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Convective flow of two immiscible viscous and couple stress permeable fluids through a vertical channel

J. C. UMAVATHI[1], J. PRATHAP KUMAR[1], Ali J. CHAMKHA[2]
[1]*Department of Mathematics, Gulbarga University, Gulbarga 585 106, Karnataka, INDIA*
[2]*Manufacturing Engineering Department, The Public Authority for Applied Education and Training, Shuweikh 70654, KUWAIT*
e-mail: achamkha@yahoo.com

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Abstract

An analytical solution is presented for fully developed laminar flow between vertical parallel plates filled with 2 immiscible viscous and couple stress fluids in a composite porous medium. The flow in the porous medium is modeled using the Brinkman equation. The viscous and Darcy dissipation terms are included in the energy equation. The transport properties of the fluids in both regions are assumed to be constant. The continuity conditions for the velocity, temperature, shear stress, and the heat flux at the interfaces between the couple stress permeable fluid layer and the viscous fluid layer are assumed. The influence of physical parameters on the flow, such as the couple stress parameter, porous parameter, Grashof number, viscosity ratio and conductivity ratio, are evaluated, and a set of graphical results is presented. An interesting and new approach is incorporated to analyze the flow for strong, weak, and comparable porosity conditions with the couple stress fluid parameter.

Key Words: Immiscible fluids; Couple stress fluids; Vertical channel; Porous medium; Perturbation method.

Introduction

The study of non-Newtonian fluids has attracted much attention because of their practical applications in engineering and industry, and particularly in the extraction of crude oil from petroleum products. In the category of non-Newtonian fluids, couple stress fluids have distinct features, such as polar effects, in addition to possessing high viscosity. The consideration of couple stress has led to the recent development of several theories of fluid micro-continua (Aero et al., 1963; Eringen, 1964; Stokes, 1966). One such couple stress theory of fluids was developed by Stokes (1966) and represents the simplest generalization of the classical theory, which allows for polar effects such as the presence of couple stresses and body couples.

Problems involving immiscible 2 fluid flow and heat transfer arise in a number of scientific and engineering disciplines. Important applications are found within the petroleum industry, geophysics, and plasma physics.

Two-fluid models consist of core (suspension of RBCs) and peripheral plasma layers. Valanis and Sun (1996), Sinha and Singh (1984), and many other authors have considered a mathematical model for blood flow in which they have assumed blood as a couple stress fluid. Two-fluid flow and heat transfer in an inclined channel containing porous and fluid layers was studied analytically by Malashetty et al. (2004). Umavathi et al. (2004, 2005, 2008) analyzed steady and unsteady flow and heat transfer of immiscible fluids in a horizontal channel.

The extension of couple stress fluids for porous mediums has become important due to several engineering applications within fields such as chemical engineering, thermal insulation systems, and nuclear waste management, in which the couple stresses provide an important parameter for deciding mass flow rate and rate of heat flow. An extensive review of convective flow and heat transfer between fluid and porous layers has been done by Prasad (1990). Srinivasan and Vafai (1994) have reported a theoretical study on 2 immiscible fluid systems in a porous medium, taking into account the non-Darcian boundary and inertia effects. Chamkha (2000) analyzed the flow of 2 immiscible fluids in porous and non-porous channels. Recently Umavathi (1999), Malashetty and Umavathi (1999), and Umavathi et al. (2005, 2007) analyzed the flow and heat transfer of couple stress fluids in a vertical channel for 1 and 2 fluid models.

Keeping in view the practical applications in the field of biomechanics and in the fluid models of the mixture of Newtonian and non-Newtonian immiscible fluids, an attempt is made to analyze a fully developed permeable couple stress and viscous fluid in a vertical channel. A new approach is established in order to know the flow nature for sparse, comparable, and dense porosity conditions with respect to the couple stress parameter.

Mathematical Formulation

A schematic diagram for the problem under consideration is shown in Figure 1. The region $-h \le y \le 0$is filled with a couple stress permeable fluid and the region $0 \le y \le h$ is occupied by a clear viscous fluid. The transport properties of both fluids are assumed to be constant. The fluids in the 2 regions are different, with different physical properties. The flow is assumed to be steady, laminar, and incompressible. The 2 walls of the channel are held at different constant temperatures, T_{w_1} and T_{w_2}, with $T_{w_1} > T_{w_2}$. The fluid is supposed to be free from external couples. Since the boundaries are infinite in the x-direction, the physical quantities of velocity and temperature depend on the y-coordinate only. We use the conditions of the hydrostatic state of the fluid and assume that the density of the fluid is a function of temperature alone (Boussinesq approximation).

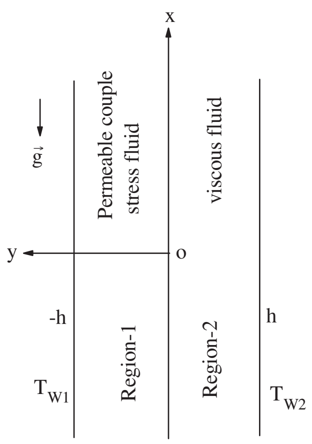

Figure 1. Physical configuration.

With the assumptions mentioned above, the governing equations of motion and energy can be written as follows:

Region 1 (permeable couple stress fluid)

$$\mu_{eff}\frac{d^2u_1}{dy^2} - \eta\frac{d^4u_1}{dy^4} + \rho_1 g\beta_1\left(T_1 - T_{W_2}\right) - \frac{\mu u_1}{s} = 0 \tag{1}$$

$$k_{eff}\,\frac{d^2T_1}{dy^2} + \mu_{eff}\left(\frac{du_1}{dy}\right)^2 + \frac{\mu u_1^2}{s} = 0 \tag{2}$$

Region 2 (clear viscous fluid)

$$\mu\frac{d^2u_2}{dy^2} + \rho_2 g\,\beta_2\left(T_2 - T_{W_2}\right) = 0 \tag{3}$$

$$k\,\frac{d^2T_2}{dy^2} + \mu_2\left(\frac{du_2}{dy}\right)^2 = 0 \tag{4}$$

where u is the x-component of fluid velocity, T is the fluid temperature, g is the magnitude of the acceleration due to gravity, η is the material constant, and β is the coefficient of thermal expansion. The suffixes 1 and 2 denote the values for Regions 1 and 2, respectively.

The boundary conditions on velocity are the no-slip conditions requiring that the velocity must vanish at the wall. Likewise, the couple stresses vanish at the boundary. The boundary conditions on temperature are the isothermal conditions. The 2 boundaries are held at constant but different temperatures. In addition, the continuity of velocity, shear stress, temperature, and heat flux at the interface between the 2 layers is assumed. The corresponding boundary and interface conditions for the above equations are as follows:

$$\begin{gathered}
u_1(-h) = 0\\
u_1(0) = u_2(0)\\
u_2(h) = 0\\
\mu_{eff}\frac{du_1}{dy} - \eta\frac{d^3u_1}{dy^3} = \mu\frac{du_2}{dy} \text{ at } y=0\\
\frac{d^2u_1}{dy^2} = 0 \text{ at } y=0\\
\frac{d^2u_1}{dy^2} = 0 \text{ at } y=-h
\end{gathered} \tag{5}$$

$$\begin{gathered}
T_1(-h) = T_{W1}\\
T_1(0) = T_2(0)\\
T_2(h) = T_{W2}\\
k_{eff}\frac{dT_1}{dy} = k\frac{dT_2}{dy} \text{ at } y=0
\end{gathered} \tag{6}$$

The conservation equations can be recast into dimensionless forms by introducing the following dimensionless variables:

$$u_1^* = \frac{u_1}{\bar{u}_1}; \quad u_2^* = \frac{u_2}{\bar{u}_1}; \quad y^* = \frac{y}{h};$$

$$\theta_1 = \frac{(T_1 - T_{W_2})}{(T_{W_1} - T_{W_2})}; \quad \theta_2 = \frac{(T_2 - T_{W_2})}{(T_{W_1} - T_{W_2})}. \tag{7}$$

The dimensionless momentum and energy equations become:

Region 1

$$\frac{d^4 u_1}{d\,y^4} - m\,a^2 \frac{d^2 u_1}{dy^2} + a^2 \sigma^2 u_1 - \frac{Gr}{Re}\,a^2 \theta_1 = 0 \tag{8}$$

$$\frac{d^2\theta_1}{dy^2} + Ec\ \mathrm{Pr}\,\frac{m}{K}\left(\frac{du_1}{dy}\right)^2 + \frac{Ec\ \mathrm{Pr}}{K}\,\sigma^2\,u_1^2 = 0 \tag{9}$$

Region 2

$$\frac{d^2 u_2}{dy^2} + \frac{Gr}{Re}\theta_2 = 0 \tag{10}$$

$$\frac{d^2\theta_2}{dy^2} + \ Ec\ \ \mathrm{Pr}\left(\frac{du_2}{dy}\right)^2 = 0 \tag{11}$$

where

$$a^2 = \frac{\mu h^2}{\eta}; \mathrm{Pr} = \frac{\mu\,C_P}{k}; Ec = \frac{\bar{u}_1^2}{C_P\,(T_{W1} - T_{W2})}; \sigma = \frac{h}{\sqrt{s}};$$

$$P = \frac{h^2}{\mu \bar{u}_1}\left(\frac{\partial\,P}{\partial\,x}\right); \quad Gr = \frac{g\,\beta\,h^3 \Delta\,T}{\nu^2}; \quad Re = \frac{\bar{u}_1 h}{\nu}; \quad m = \frac{\mu_{eff}}{\mu}; \quad K = \frac{k_{eff}}{k}$$

The corresponding dimensionless boundary and interface conditions are as follows:

$$u_1\,(-1) = 0$$

$$u_1\,(0) = u_2\,(0)$$

$$u_2\,(1) = 0$$

$$\frac{d\,u_1}{d\,y} - \frac{1}{a^2}\frac{d^3\,u_1}{d\,y^3} = \frac{1}{m}\frac{d\,u_2}{d\,y} \text{ at } y = 0 \tag{12}$$

$$\frac{d^2 u_1}{d\,y^2} = 0 \text{ at } y = 0$$

$$\frac{d^2 u_1}{d\,y^2} = 0 \text{ at } y = -1$$

$$\theta_1\,(-1) = 1$$

$$\theta_1\ (0)\ = \theta_2\,(0) \tag{13}$$

$$\theta_2(1) = 0$$

$$\frac{d\theta_1}{dy} = \frac{1}{K}\frac{d\theta_2}{dy}\ \text{at}\ y = 0$$

where the asterisks have been dropped for simplicity.

