are presented. These results show that the formulation gives accurate values for solid and hollow sections with regular curvilinear contours. On the other hand, although for the thin-walled and moderately thick-walled sections there is high accuracy in the results, for the thick-walled and solid ones the accuracy decreases gradually. Thus, it is thought that the derived formulas can be useful especially for the analysis of a wide class of closed thin-walled and moderately thick-walled cross-sections subjected to torsional loading.

Key Words: Uniform torsion, Approximate analysis, Moderately thick-walled sections, Thick-walled sections, Solid sections.

Introduction

Torsion is an important factor in the design of some load carrying elements such as shafts, curved beams, edge beams in buildings, and eccentrically loaded bridge girders. Therefore, the uniform (St. Venant) and non-uniform torsion problem of structural components has long been the subject of theoretical and practical interest in the field of solid mechanics (Chen et al., 2001).

Structural elements with very different cross-sectional shapes are widely used in various engineering structures. The exact solutions for torsion have been found for some simple cross-sectional shapes such as circle, ellipse, and triangle. In the theory of elasticity, unfortunately, it is difficult to obtain analytical solutions for complicated cross-sections. To solve general cross-sectional problems, numerical methods are usually necessary. The widely used numerical methods are the finite element method (Herrmann, 1965; Li et al., 2000; Jiang and Henshall, 2002; Darılmaz et al., 2007), the finite difference method (Ely and Zienkiewicz, 1960), and the boundary element method (Friedman and Kosmatka, 2000; Sapountzakis, 2001; Sapountzakis and Mokos, 2004). The first 2 of these methods require the whole cross-section to be discretised into elements or grids. For a complicated section, both of these methods necessitate a large number of elements or grids. The boundary element method requires discretisation of the boundary only, but in this method one has a singular integral on the boundary. Recently, mesh-free methods have developed fast as alternative solution methods (Kołodziej and Fraska, 2005; Di Paola et al., 2008).

In the solution of torsional problems the analogy methods such as membrane, electrical, and fluid flow analogies have also been used (Zhen-min and Ke-xue, 1986). In analysing complex sections, these techniques may be time consuming and the accuracy of the results depends largely on the experimental procedure followed.

The difficulty of solving accurately the torsional problem of generally shaped cross-sections has driven some researchers to simpler approximate calculations or approximate models. Lamancusa and Saravanos (1989) performed the torsional analysis of hollow square tubes with finite elements, using a 2-dimensional thermal analogy. They investigated the dependence of torsional properties on wall thickness. Serra (1996) using an approximated model and starting from the well-known Bredt's formulas (valid only for thin-walled closed sections) obtained a formulation for the calculation of the torsional problem of solid cross-sections. Wang (1998) introduced the method of eigenfunction expansion and matching to solve the torsion problem of arbitrary shaped tubes described by curved and straight pieces. Najera and Herrera (2005) presented a method to approximate the torsional rigidity of non-circular solid cross-sections encountered in mechanisms and machines. Hematiyan and Doostfatemeh (2007) presented a simple formulation for torsion analysis of moderately thick hollow tubes with polygonal shapes.

In this paper, Serra's (1996) modelling process has been extended to model arbitrarily shaped closed moderately thick-walled and thick-walled cross-sections subjected to uniform torsion. This is of considerable importance for such sections, since these find application areas especially in structural and mechanical engineering. The developed formulation can also be applied to solid cross-sections.

The outline of this presentation is as follows. In the following section the analysis model and the derived formulation are described. Then, expressions for maximum shearing stress and the angle of twist are obtained. To verify the accuracy of the model and the expressions, results for some cross-sections having analytical or numerical solutions are presented in the subsequent section. The conclusions obtained are given in the last section.

Analysis Model and Formulation

An arbitrarily shaped thick-walled closed cross-section subjected to a torsional moment T is shown in Figure 1(a). The domain and the boundaries of the cross-section are denoted by Ω, Γ_{out}, and Γ_{inn}, respectively. The point C is the centre of rotation of the cross-section and the origin of the x and y axes. The material of the cross-section is assumed to be homogeneous, isotropic, and linearly elastic with a shear modulus G. Moreover, it must be pointed out that the formulation developed in the following, which is based on Bredt's theory, does not consider the cross-sectional warping and ignores the stress concentrations.

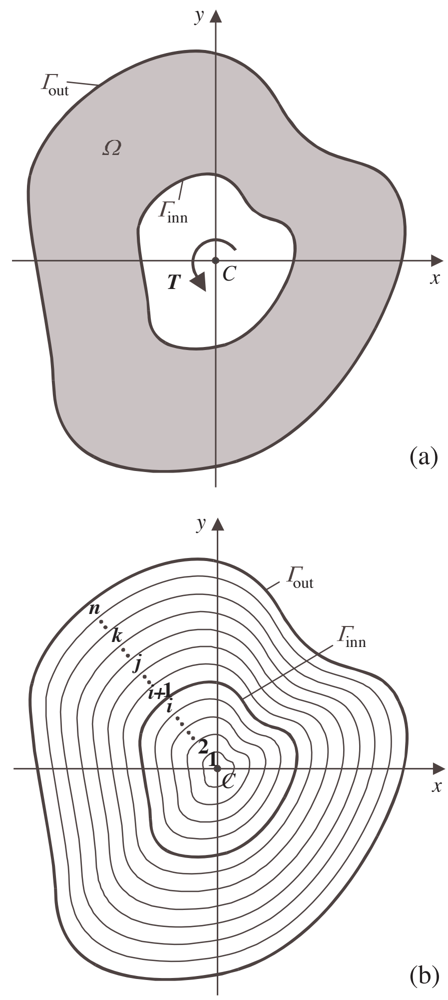

Figure 1. (a) An arbitrary shaped thick-walled cross-section under torsional moment, (b) discretisation of the section into imaginary closed strips.