Solutions

Equations (8) to (11) are coupled nonlinear ordinary differential equations and, hence, finding exact solutions is outside the scope of this paper. However, for small values of ε (= Pr.Ec), appropriate solutions can be extracted through perturbation methods. The Eckert number, Ec, is usually very small, of the order 10^{-5}, while the Prandtl number, Pr, takes a maximum of 2-digit integral values. The product of Pr and Ec is small and, hence, the regular perturbation method can be strongly justified. Adopting this technique, solutions for the velocity and temperature distributions can be assumed in the following form:

$$(u_i, \theta_i)\ =\ (u_{i0}, \theta_{i0}) + \varepsilon\,(u_{i1}, \theta_{i1}) + \cdots \tag{14}$$

where $u_{i\,0}$, θ_{i0} are solutions for the case of ε equal to zero. $u_{i\,1}$ and $\theta_{i\,1}$ are perturbed quantities relating to $u_{i\,0}$ and θ_{i0}, respectively. Substituting the above identities into Eqs. (8) to (11) and equating the coefficients of like powers of ε to 0, we obtain the zeroth and first-order equations as follows:

Region 1

Zeroth-order equations

$$\frac{d^4u_{10}}{d\,y^4} - m\,a^2\,\frac{d^2u_{10}}{dy^2} + a^2\,\sigma^2 u_{10} = \frac{Gr}{Re}\,a^2\theta_{10} \tag{15}$$

$$\frac{d^2\theta_{10}}{dy^2} = 0 \tag{16}$$

First-order equations

$$\frac{d^4u_{11}}{d\,y^4} - m\,a^2\,\frac{d^2u_{11}}{dy^2} + a^2\,\sigma^2 u_{11} = \frac{Gr}{Re}\,a^2\theta_{11} \tag{17}$$

$$\frac{d^2\theta_{11}}{dy^2} + \frac{m}{K}\left(\frac{du_{10}}{dy}\right)^2 + \frac{\sigma^2}{K}u_{10}^2 = 0 \tag{18}$$

Region 2

Zeroth-order equations

$$\frac{d^2u_{20}}{dy^2} + \frac{Gr}{Re}\theta_{20} =\ 0 \tag{19}$$

$$\frac{d^2\theta_{20}}{dy^2} = 0 \tag{20}$$

First-order equations

$$\frac{d^2u_{21}}{dy^2}+\frac{Gr}{Re}\theta_{21}=0 \tag{21}$$

$$\frac{d^2\theta_{21}}{dy^2}+\left(\frac{du_{20}}{dy}\right)^2=0 \tag{22}$$

The corresponding boundary and interface conditions (12) and (13) reduce to the following:

Zeroth-order conditions

$$\begin{gathered} u_{10}(-1)=0 \\ u_{10}(0)=u_{20}(0) \\ u_{20}(1)=0 \\ \frac{d^2u_{10}}{d\,y^2}=0 \text{ at } y=0 \\ \frac{d\,u_{10}}{d\,y}-\frac{1}{m\,a^2}\frac{d^3u_{10}}{d\,y^3}=\frac{1}{m}\frac{d\,u_{20}}{d\,y} \text{ at } y=0 \\ \frac{d^2u_{10}}{d\,y^2}=0 \text{ at } y=-1 \end{gathered} \tag{23}$$

$$\begin{gathered} \theta_{10}(-1)=1 \\ \theta_{10}(0)=\theta_{20}(0) \\ \theta_{20}(1)=0 \\ \frac{d\theta_{10}}{dy}=\frac{1}{K}\frac{d\theta_{20}}{dy} \text{ at } y=0 \end{gathered} \tag{24}$$

$$\begin{gathered} u_{11}(-1)=0 \\ u_{11}(0)=u_{21}(0) \\ u_{21}(1)=0 \\ \frac{d^2u_{11}}{d\,y^2}=0 \text{ at } y=0 \\ \frac{d\,u_{1}}{d\,y}-\frac{1}{m\,a^2}\frac{d^3u_{11}}{d\,y^3}=\frac{1}{m}\frac{d\,u_{21}}{d\,y} \text{ at } y=0 \\ \frac{d^2u_{11}}{d\,y^2}=0 \text{ at } y=0 \end{gathered} \tag{25}$$

$$\theta_{11}(-1)=0$$

$$\theta_{11}(0)=\theta_{21}(0)$$

$$\theta_{21}\,(1) = 0 \tag{26}$$

$$\frac{d\theta_{11}}{dy} = \frac{1}{K}\frac{d\theta_{21}}{dy} \text{ at } y = 0$$

Without going into detail for the sake of brevity, the solution of Eqs. (15) to (22) for $i = 1, 2$, using the boundary and interface conditions given by Eqs. (23) to (26), are given below for sparse, comparable, and dense porosity conditions following the method of solutions similar to Malashetty and Umavathi (1999).

Case 1: Sparse porosity $\left(1 - \frac{4\,\sigma^2}{m^2\,a^2} > 0\right)$

Zeroth-order solutions

$$\theta_{10} = \ C_1\,y \ + \ C_2 \tag{27}$$

$$\theta_{20} = \ C_3\,y \ + \ C_4 \tag{28}$$

$$u_{10} \ = \ B_1\,Cosh\ m_1 y + B_2\,Sinh\ m_1 y + B_3\,Cosh\ m_2 y \ + B_4\,Sinh\ m_2 y + l_1\,y + l_2 \tag{29}$$

$$u_{20} \ = \ B_5 y + B_6 + l_3\,y^3 \ + l_4 y^2 \tag{30}$$

First-order solutions

$$\begin{aligned}\theta_{11} \ = \ & r_{29}\,Cosh\ 2m_1 y + r_{30}\,Sinh\ 2m_1 y + r_{31}\,Cosh\ 2m_2 y + r_{32}\,Sinh\ 2m_2 y + \\ & r_{33}\,Cosh\ \ (m_1 + m_2)\,y \ + r_{34} Sinh\ \ (m_1 + m_2)\,y \ + r_{35}\,Cosh\ (m_1 - m_2)\,y + \\ & r_{36} Sinh\ (m_1 - m_2)\,y + r_{37}\,y\,Cosh\ m_2 y \ + r_{38}\,y\,Sinh\ m_2 y \ + r_{39}\,y\,Cosh\ m_1 y + \\ & r_{40}\,y\,Sinh\ m_1 y + r_{41}\,Cosh\ m_2 y + r_{42}\,Sinh\ m_2 y \ + r_{43}\,Cosh\ m_1 y \ + r_{44}\,Sinh\ m_1 y + \\ & r_{45}\,y^4 + r_{46}\,y^3 \ + r_{47}\,y^2 + C_5\,y + C_6\end{aligned} \tag{31}$$

$$\theta_{21} \ = \ r_{48} y^6 + r_{49} y^5 + r_{50} y^4 + r_{51} y^3 + r_{52} y^2 + C_7 y + C_8 \tag{32}$$

$$\begin{aligned}u_{11} \ = \ & B_7\,Cosh\ m_1 y + B_8\,Sinh\ m_1 y + B_9\,Cosh\ m_2 y \ + r_{10}\,Sinh\ m_2 y + l_5 Cosh\ 2m_1 y + \\ & l_6 Sinh\ 2m_1 y + l_7\,Cosh\ 2m_2 y + l_8\,Sinh\ 2m_2 y + l_9\,Cosh\ (m_1 + m_2)\,y + \\ & l_{10} Sinh\,(m_1 + m_2)\,y + l_{11}\,Cosh\ (m_1 - m_2)\,y + l_{12} Sinh\,(m_1 - m_2)\,y + \\ & l_{13}\,y\,Cosh\,m_2 y + l_{14}\,y\,Sinh\,m_2 y \ + l_{15}\,y\,Cosh\ m_1 y \ + l_{16}\,y\,Sinh\ m_1 y + \\ & l_{17}\,Cosh\ m_2 y + l_{18}\,Sinh\ m_2 y + l_{19}\,Cosh\ m_1 y \ + l_{20}\,Sinh\,m_1 y + l_{21}\,y^4 + \\ & l_{22}\,y^3 \ + l_{23}\,y^2 + l_{24}\,y + l_{25}\end{aligned} \tag{33}$$

$$u_{21} \ = \ l_{26}\,y^8 + l_{27}\,y^7 + l_{28}\,y^6 + l_{29} y^5 + l_{30} y^4 + l_{31} y^3 + l_{32} y^2 + B_{11} y + B_{12} \tag{34}$$

Case 2: Comparable porosity $\left(\frac{4\,\sigma^2}{m^2\,a^2} = 1\right)$

Zeroth-order solutions

$$\theta_{10} = \ C_1\,y \ + \ C_2 \tag{35}$$

$$\theta_{20} = \ C_3\,y \ + \ C_4 \tag{36}$$

$$u_{10} \ = \ C_5\,Cosh\ by + C_6\,Sinh\ by + C_7\,y\ Cosh\ by \ + C_8 y\,Sinh\ by + l_1\,y + l_2 \tag{37}$$

$$u_{20} \ = \ C_9 y + C_{10} + l_3\,y^3 \ + l_4 y^2 \tag{38}$$

First-order solutions

$$\begin{aligned}\theta_{11} \;=\;& r_{19}\,Cosh\,2by + r_{20}\,Sinh\,2by + r_{21}y^2\,Cosh\,2by\; + r_{22}\,y^2 Sinh\,2by + r_{23}\,y\,Cosh\,2by +\\ & r_{24}y\,Sinh\,2by + r_{25}y^2\,Cosh\,by\; + r_{26}\,y^2 Sinh\,by\; + r_{27}\,y\,Cosh\,by + r_{28}y\,Sinh\,by +\\ & r_{29}\,Cosh\,by\; + r_{30}\,Sinh\,by\; + r_{31}\,y^4 +\; r_{32}\,y^3\; + r_{33}\,y^2 + C_{11}\,y + C_{12}\end{aligned} \tag{39}$$

$$\theta_{21} \;=\; r_{34}y^6 + r_{35}y^5 + r_{36}y^4 + r_{37}y^3 + r_{38}y^2 + C_{13}y + C_{14} \tag{40}$$

$$\begin{aligned}u_{11} \;=\;& C_{15}\,Cosh\,by + C_{16}\,Sinhby + C_{17}\,y\,Cosh\,by\; + C_{18}y\,Sinh\,by + l_5\,y^2 Cosh\,2by +\\ & l_6\,y^2 Sinh\,2by + l_7\,y\,Cosh\,2by + l_8 y\,Sinh\,2by + l_9\,y^2\,Coshby +\; l_{10}\,y^2\,Sinh\,by +\\ & l_{11}\,y\,Cosh\,by + l_{12}\,y\,Sinh\,by + l_{13}\,Cosh\,2by + l_{14}\,Sinh\,2by + l_{15}\,Cosh\,by +\\ & l_{16}\,Sinh\,by + l_{17}\,y^4 + l_{18}\,y^3\; + l_{19}\,y^2 + l_{20}\,y + l_{21}\end{aligned} \tag{41}$$

$$u_{21} \;=\; l_{22}\,y^8 + l_{23}\,y^7 + l_{24}\,y^6 + l_{25}y^5 + l_{26}y^4 + l_{27}y^3 + l_{28}y^2 + C_{19}y + C_{20} \tag{42}$$

Case 3: Dense porosity $\left(1 - \frac{4\,\sigma^2}{m^2\,a^2} < 0\right)$

Zeroth-order solutions

$$\theta_{10} = \;C_1\,y\; +\; C_2 \tag{43}$$

$$\theta_{20} = \;C_3\,y\; +\; C_4 \tag{44}$$

$$\begin{aligned}u_{10} \;=\;& C_5\,Cosh\;\gamma y\;\;Cos\;\delta y + C_6\,Cosh\;\gamma y\;\;Sin\,\delta y + C_7\,Sinh\;\gamma y\;\;Cos\delta y\\ & + C_8\,Sinh\,\gamma y\;\;Sin\;\;\delta y\;\; + l_1\,y + l_2\end{aligned} \tag{45}$$

$$u_{20} \;=\; C_9 y + C_{10} + l_3\,y^3\;\; + l_4 y^2 \tag{46}$$

First order solutions

$$\begin{aligned}\theta_{11} \;=\;& r_1\,Cosh\;2\gamma y + r_2\,Cos\,2\delta y + r_3\,Sinh\;2\gamma y + r_4\,Sin\,2\delta y + r_5\;Cosh\;2\gamma y\,Cos\,2\delta y +\\ & r_6\,Sinh\;2\gamma y\,Sin\,2\delta y + r_7 Sinh\;2\gamma y\,Cos\,2\delta y + r_8\,Cosh\;2\gamma y\;\;Sin\,2\delta y +\\ & r_9\,Cosh\;\gamma y\;\;Sin\,\delta y\;\; + r_{10} Sinh\;\gamma y\,Cos\,\delta y + r_{11}\,Sinh\,\gamma y\,Sin\;\;\delta y +\\ & r_{12} Cosh\;\gamma y\;\;Cos\,\delta y + r_{13}\,y\;Sinh\;\gamma y\;\;Cos\,\delta y\;\; + r_{14} y\,Cosh\,\gamma y\,Sin\,\delta y +\\ & r_{15}\,y\,Sinh\;\gamma y\;\;Sin\,\delta y + r_{16}\,y\,Cosh\;\gamma y\,Cos\,\delta y\;\; + r_{17}\,y^4 + r_{18}\,y^3 + r_{19}\,y^2 + C_{11}\,y + C_{12}\end{aligned} \tag{47}$$

$$\theta_{21} \;=\; r_{20}y^6 + r_{21}y^5 + r_{22}y^4 + r_{23}y^3 + r_{24}y^2 + C_{13}y + C_{14} \tag{48}$$

$$\begin{aligned}u_{11} \;=\;& C_{15}\;Cosh\;\gamma y\;Cos\,\delta y + C_{16} Cosh\;\gamma y\;\;Sin\,\delta y + C_{17}\;Sinh\,\gamma y\;\;Cos\,\delta y +\\ & C_{18}\;\;Sinh\,\gamma y\;\;Sin\,\delta y + l_5 Cosh\;\;2\gamma y + l_6 Sinh\;\;2\gamma y + l_7 Cos\,2\delta y +\\ & l_8 Sin\,2\delta y + l_{50} Cosh\,2\gamma y\;\;Cos2\delta y + l_{51} Sinh\,2\gamma y\;\;Sin\,2\delta y +\\ & l_{52} Cosh\;\;2\gamma y\;\;Sin\,2\delta y + l_{53} Sinh\,2\gamma y\;\;Cos2\delta y + l_{54} Cosh\;\;\gamma y\;\;Sin\delta y +\\ & l_{55} Sinh\;\;\gamma y\;\;Cos\,\delta y + l_{56} Cosh\;\;\gamma y\;\;Cos\,\delta y + l_{57} Sinh\;\;\gamma y\;\;Sin\,\delta y +\\ & l_{58}\,y\;Cosh\;\;\gamma y\;\;Sin\,\delta y + l_{59} Sinh\;\;\gamma y\;\;Cos\,\delta y + l_{60} Cosh\;\;\gamma y\;\;Cos\,\delta y +\\ & l_{61} Sinh\;\;\gamma y\;\;Sin\,\delta y + l_{62}y^4 + l_{63}y^3 + l_{64}y^2 + l_{65}y + l_{66}\end{aligned} \tag{49}$$

$$u_{21} \;=\; l_{67}\,y^8 + l_{68}\,y^7 + l_{69}\,y^6 + l_{70}y^5 + l_{71}y^4 + l_{72}y^3 + l_{73}y^2 + C_{19}y + C_{20} \tag{50}$$

Rate of heat transfer

The rate of heat transfer through the channel wall to the fluid in non-dimensional form is given by:

$$RT_1 \;=\; \left(\frac{d\theta_1}{dy}\right)_{y=-1} \text{ and } RT_2 \;=\; \left(\frac{d\theta_2}{dy}\right)_{y=1}$$

0.1. Skin friction

The skin friction at the left and right walls in the non-dimensional form is given by:

$$\tau_1 = \left(\frac{du_1}{dy}\right)_{y=-1} \text{ and } \tau_2 = \left(\frac{du_2}{dy}\right)_{y=1}$$

The expressions for rate of heat transfer and skin friction can be obtained using the solutions for velocity and temperature in Region 1 and Region 2, respectively. The values for rate and heat transfer for different couple stress parameters and porous parameters are shown in Table 2.

All the constants defined in the above expressions are shown in the appendix.

Results and Discussion

In the previous section, an analytical solution for the problem of the convective flow of 2 immiscible fluids (couple stress permeable and viscous fluids) through a vertical channel was obtained using the regular perturbation method. The product of the Prandtl number and the Eckert number was used as a perturbation parameter. In this section, a representative set of results has been selected for presentation in graphical form.

Figures 2 to 8 for a sparse porosity condition $\left(4\,\sigma^2 m^2 < a^2\right)$, Figures 9 and 10 for a comparable porosity condition $\left(4\,\sigma^2 m^2 = a^2\right)$, and Figures 11 to 13 for a dense porosity condition $\left(4\,\sigma^2 m^2 > a^2\right)$ are presented for the effects of couple stresses, porosity, viscosity ratio, and conductivity ratio on the flow. As the couple stress parameter a increases, both the velocity and the first-order temperature increase, as seen in Figure 2 and Table 1 for a sparse porosity condition $\left(4\,\sigma^2 m^2 < a^2\right)$. The magnitude of increase is large in Region 1 (permeable couple stress fluid) compared to Region 2 (viscous fluid). However, its effect on first-order temperature is not very significant. The effects of the couple stress parameter a are quite large for small values of the dimensionless number $a = (h/l)$, where h is a typical dimension of the flow geometry and l is the material constant such that $l = \left(\sqrt{\eta/\mu}\right)$. If l is a function of the molecular dimension of a liquid, then it will vary greatly for different types of liquids. For example, the length of a polymer chain may be a million times the diameter of a water molecule. One might therefore expect the couple stresses to appear in noticeable magnitudes in liquids with very large molecules. One must therefore experiment on fluids having very different molecular sizes in order to search for the existence of couple stresses experimentally.