The whole cross-section is divided into n imaginary closed strips, all having the same thickness over a radial line, and numbered from 1 to n (Figure 1(b)). It must be noted that the numeration of the strips is started from the core of the hole and therefore, while i is the last strip of the hollow region, i+1 is the first strip of the solid part of the cross-section. The mathematical relation between any strip and the outer and inner boundaries of the section is called homothety.

Figure 2 compares the jth and nth strips of the cross-section. Suppose that P is a point on the median curve of the nth strip. For the calculation, the following definitions are used: Γ_n is the median curve of the nth strip, ρ_n is the segment CP, dS_n is the arch element over Γ_n, and A_n is the area bounded by Γ_n. Assume also that Γ_j, ρ_j, dS_j, and A_j are the same quantities related to the jth strip (Figure 2).

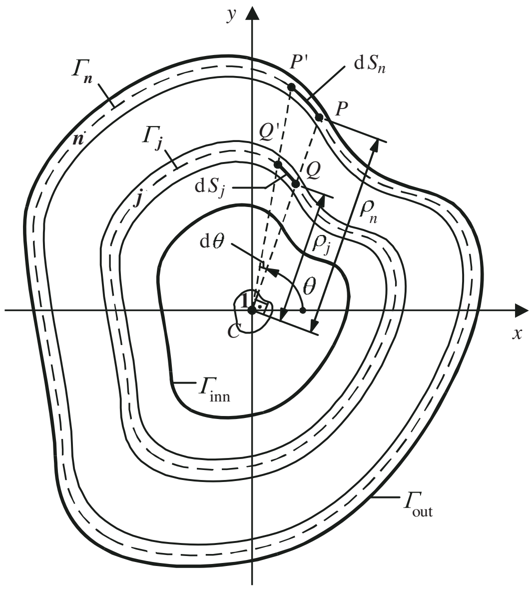

Figure 2. Two strips of the cross-section.

Due to the similarity between the $CQQ\prime$ and $CPP\prime$ triangles, one can write

$$\rho_j\left(\theta\right) = \frac{j}{n}\rho_n\left(\theta\right), dS_j = \frac{j}{n}dS_n \qquad (1\text{a, b})$$

$$(j = i+1, i+2, \ldots, n)$$

Moreover, from the knowledge of differential geometry, the areas surrounded by the jth and nth strips are

$$A_j = \frac{1}{2}\oint_0^{2\pi}\rho_j^2\left(\theta\right)d\theta, A_n = \frac{1}{2}\oint_0^{2\pi}\rho_n^2\left(\theta\right)d\theta \qquad (1\text{c, d})$$

and taking into account Eq. (1a), one obtains

$$\frac{A_j}{A_n} = \left(\frac{j}{n}\right)^2 \rightarrow A_j = \left(\frac{j}{n}\right)^2 A_n \qquad (1\text{e})$$

From Figure 3(a) and considering the triangle CPN' in Figure 3(b), for the thickness δ_j of the jth strip, the following expression is obtained:

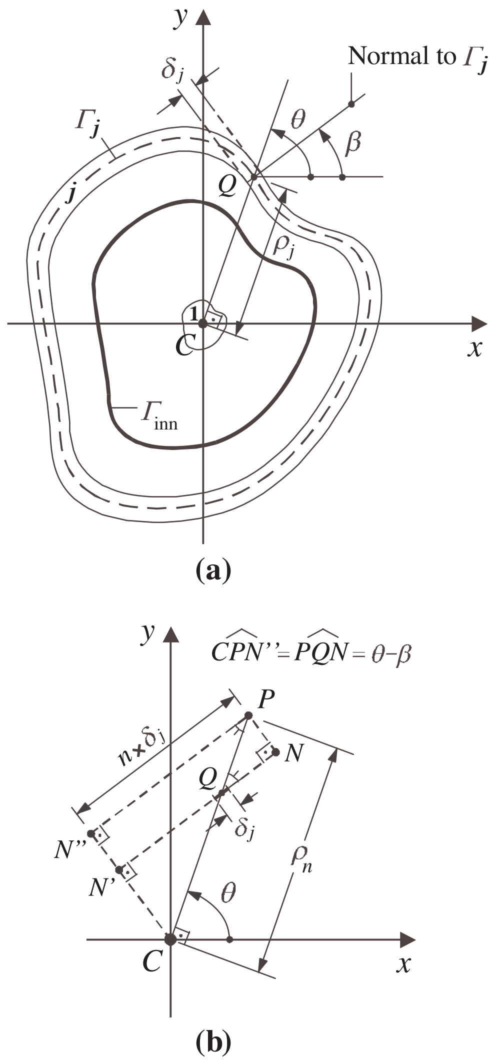

Figure 3. (a) The jth strip of the section, (b) the calculation of the thickness, δ_j, of this strip.

$$\cos\left(\theta-\beta\right) = \frac{n\delta_j}{\rho_n\left(\theta\right)} \rightarrow$$

$$\delta_j = \delta = \frac{\rho_n\left(\theta\right)}{n}\cos\left(\theta-\beta\right) \qquad (2)$$

Let us consider now that the jth and kth strips take the torsional moments ΔT_j and ΔT_k from the total moment T. At this stage, using Bredt's second formula, the angles of twist ω_j and ω_k of the jth and kth strips can be written, respectively, as

$$\omega_j = \frac{\Delta T_j}{4GA_j^2} \oint_{S_j} \frac{dS_j}{\delta} \tag{3a}$$

$$(j = i+1, i+2, \ldots, n)$$

$$\omega_k = \frac{\Delta T_k}{4GA_k^2} \oint_{S_k} \frac{dS_k}{\delta} \tag{3b}$$

$$(k = i+1, i+2, \ldots, n)$$

In these equations S_j and S_k are the perimeters of the jth and kth strips, respectively. On the other hand, according to the compatibility condition of the problem, one can write

$$\omega_j = \omega_k = \omega \tag{4}$$

$$(j \neq k, j, k = i+1, i+2, \ldots, n)$$

which implies that all the strips rotate at the same angle ω. Now, substituting Eqs. (3a) and (3b) into Eq. (4),

$$\frac{\Delta T_j}{4GA_j^2} \oint_{S_j} \frac{dS_j}{\delta} = \frac{\Delta T_k}{4GA_k^2} \oint_{S_k} \frac{dS_k}{\delta} \tag{5a}$$

$$(j \neq k, j, k = i+1, i+2, \ldots, n)$$

and considering Eq. (1b), one obtains

$$\frac{\Delta T_j}{A_j^2} j = \frac{\Delta T_k}{A_k^2} k \tag{5b}$$

Substitution of Eq. (1e) into Eq. (5b) gives

$$\frac{\Delta T_j}{j^3} = \frac{\Delta T_k}{k^3} \tag{6}$$

Now, when the index k is taken equal to the i+1, Eq. (6) gives

$$\Delta T_j = \left(\frac{j}{i+1}\right)^3 \Delta T_{i+1} \tag{7a}$$

and

$$\Delta T_n = \left(\frac{n}{i+1}\right)^3 \Delta T_{i+1} \tag{7b}$$

On the other hand, torsional moment equilibrium of the cross-section yields

$$T = \sum_{j=i+1}^{n} \Delta T_j \tag{8}$$

Substituting Eq. (7a) into Eq. (8), and considering that

$$\sum_{j=i+1}^{n} j^3 = \sum_{j=0}^{n} j^3 - \sum_{j=0}^{i} j^3 = \left[\frac{n(n+1)}{2}\right]^2 - \left[\frac{i(i+1)}{2}\right]^2 \tag{9}$$

one obtains

$$\Delta T_{i+1} = \frac{4T(i+1)^3}{[n(n+1)]^2 - [i(i+1)]^2} \tag{10}$$

and then substituting this expression into Eq. (7b), one arrives at the following expression

$$\Delta T_n = \frac{4Tn^3}{[n(n+1)]^2 - [i(i+1)]^2} \tag{11}$$

which is the torsional moment carried by the nth strip of the cross-section.

Calculation of Maximum Shear Stress and the Angle of Twist

In this section, firstly the expression of shearing stresses on the boundary of a general cross-section is derived and then the angle of twist per unit length is obtained.

Shear stresses on the outer boundary of the cross-section

For the calculation of the shear stresses on the outer boundary of the section, now consider the outer strip, i.e. the nth strip of the section. This strip carries the torsional moment ΔT_n. At this stage it is convenient to use Bredt's first formula in the following:

$$\tau_n = \frac{\Delta T_n}{2\delta_n A_n} \tag{12}$$

By substituting the expressions of ΔT_n and δ_n from Eqs. (11) and (2) into the above equation, one obtains

$$\tau_n(\theta) = \frac{2T}{\rho_n(\theta) A_n \cos(\theta - \beta)_n} \times \frac{n^4}{[n(n+1)]^2 - [i(i+1)]^2} \tag{13}$$

When the section is divided ideally into the sufficiently high number of strips, i.e. the number n is high enough, such as 1000, 10,000, 50,000, one has

$$\left.\begin{array}{r}\rho_n \to \bar{\rho}, A_n \to \bar{A}\\ \cos(\theta-\beta)_n \to \cos\overline{(\theta-\beta)}\end{array}\right\} \tag{14}$$

where $\bar{\rho}$: radius of the boundary (segment CP, Figure 2), $\bar{A}$: area of the section, and $\cos\overline{(\theta-\beta)}$: value of $\cos(\theta-\beta)$ on the point P (Figure 2). Therefore, the expression for the shearing stress τ on the boundary becomes

$$\tau(\theta) = \frac{2T}{\bar{\rho}\bar{A}\cos\overline{(\theta-\beta)}} \frac{n^4}{[n(n+1)]^2 - [i(i+1)]^2} \tag{15}$$

If the cross-section is a solid section, i.e. if it does not have any hole, in this case $i = 0$, and taking $n \to \infty$, one has

$$\lim_{n\to\infty} \frac{n^4}{[n(n+1)]^2} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 \to 1 \tag{16}$$

and hence Eq. (15) takes the following form

$$\tau(\theta) = \frac{2T}{\bar{\rho}\bar{A}\cos\overline{(\theta-\beta)}} \tag{17}$$

which coincides with Serra's (1996) shearing stress expression. By using Eqs. (15) and (17), respectively, the shearing stress at any point on the contour, and at the same time the maximum shearing stress, of a closed moderately thick-walled, thick-walled, or solid cross-section can be calculated.

Angle of twist of the cross-section

To arrive at an expression for the angle of twist ω of the cross-section, let us consider Eq. (3a)

$$\omega_j = \frac{\Delta T_j}{4GA_j^2} \oint_{S_j} \frac{dS_j}{\delta} \quad (j = i+1, i+2, \ldots, n)$$

From the differential geometry, the infinitesimal arch element $\mathrm{d}S_j$ can be written in polar coordinates as

$$dS_j = \sqrt{\rho_j'^2 + \rho_j^2}\, d\theta \tag{18}$$

where the prime denotes derivation with respect to θ, and hence Eq. (3a) becomes

$$\omega_j = \frac{\Delta T_j}{4GA_j^2} \oint_{\theta=0}^{\theta=2\pi} \frac{\sqrt{\rho_j'^2 + \rho_j^2}}{\frac{\rho_n}{n}\cos(\theta-\beta)} d\theta \tag{19}$$