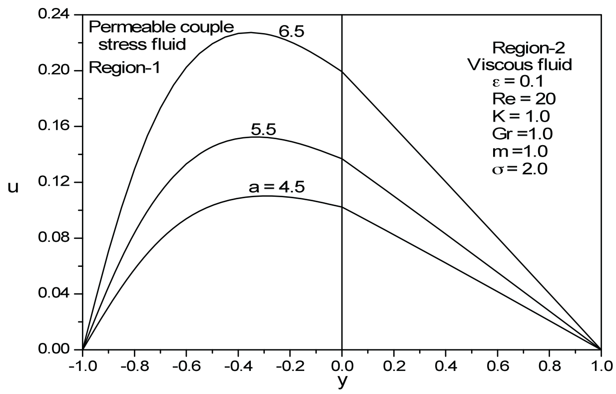

Figure 2. Velocity profiles for different values of the couple stress parameter 'a' for sparse porosity.

The effect of the porous parameter σ on the flow is displayed in Figures 3 and 4. The effect of porosity is to suppress both the velocity and first-order temperature fields in both the regions. For the large porous parameter, the frictional drag resistance against the flow in the porous region is very large and, as a result, the velocity and first-order temperature is small in the permeable couple stress fluid region. The flow decreases as the porous parameter σ increases in the viscous fluid region, also due to the coupling effect. The effects of the viscosity ratio m on the velocity and first-order temperature fields are shown in Figures 5 and 6, respectively, for a sparse porosity condition. As the viscosity ratio m increases, the velocity and first-order temperature decrease in both flow regions. As $m$becomes large, the velocity profiles in Region 1 become flat. The effect of the conductivity ratio K on the flow is shown in Figures 7 and 8. As the conductivity ratio K increases, the velocity decreases in both flow regions. The total temperature (both zeroth and first-order temperature) increases with an increase in the value of K.

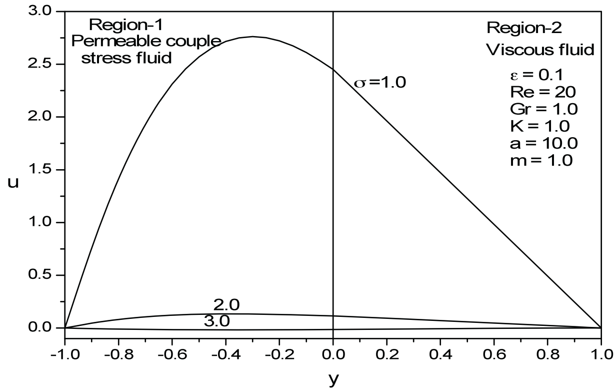

Figure 3. Velocity profiles for different values of the porous parameter σ for sparse porosity.

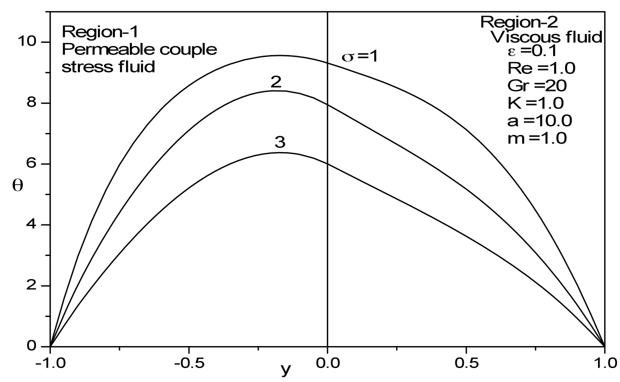

Figure 4. Temperature profiles for different values of the porous parameter σ for sparse porosity.

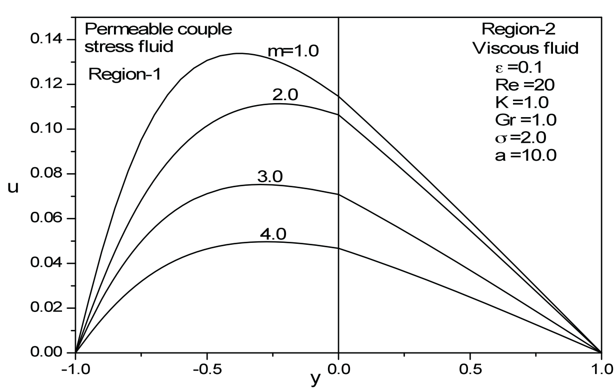

Figure 5. Velocity profiles for different values of the ratio of viscosities 'm' for sparse porosity.

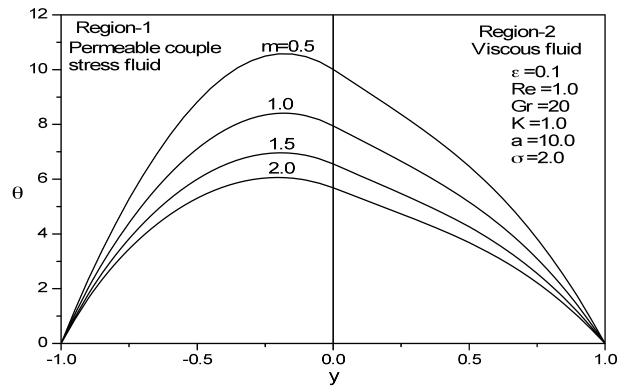

Figure 6. Temperature profiles for different values of the ratio of viscosities 'm' for sparse porosity.

The effect of the couple stress parameter $a$on the flow is displayed in Figures 9 and 10 for a comparable porosity condition $\left(4\,\sigma^2 m^2 = a^2\right)$. As the couple stress parameter a increases, the velocity and temperature decrease in both flow regions. The magnitude of the velocity in Region 1 is very small compared to that in

Region 2. The effect of the conductivity ratio K is to promote the velocity for a comparable porosity condition, as shown in Table 2, which is the reversal effect observed for a sparse porosity condition. The effect of K on the total temperature for a comparable porosity condition remains the same as that for a sparse porosity condition. The effects of the viscosity ratio on the velocity and first-order temperature for a comparable porosity condition show a similar nature to those of a sparse porosity condition, which are not displayed graphically.

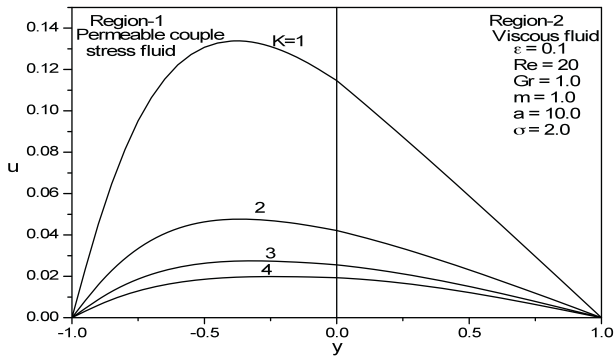

Figure 7. Velocity profiles for different values of the ratio of thermal conductivity K for sparse porosity.

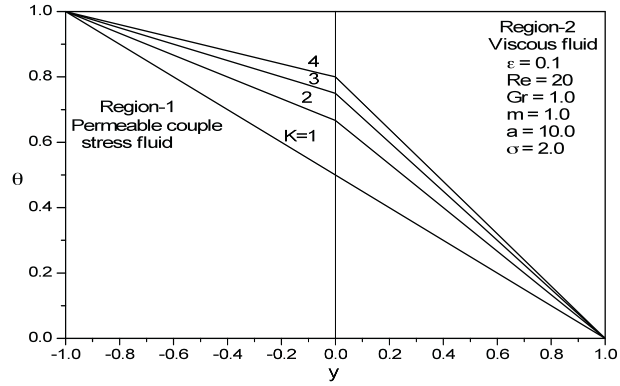

Figure 8. Temperature profiles for different values of the ratio of thermal conductivities K for sparse porosity.

The effect of the couple stress parameter a for dense porosity condition is shown in Figures 11 and 12. As $a$increases, the velocity decreases in both flow regions, whereas the first-order temperature increases. The magnitude of the first-order temperature is very large in Region 1 compared to Region 2. It is also interesting to note that the first-order temperature increases as the porous parameter σ increases for a dense porosity condition, and the magnitude is very large in Region 1 compared to Region 2, as seen in Figure 13. The effect of the porous medium parameter σ on the velocity for a dense porosity condition remains the same as that for a sparse porosity condition. The effects of the viscosity ratio and the conductivity ratio on the flow for a dense porosity condition show similar effects as for a sparse porosity condition and, hence, they not shown in figures.

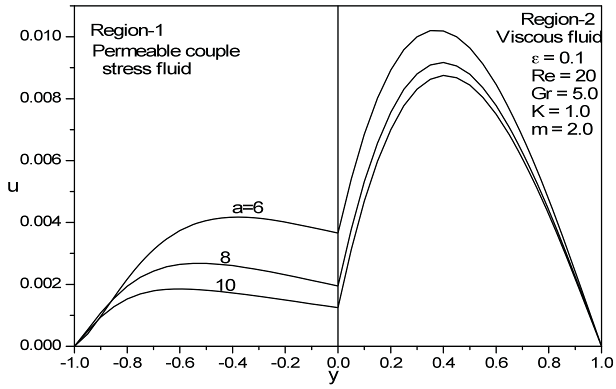

Figure 9. Velocity profiles for different values of the couple stress parameter 'a' for comparable porosity.

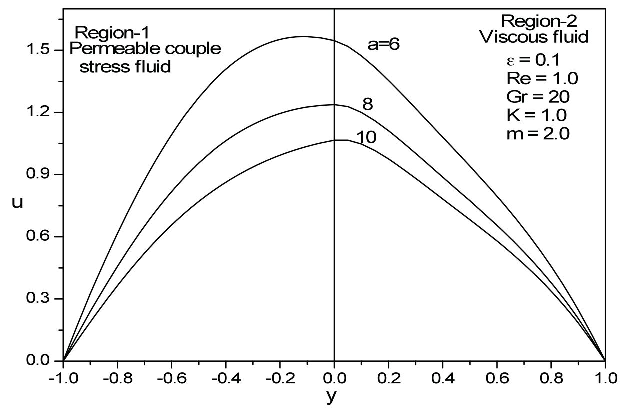

Figure 10. Temperature profiles for different values of the couple stress parameter 'a' for comparable porosity.