By substituting Eqs. (1a), (1e), and (7a) into Eq. (19) and taking into account Eq. (10), one arrives at

$$\omega = \frac{T}{GA_n^2} \frac{n^4}{[n(n+1)]^2 - [i(i+1)]^2} \times \oint_0^{2\pi} \frac{\sqrt{\rho_n'^2 + \rho_n^2}}{\rho_n \cos(\theta-\beta)} d\theta \tag{20}$$

Again, considering that the cross-section is divided into a large number of strips, one can write the above equation in the following form

$$\omega = \frac{T}{G\bar{A}^2} \frac{n^4}{[n(n+1)]^2 - [i(i+1)]^2} \times \oint_0^{2\pi} \frac{\sqrt{\bar{\rho}'^2 + \bar{\rho}^2}}{\bar{\rho}\cos\overline{(\theta-\beta)}} d\theta \tag{21}$$

In the case of a solid section i becomes zero, and taking the limit as $n \to \infty$, Eq. (21) takes finally the following form:

$$\omega = \frac{T}{G\bar{A}^2} \oint_0^{2\pi} \frac{\sqrt{\bar{\rho}'^2 + \bar{\rho}^2}}{\bar{\rho}\cos\overline{(\theta-\beta)}} d\theta \tag{22}$$

which coincides with Serra's (1996) angle of rotation expression.

Application to Several Cross-Sections and Discussion

To verify the accuracy and efficiency of the derived expressions of (15), (17), (21), and (22), several cross-sections having analytical or numerical solutions are chosen. Firstly 6 solid sections and then 3 hollow sections are considered.

Solid cross-sections

a) Circular cross-section with radius r, Figure 4: For this section $\bar{\rho} = r$, $\bar{A} = \pi r^2$, and $\theta = \beta \to \cos\overline{(\theta-\beta)} = 1$. Thus, from Eqs. (17) and (22) one obtains

$$\tau_{\max}^{\text{appr}} = \frac{2T}{\pi r^3}, w^{\text{appr}} = \frac{2T}{G\pi r^4}$$

which are equal to the exact solutions.

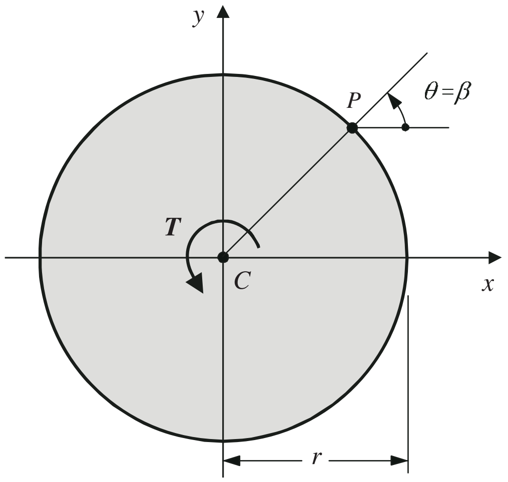

Figure 4. Solid circular cross-section.

b) Elliptical cross-section with diameters 2a and 2b, (a > b), Figure 5: For the ellipse $\bar{A} = \pi ab$, and in this section maximum shearing stresses occur at the end points, P_t and P_b, of the minor axis, Figure 5. At these points $\bar{\rho} = b$ and $\theta = \beta \rightarrow \cos\overline{(\theta - \beta)} = 1$; therefore, from Eq. (17) one finds

$$\tau_{\max}^{\text{appr}} = \frac{2T}{\pi ab^2}$$

which is equal to the exact value (İnan, 1988). As for the angle of twist, by the properties of an ellipse, the following expressions can be written

$$\rho(\theta) = \left[\frac{a^2 \left(1 + \tan^2 \theta\right)}{1 + (a/b)^2 \tan^2 \theta} \right]^{1/2} \tag{23a}$$

$$\cos(\theta - \beta) = \frac{\left[1 + (a/b)^2 \tan^2 \theta\right]}{\left\{ \left(1 + \tan^2 \theta\right) \left[1 + (a/b)^4 \tan^2 \theta\right] \right\}^{1/2}} \tag{23b}$$

and therefore from Eq. (22)

$$\omega = \frac{T}{G\,(\pi ab)^2} \times \int_0^{2\pi} \frac{\left(1 + \tan^2 \theta\right) \left[1 + (a/b)^4 \tan^2 \theta\right]}{\left[1 + (a/b)^2 \tan^2 \theta\right]^2} d\theta = \frac{T}{G\left(\dfrac{\pi a^3 b^3}{a^2 + b^2}\right)}$$

which is equal to the exact solution of the theory of elasticity (İnan, 1988).

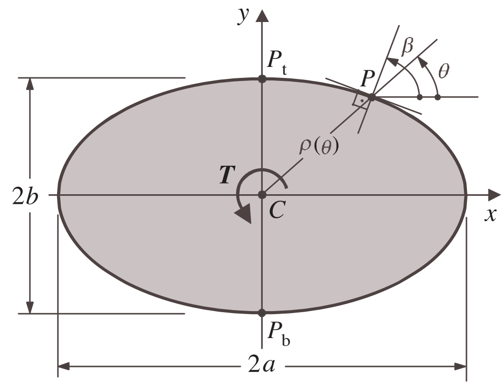

Figure 5. Solid elliptical cross-section.

c) Regular octagonal cross-section with a side of a, Figure 6: In this section, maximum stress is obtained at the midpoints of the sides where $\bar{\rho} = 1.2071a$ and $\cos\overline{(\theta - \beta)} = 1$. For the section $\bar{A} = 4a\bar{\rho} = 4.8284a^2$ and therefore

$$\tau_{\max}^{\text{appr}} = \frac{2T}{(1.2071a)\,(4.8284a^2)\,1} \cong 0.34315 \frac{T}{a^3}$$

The theory of elasticity gives the following maximum stress value for the section (İnan, 1988):

$$\tau_{\max}^{\text{exact}} = 0.38643 \frac{T}{a^3}$$

Thus, the relative error $(\tau_{\max}^{\text{exact}} - \tau_{\max}^{\text{appr}})/\tau_{\max}^{\text{exact}}$ is obtained as 11.20%.