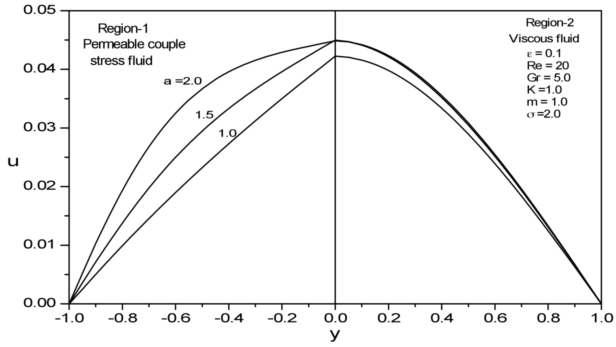

Figure 11. Velocity profiles for different values of the couple stress parameter 'a' for dense porosity.

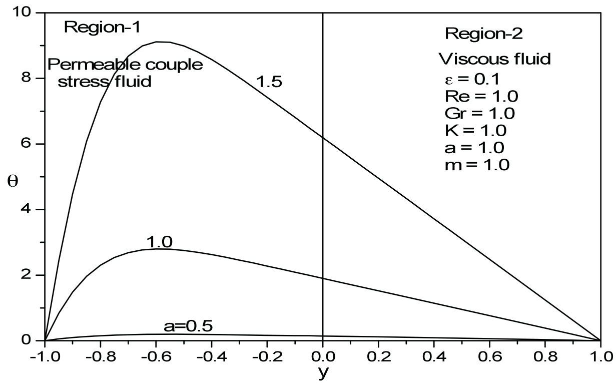

Figure 12. Temperature profiles for different values of the porous parameter σ.

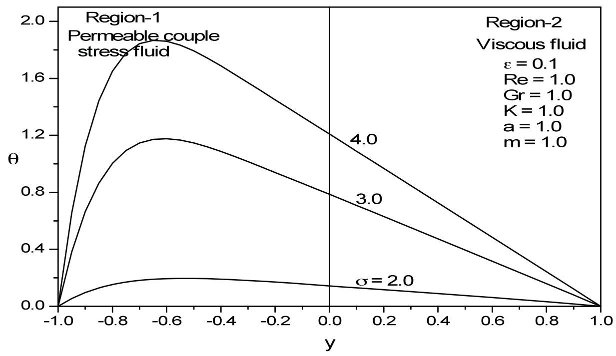

Figure 13. Temperature profiles for different values of the porous parameter σ.

Table 1. Temperature for different values of couple stress parameter a for dense porosity.

y	$a = 4.5$	$a = 5.5$	$a = 6.5$
-1	0	0	0
-0.8	3.48295	3.55654	3.60654
-0.6	6.00625	6.08184	6.13064
-0.4	7.65199	7.7062	7.74083
-0.2	8.34533	8.37111	8.38793
0	7.94094	7.94629	7.95033
0.2	6.92964	6.92563	6.92354
0.4	5.84395	5.83265	5.82573
0.6	4.47895	4.46483	4.45598
0.8	2.59224	2.58154	2.57477
1	0	0	0

Table 2. Velocity for different values of porous parameter σ for comparable porosity.

Y	$\sigma = 4.5$	$\sigma = 5.5$	$\sigma = 6.5$
-1	0	0	0
-0.8	1.7E-4	1E-3	0.00145
-0.6	7.7E-4	0.00197	0.00278
-0.4	0.00141	0.00274	0.00384
-0.2	0.00194	0.00326	0.00465
0	0.00237	0.00361	0.00528
0.2	0.0031	0.00449	0.00603
0.4	0.00302	0.0043	0.00557
0.6	0.00235	0.00331	0.00421
0.8	0.00127	0.00179	0.00226
1	0	0	0

The rate of heat transfer and skin friction were computed for different values of the couple stress parameter a and porous parameter σ, and the results are tabulated in Tables 3, 4, and 5 for sparse, comparable, and dense porosity, respectively. As the couple stress parameter a and porous parameter σ increase, skin friction at both the left and right walls decreases substantially in magnitude. The rate of heat transfer remains almost invariant at both walls for the effects of couple stress and porous parameters. For comparable porosity, skin friction increases in magnitude at the left wall and decreases in magnitude at the right wall as the couple stress parameter a increases, as observed in Table 4. In this case, the rate of heat transfer is almost invariant. Table 5 displays the effects of couple stress and porous medium parameters on the skin friction and the rate of heat transfer. For dense porosity as couple stress parameter$a$ increases, skin friction increases at both the walls in magnitude, whereas it decreases at both walls in magnitude as the porous parameter σ increases. The rate of heat transfer decreases at the left wall but increases at the right wall for the effects of couple stress parameter a. A similar effect is observed for the rate of heat transfer for varying porous parameter σ.

Table 3. Skin friction and rate of heat transfer for different values of couple stress parameter a and porous parameter σ for sparse porosity.

	Couple Stress parameter a			Porous Parameter σ		
	6.0	8.0	10.0	1.0	2.0	3.0
SF1	6.163689	0.665718	0.476056	3.670874	0.467847	0.064475
SF2	-1.723330	-0.173956	-0.118822	-1.085079	-0.106880	-0.017831
RT1	-0.499968	-0.499969	-0.499969	-0.498343	-0.499969	-0.499984
RT2	-0.500009	-0.500009	-0.500009	-0.500043	-0.500009	-0.500006

Table 4. Skin friction and rate of heat transfer for different values of couple stress parameter a for comparable porosity.

	Couple Stress parameter a		
	6.0	8.0	10.0
SF1	0.005380	0.009785	0.011409
SF2	-0.024315	-0.022787	-0.022092
RT1	-0.499931	-0.500084	-0.503395
RT2	-0.500041	-0.500035	-0.500031

Table 5. Skin friction and rate of heat transfer for different values of couple stress parameter a and porous parameter $\sigma$for dense porosity.

	Couple Stress parameter a			Porous Parameter σ		
	0.5	1.0	1.5	2.0	3.0	4.0
SF1	0.063854	0.081293	0.103200	0.081293	0.045546	0.030107
SF2	-0.052621	-0.063112	-0.066318	-0.063112	-0.044145	-0.033801
RT1	-0.499268	-0.492090	-0.380949	-0.492090	-0.445322	-0.404377
RT2	-0.500547	-0.501039	-0.512039	-0.501039	-0.504981	-0.507592

Conclusion

The problem of fully developed laminar flow between vertical parallel plates filled with an immiscible couple stress permeable fluid and a clear viscous fluid was analyzed. The basic equations governing the flow were solved analytically using the regular perturbation method. The effect of couple stress parameter a on the flow was promoted for a sparse porosity condition and suppressed for a comparable porosity condition, and the velocity decreased. Meanwhile, the first-order temperature increased for a dense porosity condition. The effect of the porous parameter was predicted to reduce the flow for sparse and dense porosity conditions, but the first-order temperature increased as the porosity increased for the dense porosity condition. Increasing the conductivity ratio reduced the velocity and increased the temperature for the sparse porosity condition, but the velocity and temperature were promoted for comparable and dense porosity conditions. Skin friction decreased at the left wall and increased at the right wall for sparse porosity, whereas the reversal effect was observed for comparable and dense porosity for variations of couple stress parameter. The porous parameter σ decreased the skin friction at both of the walls in all cases. The rate of heat transfer remained invariant for the effects of the couple stress parameter and porous parameter for sparse porosity, whereas for comparable porosity, it increased at the left wall and decreased at the right wall. The rate of heat transfer decreased at the left wall and increased at the right wall for both the couple stress and permeable parameters. Thus, one can conclude that flow can be effectively controlled by choosing different types of liquids with different material constants and also by choosing different pore structures.

Nomenclature

a	couple stress parameter
b	ratio of the coefficients of thermal expansion
C_p	specific heat at constant pressure
Ec	Eckert number
h	width of Regions 1 and 2
K	ratio of thermal conductivities
k	thermal conductivity
k_{eff}	effective thermal conductivity of Region 2
P	non-dimensional pressure gradient
Pr	Prandtl number
Re	Reynolds number
s	permeability of the porous medium
T	fluid temperature
$T_{w1},\ T_{w2}$	temperatures of the boundaries
u	axial velocity
$\bar{u}_1$	average velocity
x, y	space coordinates

Greek symbols

σ	porous parameter
η	material constant
μ	dynamic viscosity
μ_{eff}	effective viscosity of Region-2
ρ	density of the fluid
ν	kinematics viscosity
ε	product of Prandtl number and Eckert number
θ	non-dimensional temperature

References

Aero, E.L., Bullygin, A.N. and Kuvshinski, E.V., "Asymmetric Hydrodynamics", Appl. Math. Mech., 333, 29-36, 1963.

Chamkha, A.J., "Flow of Two-Immiscible Fluids in Porous and Non-Porous Channels", ASME J. of Fluid Engineering, 122, 117-124, 2000.

Eringen, A.C., "Simple Microfluids", Int. J. Engng. Sci., 2, 205-215, 1964.

Malshetty, M.S. and Umavathi, J.C., "Oberbeck Convection Flow of a Couple Stress Fluid Through a Vertical Porous Stratum", Int. J. Non-Linear Mech., 34, 1037-1045, 1999.

Malashetty, M.S., Umavathi, J.C. and Prathap Kumar, J., "Two Fluid Flow and Heat Transfer in an İnclined Channel Containing Porous and Fluid Layer", Heat and Mass Transfer, 40, 871-876, 2004.