For the angle of twist, after determination of $\rho(\theta)$ in each of the 8 regions of the section, Eq. (22) yields

$$\omega^{\text{appr}} = \frac{T}{G\,(4.8284a^2)^2} \left[\int_{-\pi/8}^{\pi/8} \frac{\sqrt{\bar{\rho}'^2 + \bar{\rho}^2}}{\bar{\rho}\cos\overline{(\theta - \beta)}}\, d\theta + \cdots + \int_{13\pi/8}^{15\pi/8} \frac{\sqrt{\bar{\rho}'^2 + \bar{\rho}^2}}{\bar{\rho}\cos\overline{(\theta - \beta)}} d\theta \right]$$

$$= \frac{T}{G\,(4.8284a^2)^2} \left[0.828 + 0.828 + \cdots + 0.828\right]$$

$$= \frac{T}{G\,(4.8284a^2)^2} 8 \times 0.828 = 0.2841\, \frac{T}{Ga^4}$$

and exact solution is (İnan, 1988)

$$\omega^{\text{exact}} = 0.2733 \frac{T}{Ga^4}$$

The relative error for twist $|(\omega^{\text{exact}} - \omega^{\text{appr}})/\omega^{\text{exact}}|$ is obtained as 3.95%.

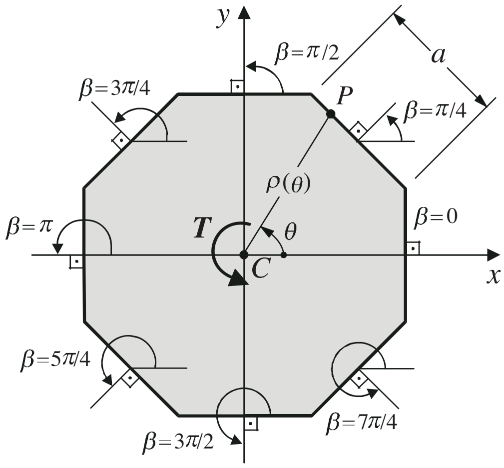

Figure 6. Solid regular octagonal cross-section.

d) Regular hexagonal, square, and equilateral triangular cross-sections, Figure 7: For these sections $\tau_{\max}^{\text{appr}}$ and ω^{appr} values are calculated and compared with the exact values (İnan, 1988) in the Table. As can be seen, from hexagon to triangle, the relative errors increase for both τ and ω.

Hollow cross-sections

a) Circular section with a co-centric circular hole: Consider a hollow circular cross-section with outer radius $r_{out} = r$ and inner radius $r_{inn} = 0.5\ r$, shown in Figure 8. Let us divide the cross-section into $n =$ 10,000 imaginary closed strips. Since the ratio of inner and outer radii is 0.5, i becomes 5000. As stated earlier i is the number of the last strip of the hollow region. Moreover, for this cross-section $\bar{\rho} = r$, $\cos\overline{(\theta - \beta)} = 1$, and $\bar{A} = \pi r^2$. Now, using Eqs. (15) and (21) one obtains

$$\tau_{\max}^{\text{appr}} = 1.06646\frac{2T}{\pi r^3}, \omega^{\text{appr}} = 1.06646\frac{2T}{G\pi r^4}$$

which are almost equal to the exact values of

$$\tau_{\max}^{\text{exact}} = 1.06666\frac{2T}{\pi r^3}, \omega^{\text{exact}} = 1.06666\frac{2T}{G\pi r^4}$$

b) Elliptical section with a co-centric elliptical hole: Consider an elliptical section with outer diameters $2a_{out}$, $2b_{out}$ and inner diameters $2a_{inn}$, $2b_{inn}$ (Figure 9). For such a cross-section, with $a_{inn}\ /\ a_{out} = b_{inn}\ /\ b_{out} = k < 1$, the theory of elasticity gives the following results (Timoshenko, 1970; İnan, 1988; Sadd, 2005):

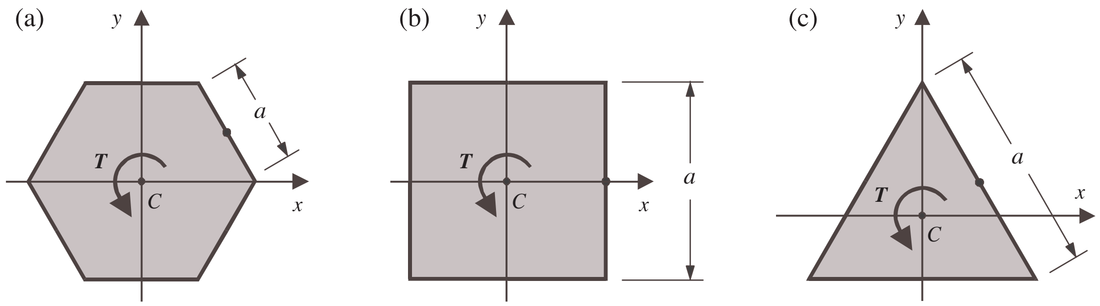

Figure 7. Some other solid sections; (a) regular hexagon, (b) square, (c) equilateral triangle.

Table. Comparison of approximate and exact $\tau_{\max}$ and ω values for hexagonal, square, and equilateral triangular cross-sections.