Prasad, V., "Convection Flow Interaction and Heat Transfer between Fluid and Porous Layers", Proc. NATO Advanced Study Institute on Convective Heat and Mass Transfer in Porous Media, Turkey, 1990.

Sinha, P. and Singh, C., "Effects of Couple Stress on the Blood Flow through an Artery with Mild Stenosis", Biorheology, 21, 303-312 1984.

Srinivasan, V. and Vafai, K., "Analysis of Linear Encroachment in Two-Immiscible Fluid Systems", ASME J. of Fluid Engineering, 116, 135-139, 1994.

Stokes, V.K., "Couple Stresses in Fluids", Phys. of Fluids, 9, 1709-1715, 1966.

Umavathi, J.C., "Free Convection Flow of Electrically Conducting Couple Stress Fluid in a Vertical Channel", AMSE 68, 1-24, 1999.

Umavathi, J.C., Chamkha, A.J., Manjula, M.H. and Al-Mudhaf, A., "Flow and Heat Transfer of a Couple-Stress Fluid Sandwiched between Viscous Fluid Layers", Canadian J. of Physics, 83, 705-720, 2005.

Umavathi, J.C., Chamkha, A.J., Abdul Mateen and Al-Mudhaf, A., "Unsteady Two Fluid Flow and Heat Transfer in a Horizontal Channel", Heat Mass Transfer, 42, 81-90, 2005.

Umavathi, J.C., Malashetty, M.S. and Abdul Mateen, "Fully Developed Flow and Heat Transfer in a Horizontal Channel Containing Electrically Conducting Fluid Sandwiched between Two Fluid Layers", Int. J. Appl. Mechanical Engineering, 9, 781-794, 2004.

Umavathi, J.C., Manjula, M.H., Liu, I.C. and Pop, I., "Flow and Heat Transfer of Couple Stress and Viscous Fluids in a Vertical Channel", Int. J. Appl. Mech. and Engng., 12, 537-555, 2007.

Umavathi, J.C., Prathap Kumar, J. and Chamkha, A.J., "Flow and Heat Transfer of a Micropolar Fluid Sandwiched between Viscous Fluid Layers", Canadian J. of Physics, 86, 961-973, 2008.

Valanis, K.C. and Sun, C.T., "Poiseuille Flow of a Fluid with a Couple Stress with Applications to Blood Flow", Biorheology, 6, 85-97, 1969.

Appendix

Case 1. $\left(1-\frac{4\,\sigma^2}{m^2\,a^2}\right) \quad > 0$

$$A_1 = \frac{Gr\,a^2}{Re};\quad A_2 = \frac{-\,m}{K};\quad A_3 = \frac{-\,\sigma^2}{K};\quad B_1 = \frac{D_9 - D_8 B_4}{D_7}$$

$$B_2 = \frac{D_4 B_1 -\ m_2^2\,Sinh\,m_2 B_4}{m_1^2\,Sinh\,m_1};\quad B_3 = \frac{l_1 - l_2 +\ Sinh\,m_2 B_4 + Sinh\,m_1 B_2 - Cosh\,m_1 B_1}{Cosh\,m_2}$$

$$B_4 = \frac{D_9\,D_{10} - D_7 D_{12}}{D_8\,D_{10} - D_7 D_{11}};\quad B_5 = -\left(l_3 + l_4 + B_6\right);\quad B_6 = \left(B_1 + B_3 + l_2\right)$$

$$B_7 = \frac{e_{12} + Sinh\,m_1 B_8 + Sinh\,m_2\,B_{10}}{e_{11}};\quad B_8 = \frac{e_{21} - e_{20} B_{10}}{e_{19}}$$

$$B_9 = \frac{e_1 - Cosh\,m_1 B_7 + Sinh\,m_1 B_8 + Sinh\,m_2\,B_{10}}{Cosh\,m_2};\quad B_{10} = \frac{e_{18}\,e_{19} - e_{16}\,e_{21}}{e_{17}\,e_{19} - e_{16}\,e_{20}}$$

$$B_{11} = e_2 - B_{12};\quad B_{12} = B_7 + B_9 - e_3;\quad C_1 = \frac{-1}{1+K};\quad C_2 = \frac{K}{1+K}$$

$$C_3 = \frac{-K}{1+K};\quad C_4 = \frac{K}{1+K};\quad C_5 = C_6 - D_{13};\quad C_6 = \frac{(K\,D_{13} + D_{17})}{1+K}\quad C_7 = D_{14} - C_8;\quad C_8 = C_6 - D_{15};$$

$$D_1 = \left(m\,m_1\,a^2 - m_1^3\right);\quad D_2 = \left(m\,m_2\,a^2 - m_2^3\right)\quad D_3 = \frac{m_2^2\,Cosh\,m_1 - m_1^2\,Cosh\,m_2}{m_2^2};$$

$$D_4 = \left(m_1^2\,Cosh\,m_1 - m_1^2\,Cosh\,m_2\right);\quad D_5 = \frac{m_2^2\,a^2 - m_1^2\,a^2}{m_2^2}$$

$$D_6 = \frac{-\left(a^2\,l_3 + a^2\,l_4 + a^2\,l_1\,m + a^2\,l_2\right)}{D_1};\quad D_7 = \frac{\left(D_3 D_1 + D_5\,Sinh\,m_1\right)}{D_1}$$

$$D_8 = \frac{\left(D_2\,Sinh\,m_1 - D_1\,Sinh\,m_2\right)}{D_1};\quad D_9 = \left(l_1 - l_2 + D_6 Sinh\,m_1\right)$$

$$D_{10} = \frac{\left(D_4 D_1 + D_5\,m_1^2\,Sinh\,m_1\right)}{D_1};\quad D_{11} = \frac{\left(D_2\,m_1^2\,Sinh\,m_1 - D_1\,m_2^2\,Sinh\,m_2\right)}{D_1}$$

$$D_{12} = m_1^2\,Sinh\,m_1 D_6$$

$$D_{13} = -\begin{pmatrix} r_{29}\,Cosh\,2\,m_1 - r_{30}\,Sinh\,2\,m_1 + r_{31}\,Cosh\,2\,m_2 - r_{32}\,Sinh\,2\,m_2 + r_{33}\,Cosh\,(m_1+m_2) \\ -r_{34}\,Sinh\,(m_1+m_2) + r_{35}\,Cosh\,(m_1-m_2) - r_{36}\,Sinh\,(m_1-m_2) - r_{37} Cosh\,m_2 + \\ r_{38}\,Sinh\,m_2 - r_{39}\,Cosh\,m_1 + r_{40}\,Sinh\,m_1 + r_{41}\,Cosh\,m_2 - r_{42}\,Sinh\,m_2 + r_{43}\,Cosh\,m_1 \\ -r_{44}\,Sinh\,m_1 + r_{45} - r_{46} + r_{47} \end{pmatrix}$$

$$D_{14} = -\left(r_{48} + r_{49} + r_{50} + r_{51} + r_{52}\right);\quad D_{15} = -\left(r_{29} + r_{31} + r_{33} + r_{35} + r_{41} + r_{43}\right)$$

$$D_{16} = \begin{pmatrix} 2\,m_1\,K\,r_{30} + 2\,m_2\,K\,r_{32} + (m_1+m_2)\;K\,r_{34} + (m_1-m_2)\;K\,r_{36} + K\,r_{37} + K\,r_{39} \\ +m_2\,K\,r_{42} + m_1\,K\,r_{44} \end{pmatrix}$$

$$D_{17} = \left(D_{14} + D_{15} - D_{16}\right)$$

$$e_1 = -\begin{pmatrix} l_5\,Cosh\,2m_1 - l_6 Sinh\,2m_1 + l_7\,Cosh\,2m_2 - l_8 Sinh\,2m_2 + l_9\,Cosh\,(m_1+m_2) \\ -l_{10} Sinh\,(m_1+m_2) + l_{11}\,Cosh\,(m_1-m_2) - l_{12} Sinh\,(m_1-m_2) - l_{13}\,Cosh\,m_2 \\ +l_{14} Sinh\,m_2 - l_{15}\,Cosh\,m_1 + l_{16} Sinh\,m_1 + l_{17}\,Cosh\,m_2 - l_{18} Sinh\,m_2 + l_{19}\,Cosh\,m_1 \\ -l_{20} Sinh\,m_1 + l_{21} - l_{22} + l_{23} - l_{24} + l_{25} \end{pmatrix}$$

$$e_2 = -\left(l_{26} + l_{27} + l_{28} + l_{29} + l_{30} + l_{31} + l_{32}\right);\quad e_3 = -\left(l_5 + l_7 + l_9 + l_{11} + l_{17} + l_{19} + l_{25}\right)$$

$$e_4 = \left(m_1^3 - m\,m_1\,a^2\right);\quad e_5 = \left(m_2^3 - m\,m_2\,a^2\right)$$

$$e_6 = \begin{pmatrix} 2\,m\,a^2\,m_1 l_6 + 2\,m\,a^2\,m_2 l_8 + m\,a^2 l_{10}\,(m_1+m_2) + m\,a^2 l_{12}\,(m_1-m_2) \\ +m\,a^2 l_{13} + m\,a^2 l_{15} + m\,m_2 a^2 l_{18} + m\,m_1 a^2 l_{20} + m\,a^2 l_{24} - 8\,m_1^3 l_6 \\ -8\,m_2^3\,l_8 - l_{10}\,(m_1+m_2)^3 - l_{12}\,(m_1-m_2)^3 - 3\,m_2^2\,l_{13} - 3\,m_1^2 l_{15} - m_2^3\,l_{18} \\ -m_1^3\,l_{20} - 6\,l_{22} \end{pmatrix}$$

$$e_7 = -\begin{pmatrix} 4\;m_1^2\,l_5\,Cosh\,2m_1 - 4\;m_1^2\;l_6\,Sinh\,2m_1 + 4\;m_2^2\,l_7\,Cosh\,2m_2 - 4\;m_2^2\,l_8\,Sinh\,2m_2 \\ +l_9\;(m_1+m_2)^2\,Cosh\,(m_1+m_2) - l_{10}\,(m_1+m_2)^2\,Sinh\,(m_1+m_2) + l_{11}\,(m_1-m_2)^2 \\ Cosh\,(m_1-m_2) - l_{12}\;(m_1-m_2)^2\,Sinh\,(m_1-m_2) - l_{13}\,m_2^2 Cosh\,m_2 - 2\,m_2\,l_{13} Sinh m_2 \\ +l_{14}\,m_2^2\,Sinh\,m_2 + 2\,m_2\,l_{14} Cosh m_2 - m_1^2\,l_{15}\,Cosh\,m_1 - 2\,m_1\,l_{15} Sinh m_1 + m_1^2 l_{16} Sinh\,m_1 \\ +2\,m_1\,l_{16} Cosh m_1 + m_2^2\,l_{17}\,Cosh\,m_2 - m_2^2\,l_{18} Sinh\,m_2 + m_1^2\,l_{19}\,Cosh\,m_1 \\ -m_1^2\,l_{20} Sinh\,m_1 + 12\,l_{21} - 6\,l_{22} + 2\,l_{23} \end{pmatrix}$$

$$e_8 = -\begin{pmatrix} 4\;m_1^2\,l_5 + 4\;m_1^2\;l_7 + l_9\;(m_1+m_2)^2 + l_{11}\;(m_1-m_2)^2 \\ +2\,m_2\,l_{14} + 2\,m_1\,l_{16} + m_2^2\,l_{17} + m_1^2\,l_{19} + 2\,l_{23} \end{pmatrix}$$

$$e_9 = \frac{e_8}{m_2^2}; \quad e_{10} = \frac{m_1^2}{m_2^2}; \quad e_{11} = (Cosh\, m_1 - e_{10}\, Cosh\, m_2)\,; \quad e_{12} = (e_1 - e_9\, Cosh\, m_2)$$

$$e_{13} = \left(m_1^2\, Cosh\, m_1 - e_{10}\, m_2^2\, Cosh\, m_2\right); \quad e_{14} = \left(e_7 - e_9\, m_2^2\, Cosh\, m_2\right); e_{15} = \left(e_2 + e_3 - \frac{e_6}{a^2} - e_9\right)$$

$$e_{16} = \left(m_1^2\, e_{11}\, Sinh\, m_1 - e_{13}\, Sinh\, m_1\right); \quad e_{17} = \left(m_2^2\, e_{11}\, Sinh\, m_2 - e_{13}\, Sinh\, m_2\right)$$

$$e_{18} = (e_{12}\, e_{13} - e_{11}\, e_{14})\,; \quad e_{19} = \left(\frac{e_4}{a^2} e_{11} - (1 - e_{10})\ Sinh\, m_1\right)$$

$$e_{20} = \left(\frac{e_5}{a^2} e_{11} - (1 - e_{10})\ Sinh\, m_2\right); \quad e_{21} = (e_{12}\,(1 - e_{10}) - e_{11} e_{15})\,; \quad l_1 = \frac{A_1 C_1}{a^2\, \sigma^2}$$

$$l_2 = \frac{A_1 C_2}{a^2\, \sigma^2}; \quad l_3 = \frac{-\, Gr\, C_3}{6\, Re}; \quad l_4 = \frac{-\, Gr\, C_4}{2\, Re}; \quad l_5 = \frac{A_1 r_{29}}{a^2\, \sigma^2\, S_1}$$

$$l_6 = \frac{A_1 r_{30}}{a^2\, \sigma^2\, S_1}; \quad l_7 = \frac{A_1 r_{31}}{a^2\, \sigma^2\, S_2}; \quad l_8 = \frac{A_1 r_{32}}{a^2\, \sigma^2\, S_2}; \quad l_9 = \frac{A_1 r_{33}}{a^2\, \sigma^2\, S_3}$$

$$l_{10} = \frac{A_1 r_{34}}{a^2\, \sigma^2\, S_3}; \quad l_{11} = \frac{A_1 r_{35}}{a^2\, \sigma^2\, S_4}; \quad l_{12} = \frac{A_1 r_{36}}{a^2\, \sigma^2\, S_4}; \quad l_{13} = \frac{A_1 r_{37}}{P_1}$$

$$l_{14} = \frac{A_1 r_{38}}{P_1}; \quad l_{15} = \frac{A_1 r_{39}}{P_3}; \quad l_{16} = \frac{A_1 r_{40}}{P_3}$$

$$l_{17} = \left[\frac{-4\, A_1 r_{38}\, m_2^3}{P_2} + \frac{2\, a^2\, A_1\, m\, m_2 r_{38}}{P_2} + \frac{A_1\, r_{41}}{a^2 \sigma^2\, S_5}\right]$$

$$l_{18} = \left[\frac{-4\, A_1 r_{37}\, m_2^3}{P_2} + \frac{2\, a^2\, A_1\, m\, m_2 r_{37}}{P_2} + \frac{A_1\, r_{42}}{a^2 \sigma^2\, S_5}\right]$$

$$l_{19} = \left[\frac{-4\, A_1 r_{40}\, m_1^3}{P_4} + \frac{2\, a^2\, A_1\, m\, m_1 r_{40}}{P_4} + \frac{A_1\, r_{43}}{a^2 \sigma^2\, S_7}\right]$$

$$l_{20} = \left[\frac{-4\, A_1 r_{39}\, m_1^3}{P_4} + \frac{2\, a^2\, A_1\, m\, m_1 r_{39}}{P_4} + \frac{A_1\, r_{44}}{a^2 \sigma^2\, S_7}\right]$$

$$l_{21} = \frac{A_1 r_{45}}{a^2\, \sigma^2}; \quad l_{22} = \frac{A_1 r_{46}}{a^2\, \sigma^2}; \quad l_{23} = \left[\frac{12\, A_1\, m\, r_{45}}{a^2\, \sigma^4} + \frac{A_1\, r_{47}}{a^2\, \sigma^2}\right]$$

$$l_{24} = \left[\frac{6\, A_1\, m\, r_{46}}{a^2\, \sigma^4} + \frac{A_1\, C_5}{a^2\, \sigma^2}\right]; \quad l_{25} = \left[\frac{-24\, A_1 r_{45}}{a^4\, \sigma^4} + \frac{24\, A_1\, m^2 r_{45}}{a^2\, \sigma^6} + \frac{2 A_1 m\, r_{47}}{a^2 \sigma^4} + \frac{A_1\, C_6}{a^2\, \sigma^2}\right]$$

$$l_{26} = \frac{-\, Gr\, r_{48}}{56\, Re}; \quad l_{27} = \frac{-\, Gr\, r_{49}}{42\, Re}; \quad l_{28} = \frac{-\, Gr\, r_{50}}{30\, Re}; \quad l_{29} = \frac{-\, Gr\, r_{51}}{20\, Re}$$

$$l_{30} = \frac{-\, Gr\, r_{52}}{12\, Re}; \quad l_{31} = \frac{-\, Gr\, C_7}{6\, Re}; \quad l_{32} = \frac{-\, Gr\, C_8}{2\, Re}$$

$$m_1 = \left[\frac{m\, a^2\ + m\, a^2 \sqrt{1 - \frac{4\, \sigma^2}{m^2\, a^2}}}{2}\right]^{1/2}; \quad m_2 = \left[\frac{m\, a^2\ - m\, a^2 \sqrt{1 - \frac{4\, \sigma^2}{m^2\, a^2}}}{2}\right]^{1/2}$$

$$P_1 = \left(m_2^4 - m\, a^2\, m_2^2 + a^2 \sigma^2\right); \quad P_2 = \left(m_2^4 - m\, a^2\, m_2^2 + a^2 \sigma^2\right)^2$$

$$P_3 = \left(m_1^4 - m\,a^2\,m_1^2 + a^2\sigma^2\right); \quad P_4 = \left(m_1^4 - m\,a^2\,m_1^2 + a^2\sigma^2\right)^2$$

$$r_1 = \left(A_2\,B_1^2\,m_1^2 + A_3\,B_2^2\right); \quad r_2 = \left(A_2\,B_2^2\,m_1^2 + A_3\,B_1^2\right)$$

$$r_3 = \left(A_2\,B_3^2\,m_2^2 + A_3\,B_4^2\right); \quad r_4 = \left(A_2\,B_4^2\,m_2^2 + A_3\,B_3^2\right)$$

$$r_5 = \left(A_2\,B_1\,B_2\,m_1^2 + A_3\,B_1\,B_2\right) \quad ; \quad r_6 = \left(A_2\,B_1\,B_3\,m_1 m_2 + A_3\,B_4\,B_2\right)$$

$$r_7 = \left(A_2\,B_1\,B_4\,m_1 m_2 + A_3\,B_3\,B_2\right); \quad r_8 = \left(A_2\,B_2\,B_3\,m_1 m_2 + A_3\,B_4\,B_1\right)$$

$$r_9 = \left(A_2\,B_2\,B_4\,m_1 m_2 + A_3\,B_1\,B_3\right); \quad r_{10} = \left(A_2\,B_4\,B_3\,m_2^2 + A_3\,B_4\,B_3\right)$$

$$r_{11} = 2\,A_3 B_3 l_1; \quad r_{12} = 2\,A_3 B_4 l_1; \quad r_{13} = 2\,A_3 B_1 l_1; \quad r_{14} = 2\,A_3 B_2 l_1$$