Cross-section	Max. shearing stress		Angle of twist		Relative errors, %	
	$\tau_{\max}^{\text{appr}}$	$\tau_{\max}^{\text{exact}}$	ω^{appr}	ω^{exact}	For τ	For ω
Hexagon, Figure 7(a)	0.8888 T/a^3	1.0193 T/a^3	1.0266 T/Ga^4	0.9628 T/Ga^4	12.808	6.628
Square, Figure 7(b)	4 T/a^3	4.8077 T/a^3	8 T/Ga^4	7.092 T/Ga^4	16.80	12.803
Triangle, Figure 7(c)	16 T/a^3	20 T/a^3	55. 424 T/Ga^4	46.188 T/Ga^4	20	20

$$\tau_{\max}^{\text{exact}} = \frac{2T}{\pi\, a_{out} b_{out}^2 \,(1-k^4)} \tag{24a}$$

$$\omega^{\text{exact}} = \frac{T}{G\pi \dfrac{a_{out}^3 b_{out}^3}{a_{out}^2 + b_{out}^2}\left(1-k^4\right)} \tag{24b}$$

Now, let us select a special case of $k = 0.60$. When the section is divided into $n = 10{,}000$ imaginary closed strips, i becomes 6000. For the section $\bar{\rho} = b_{out}$, $\cos\overline{(\theta-\beta)} = 1$, and $\bar{A} = \pi a_{out} b_{out}$. From Eqs. (15) and (21) one can obtain the τ and ω values as

$$\tau_{\max}^{\text{appr}} = 1.14869 \frac{2T}{\pi\, a_{out} b_{out}^2},$$

$$\omega^{\text{appr}} = 1.14869 \frac{T}{G\pi \dfrac{a_{out}^3 b_{out}^3}{a_{out}^2 + b_{out}^2}}$$

which are almost equal to the exact values obtained from Eqs. (24a) and (24b)

$$\tau_{\max}^{\text{exact}} = 1.14889 \frac{2T}{\pi\, a_{out} b_{out}^2},$$

$$\omega^{\text{exact}} = 1.14889 \frac{T}{G\pi \dfrac{a_{out}^3 b_{out}^3}{a_{out}^2 + b_{out}^2}}$$

c) Hollow square cross-section: As the last example, a hollow square section with uniform thickness, shown in Figure 10, is considered. Structural elements having this type of cross-section are widely used in numerous engineering systems, such as truss and framed structures, automotive chassis, mechanisms, and robot arms (Lamancusa and Saravanos, 1989).

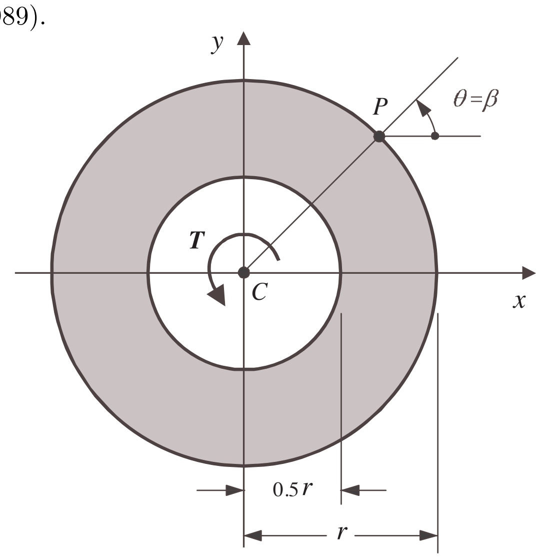

Figure 8. Hollow circular cross-section.

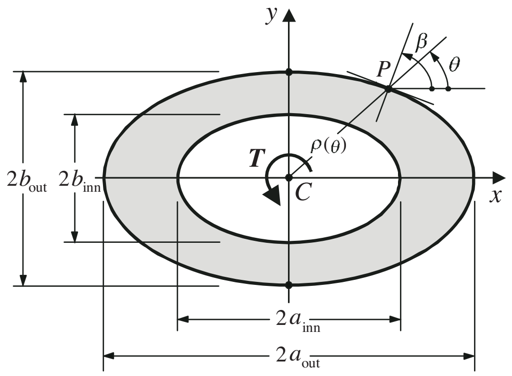

Figure 9. Elliptic section with elliptic hole.

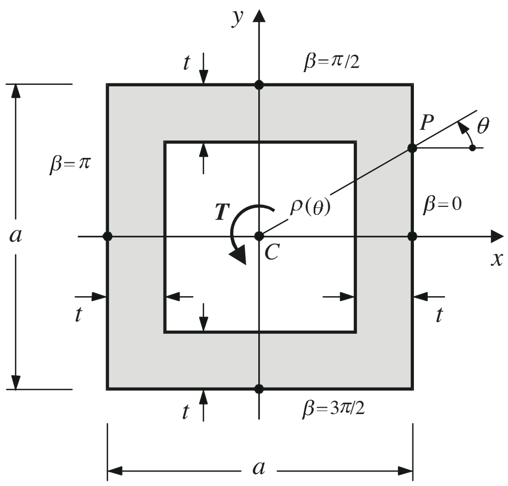

Figure 10. Hollow square cross-section.

The outer length of each edge and the wall thickness of the section are a and t, respectively. Analyses are carried out for the range of $0 \le t \le 0.5$. In this case, the calculated section is situated between 2 extreme sections: negligibly thin-walled and solid sections.

It must be pointed out that, while the inner corners and outer mid-length points of this section are both locations of high stress, disregarding the stress concentration on the inner corners, only mid-length stresses are considered here. Investigation of the stresses at the inner corners is outside the scope of this analysis.

As is well known, one can express the maximum shearing stress and the angle of twist of any cross-section, in compact form, as

$$\tau_{\max} = \frac{T}{W_T}, \omega = \frac{T}{GK_T} \tag{25a, b}$$

in which W_T and GK_T are the torsional strength moment and the torsional rigidity of the section, respectively. Making use of Eqs. (15) and (21), and taking into account the specific properties of the considered cross-section, we can write W_T and GK_T in non-dimensional forms as follows:

$$\frac{W_T}{a^3} = \frac{T}{\tau_{\max} a^3} = \frac{[n(n+1)]^2 - [i(i+1)]^2}{4n^4} \tag{26a}$$

$$\frac{GK_T}{Ga^4} = \frac{T}{\omega a^4} = \frac{[n(n+1)]^2 - [i(i+1)]^2}{n^4 \oint\limits_0^{2\pi} \frac{\sqrt{\bar{\rho}'^2 + \bar{\rho}^2}}{\bar{\rho}\cos(\theta - \beta)} d\theta} \tag{26b}$$

For this type of section, Lamancusa and Saravanos (1989) give second, third, and fourth-order polynomial expressions for the non-dimensional torsional strength moment and the torsional rigidity expressions, which are derived from the performed finite element analyses. In the following, only third-order polynomials are given:

$$\frac{W_T}{a^3} = \frac{T}{\tau_{\max} a^3} = 1.864\left(\frac{t}{a}\right) - 5.340\left(\frac{t}{a}\right)^2 + 4.984\left(\frac{t}{a}\right)^3 \tag{27a}$$

$$\frac{GK_T}{Ga^4} = \frac{T}{\omega a^4} = 0.978\left(\frac{t}{a}\right) - 2.309\left(\frac{t}{a}\right)^2 + 1.826\left(\frac{t}{a}\right)^3 \tag{27b}$$

On the other hand, from the thin-wall approximation theory (Bredt's theory) the following equations are obtained readily for these 2 quantities:

$$\frac{W_T}{a^3} = \frac{T}{\tau_{\max} a^3} = \frac{2At}{a^3} = 2\frac{t}{a}\left(1 - \frac{t}{a}\right)^2 \tag{28a}$$

$$\frac{GK_T}{Ga^4} = \frac{T}{\omega a^4} = \frac{4GA^2}{Ga^4 \oint \frac{dS}{t}} = \frac{t}{a}\left(1 - \frac{t}{a}\right)^3 \tag{28b}$$

where A is the area enclosed by the median curve.

Equations (26a), (27a), (28a), and Eqs. (26b), (27b), (28b) are plotted in Figure 11(a) and (b), respectively. It is clear from Figure 11(a) that all 3 methods show good agreement for low thickness/width ratios, but the thin-wall formula (Eq. (28a)) especially and presented formula (Eq. 26(a)) to some extent, deviate from the finite element method results (Eq. (27(a)) at high thicknesses. For the solid section, the exact value of dimensionless torsional strength moment, $(W_T/a^3)\times 10^3$, is 208, and therefore the relative errors of the thin-wall approximation, presented formulation, and the finite element method are 20.19%, 20.19%, and 5.77%, respectively. As for non-dimensional torsional rigidity, it is deduced from Figure 11(b) that, again, the presented formulation, finite element method, and thin-wall approximation are in good agreement for thin-walled sections. For the thick-walled and solid sections, while the results of the presented formula (Eq. (26b)) are similar to those of the finite element

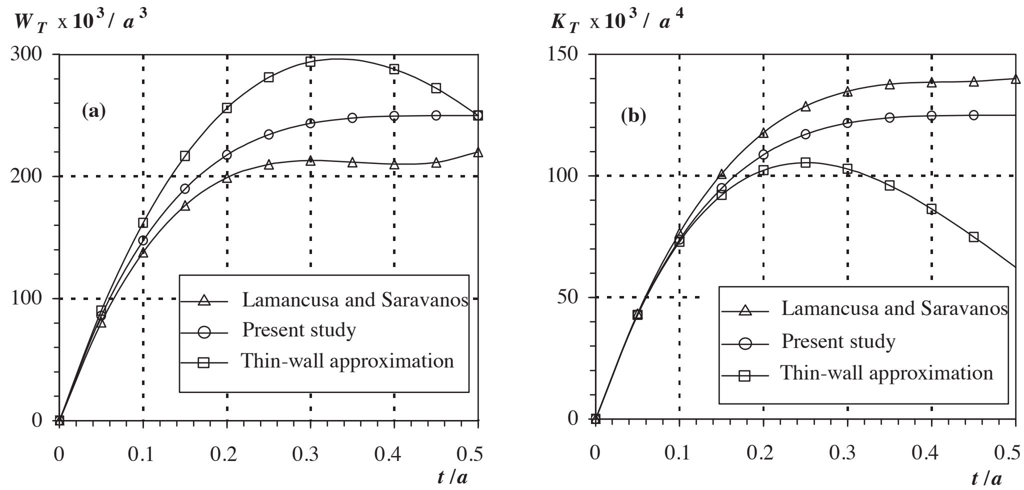

Figure 11. For hollow square section, variation of dimensionless quantities with t/a ratio; (a) torsional strength moment, (b) torsional rigidity.

method (Eq. (27b)), the results of the thin-wall approximation (Eq. (28b)) display significant discrepancy from the results of the finite element method. For the solid square cross-section, the exact value of the non-dimensional torsional rigidity, $(GK_T/Ga^4)\times 10^3$, is 141, and therefore the relative errors of the thin-wall approximation, presented formulation, and the finite element method are 55.67%, 11.35%, and 0.71%, respectively.

From the above-presented solid and hollow section examples, we can make the following remarks:

- In the case of circular and elliptical cross-sections, which have regular curvilinear contours, exact values are obtained for both shearing stresses and angles of twist.
- For the sections with contours formed by straight sides and having obtuse angles, such as regular octagonal and hexagonal cross-sections, good approximation is obtained with the exact values.
- In the case of cross-sections with contours formed by straight sides and having right or acute angles, such as square and triangular sections, approximated theory gives less good results. This is caused partially by no consideration of warping effect in the presented formulation.
- For the closed thin-walled and moderately thick-walled sections there is high accuracy, but for the thick-walled ones the accuracy is somewhat low.
- Since the sharp dead corners have very little contribution to the total torsional strength of the cross-section, and because the approximated theory is unable to take into account this aspect, the following relations apply: $\tau_{\max}^{\text{appr}} \leq \tau_{\max}^{\text{exact}}$ and $\omega^{\text{appr}} \geq \omega^{\text{exact}}$.

Conclusions and Future Work

One of the main problems that must be solved in the design of some load carrying elements subjected to torsion is to determine the maximum shear stresses and the angles of twist under a given torsional moment. The approximate model and the mathematical formulation presented in this paper allow for conducting uniform torsion analysis of closed moderately thick-walled, thick-walled, and solid cross-sections. Similar to thin-walled theory, stress concentrations have been ignored in the formulation. Easily and quickly calculable expressions of maximum shearing stress and the angle of twist have been derived for both hollow and solid cross-sections. The maximum shearing stress and the angle of twist values of a series of sample cross-sections have been calculated and compared with the analytical or numerical values. The results have shown that the obtained formulas give exact values for the sections having regular curvilinear contours. Moreover, results obtained for hollow sections demonstrated that the model is more suitable for sections with small and moderate wall thicknesses than for those having high wall thickness values. In conclusion, the model and the formulation presented in this paper are efficient and can help engineers to solve the uniform torsion problem of a wide variety of cross-sections encountered in engineering practice.

To improve the presented formulation, the authors are working on a new study that takes into account the cross-sectional warping effect. Another possible extension of the study is the torsion analysis of composite sections made of 2 or more different materials, which is left for a future study.

Nomenclature

A	area of the cross-section,
A_j	area confined by the jth strip,
G	modulus of shear,
GK_T	torsional rigidity of the cross-section,
a, b, t	dimensions related to the sample cross-sections,
i	index,
n	number of imaginary strips,
r	radius of circular cross-section,
T	torsional moment,
W_T	torsional strength moment of the cross-section,
β	angle between the horizontal line and any median curve,
δ	thickness of a strip,
ΔT_j	torsional moment carried by the jth strip,
Γ_{inn}	inner boundary of the cross-section,
Γ_j	median curve of the jth strip,
Γ_{out}	outer boundary of the cross-section,
Ω	domain of the cross-section,
ρ	distance from the centre of rotation,
τ	shear stress,
θ	angle measured from the x axis,
ω	angle of twist.

References

Chen, T., Benveniste, Y. and Chuang, P.C., "Torsion of Compound Cross-Sections with Imperfect Interface", Acta Mechanica, 152, 139-163, 2001.

Darılmaz, K., Orakdogen, E. and Girgin, K., "Torsional Rigidity of Arbitrarily Shaped Composite Sections by Hybrid Finite Element Approach", Steel and Composite Structures, 7, 241-251, 2007.

Di Paola, M., Pirrotta, A. and Santoro, R., "Line Element-less Method (LEM) for Beam Torsion Solution (Truly No-Mesh Method)", Acta Mechanica, 195, 349-364, 2008.

Ely, J.F. and Zienkiewicz, O.C., "Torsion of Compound Bars - A Relaxation Solution", International Journal of Mechanical Sciences, 1, 356-365, 1960.

Friedman, Z. and Kosmatka, J.B., "Torsion and Flexure of A Prismatic Isotropic Beam Using the Boundary Element Method", Computers and Structures, 74, 479-494, 2000.

Hematiyan, M.R. and Doostfatemeh, A., "Torsion of Moderately Thick Hollow Tubes with Polygonal Shapes", Mechanics Research Communications, 34, 528-537, 2007.

Herrmann, R.L., "Elastic Torsion Analysis of Irregular Shapes", Journal of Engineering Mechanics Division, Proc. ASCE, 9(EM6), 11-19, 1965.

İnan, M., Strength of Materials, Foundation of İstanbul Technical University, İstanbul, 1988, (in Turkish).

Jiang, W.G. and Henshall, J.L., "A Coupling Cross-Section Finite Element Model for Torsion Analysis of Prismatic Bars", European Journal of Mechanics, A-Solids, 21, 513-522, 2002.

Kołodziej, J.A. and Fraska, A., "Elastic Torsion of Bars Possessing Regular Polygon in Cross-Section Using BCM", Computers and Structures, 84, 78-91, 2005.

Lamancusa, J.S. and Saravanos, D.A., "The Torsional Analysis of Bars with Hollow Square Cross-Sections", Finite Elements in Analysis and Design, 6, 71-79, 1989.

Li, Z., Ko, J.M. and Ni, Y.Q., "Torsional Rigidity of Reinforced Concrete Bars with Arbitrary Sectional Shape", Finite Elements in Analysis and Design, 35, 349-361, 2000.

Najera, A. and Herrera, J.M., "Torsional Rigidity of Non-Circular Bars in Mechanisms and Machines", Mechanism and Machine Theory, 40, 638-643, 2005.

Sadd, M.H., Elasticity: Theory, Applications and Numerics, Elsevier, Burlington MA, 2005.

Sapountzakis, E.J. and Mokos, V.G., "Nonuniform Torsion of Bars of Variable Cross Section", Computers and Structures, 82, 703-715, 2004.

Sapountzakis, E.J., "Nonuniform Torsion of Multi-material Composite Bars by the Boundary Element Method", Computers and Structures, 79, 2805-2816, 2001.

Serra, M., "Approximated Calculus of Torsional Rigidity of Beams with Solid Cross-Section", Computers and Structures, 62, 771-774, 1996.

Timoshenko, S.P. and Goodier, J.N., Theory of Elasticity, McGraw-Hill, New York, 1970.

Wang, C.Y., "Torsion of Tubes of Arbitrary Shape", International Journal of Solids and Structures, 35, 719-731, 1998.

Zhen-min, W. and Ke-xue, Z., "Eddy Current Analogy of Torsion Problem", Applied Mathematics and Mechanics, English Edition, 7, 595-608, 1986.