$$r_{15} = 2\,\left(A_2\,B_2\,l_1\,m_1 + A_3\,B_1\,l_2\right); \quad r_{16} = 2\,\left(A_2\,B_4\,l_1\,m_2 + A_3\,B_3\,l_2\right)$$

$$r_{17} = 2\,\left(A_2\,B_3\,l_1\,m_2 + A_3\,B_4\,l_2\right) \quad ; \quad r_{18} = 2\,\left(A_2\,B_1\,l_1\,m_1 + A_3\,B_2\,l_2\right)$$

$$r_{19} = A_3 l_1^2; \quad r_{20} = 2\,A_3 l_2 l_1; \quad r_{21} = 2\,\left(A_2\,l_1^2 + A_3\,l_2^2\right)$$

$$r_{22} = \frac{(r_1 + r_2)}{2}; \quad r_{23} = \frac{(r_3 + r_4)}{2}; \quad r_{24} = (r_6 + r_9); \quad r_{25} = (r_9 - r_6)$$

$$r_{26} = (r_7 + r_8); \quad r_{27} = (r_7 - r_8); \quad r_{28} = \frac{(r_2 - r_1 + r_4 - r_3 + 2r_{21})}{2}$$

$$r_{29} = \frac{r_{22}}{4\,m_1^2}; \quad r_{30} = \frac{r_5}{4\,m_1^2}; \quad r_{31} = \frac{r_{23}}{4\,m_2^2}; \quad r_{32} = \frac{r_{10}}{4\,m_2^2}$$

$$r_{33} = \frac{r_{24}}{(m_1 + m_2)^2}; \quad r_{34} = \frac{r_{26}}{(m_1 + m_2)^2}; \quad r_{35} = \frac{r_{25}}{(m_1 - m_2)^2}; \quad r_{36} = \frac{r_{27}}{(m_1 - m_2)^2}$$

$$r_{37} = \frac{r_{11}}{m_2^2}; \quad r_{38} = \frac{r_{12}}{m_2^2}; \quad r_{39} = \frac{r_{13}}{m_1^2}; \quad r_{40} = \frac{r_{14}}{m_1^2}$$

$$r_{41} = \left(\frac{r_{16}}{m_2^2} - \frac{2\,r_{12}}{m_2^3}\right); \quad r_{42} = \left(\frac{r_{17}}{m_2^2} - \frac{2\,r_{11}}{m_2^3}\right); \quad r_{43} = \left(\frac{r_{15}}{m_1^2} - \frac{2\,r_{14}}{m_1^3}\right); \quad r_{44} = \left(\frac{r_{18}}{m_1^2} - \frac{2\,r_{13}}{m_1^3}\right)$$

$$r_{45} = \frac{r_{19}}{12}; \quad r_{46} = \frac{r_{20}}{6}; \quad r_{47} = \frac{r_{28}}{2}; \quad r_{48} = \frac{-9\,l_3^2}{30}; \quad r_{49} = \frac{-12\,l_3\,l_4}{20}$$

$$r_{50} = \frac{-\left(4\,l_4^2 + 6\,B_5\,l_3\right)}{12}; \quad r_{51} = \frac{-4\,l_4\,B_5}{6}; \quad r_{52} = \frac{-\,B_5^2}{2}$$

$$S_1 = \left(\frac{a^2\sigma^2 - 4\,m\,m_1^2\,a^2 + 16\,m_1^4}{a^2\sigma^2}\right); \quad S_2 = \left(\frac{a^2\sigma^2 - 4\,m\,m_2^2\,a^2 + 16\,m_2^4}{a^2\sigma^2}\right)$$

$$S_3 = \left(\frac{a^2\sigma^2 - 4\,m\,\left(m_1^2 + m_2^2\right)\,a^2 + 16\,\left(m_1^4 + m_2^4\right)}{a^2\sigma^2}\right)$$

$$S_4 = \left(\frac{a^2\sigma^2 - 4\,m\,\left(m_1^2 - m_2^2\right)\,a^2 + 16\,\left(m_1^4 - m_2^4\right)}{a^2\sigma^2}\right)$$

$$S_5 = \left(\frac{a^2\sigma^2 - m\,m_2^2\,a^2 + m_2^4}{a^2\sigma^2}\right); \quad S_7 = \left(\frac{a^2\sigma^2 - m\,m_1^2\,a^2 + m_1^4}{a^2\sigma^2}\right)$$

Case 2. $1 = \frac{4\sigma^2}{m^2 a^2}$

$$A_1 = \frac{Gr\,a^2}{Re};\quad A_2 = \frac{-m}{K};\quad A_3 = \frac{-\sigma^2}{K};\quad b = a\sqrt{\frac{m}{2}}\quad B_1 = \frac{D_9 - D_8 B_4}{D_7};\quad B_2 = \frac{D_4 B_1 - m_2^2\,Sinh\,m_2 B_4}{m_1^2\,Sinh\,m_1}$$

$$B_3 = \frac{l_1 - l_2 + Sinh\,m_2 B_4 + Sinh\,m_1 B_2 - Cosh\,m_1 B_1}{Cosh\,m_2};\quad B_4 = \frac{D_9\,D_{10} - D_7 D_{12}}{D_8\,D_{10} - D_7 D_{11}}$$

$$B_5 = -\,(l_3 + l_4 + B_6)\,;\quad B_6 = (B_1 + B_3 + l_2)$$

$$B_7 = \frac{e_{12} + Sinh\,m_1 B_8 + Sinh\,m_2\,B_{10}}{e_{11}};\quad B_8 = \frac{e_{21} - e_{20} B_{10}}{e_{19}}$$

$$B_9 = \frac{e_1 - Cosh\,m_1 B_7 + Sinh\,m_1 B_8 + Sinh\,m_2\,B_{10}}{Cosh\,m_2};\quad B_{10} = \frac{e_{18}\,e_{19} - e_{16}\,e_{21}}{e_{17}\,e_{19} - e_{16}\,e_{20}}$$

$$B_{11} = e_2 - B_{12};\quad B_{12} = B_7 + B_9 - e_3;\quad C_1 = \frac{-1}{1+K};\quad C_2 = \frac{K}{1+K}$$

$$C_3 = \frac{-K}{1+K};\quad C_4 = \frac{K}{1+K};\quad C_5 = \frac{(D_1 + C_6\,Sinh\,b + C_7\,Cosh\,b - C_8\,Sinh\,b)}{Cosh\,b}$$

$$C_6 = \frac{(D_{22} - D_{21}\,C_8)}{D_{12}};\quad C_7 = \frac{(\,D_{18} - D_{17}\,C_8)}{D_{16}};\quad C_8 = \frac{D_{12}\,D_{20} + b^2 D_8\,D_{22}}{D_{12}\,D_{19} + b^2 D_8\,D_{21}}$$

$$C_9 = D_2 - C_{10};\quad C_{10} = C_5 - D_3;\quad C_{11} = \frac{D_{23} + D_{24} - D_{25} - D_{26}}{1+K}$$

$$C_{12} = C_{11} - D_{23};\quad C_{13} = D_{24} - C_{14};\quad C_{14} = D_{25} + C_{12}$$

$$C_{15} = \frac{(e_1 + C_{16}\,Sinh\,b + C_{17}\,Cosh\,b - C_{18}\,Sinh\,b)}{Cosh\,b};\quad C_{16} = \frac{(\,e_{26} - e_{25}\,C_{18})}{e_{15}}$$

$$C_{17} = \frac{(\,e_{21} - e_{20}\,C_{18})}{e_{19}};\quad C_{18} = \frac{(\,e_{15}\,e_{24} - e_{22}\,e_{26})}{(\,e_{15}\,e_{23} - e_{22}\,e_{25})};\quad C_{19} = e_4\,C_{16} + e_5\,C_{17} - e_6$$

$$C_{20} = C_{19} - e_2;\quad D_1 = (l_1 - l_2)\,;\quad D_2 = -\,(l_3 + l_4)\,;\quad D_3 = -l_2$$

$$D_4 = \left(m\;a^2\mathbf{b} - b^3\right);\quad D_5 = \left(m\;a^2 - 3\,b^2\right);\quad D_6 = -\,m\,l_1\,a^2;\quad D_7 = b^2\,Cosh\,b$$

$$D_8 = b^2\,Sinh\,b\quad;\quad D_9 = \left(-b^2\,Cosh\,b - 2\,b\,Sinh\,b\right);\quad D_{10} = \left(b^2\,Sinh\,b + 2\,b\,Cosh\,b\right)$$

$$D_{11} = \left(b^2\,D_{10} - 2\,b\,D_7\right);\quad D_{12} = (D_8\;Cosh\,b - D_7 Sinh\,b)$$

$$D_{13} = (D_7\;Cosh\,b + D_9 Cosh\,b)\,;\quad D_{14} = (D_7\,Sinh\,b - D_{10}\,Cosh\,b)\,;\quad D_{15} = D_1\,D_7$$

$$D_{16} = b^2 D_9\,D_{12} - b^2 D_8\,D_{13};\quad D_{17} = D_{11}\,D_{12} + b^2 D_8\,D_{14} D_{18} = b^2 D_8\,D_{15}$$

$$D_{19} = D_{11} - \frac{b^2 D_9\,D_{17}}{D_{16}}\quad;\quad D_{20} = -\frac{b^2 D_9\,D_{18}}{D_{16}};\quad D_{21} = D_{14} + \frac{D_{13}\,D_{17}}{D_{16}};\quad D_{22} = D_{15} + \frac{D_{13}\,D_{18}}{D_{16}}$$

$$D_{23} = -\left(\begin{array}{l} r_{19}\,Cosh\,2\,b - r_{20}\,Sinh\,2b + r_{21}\,Cosh\,2\,b - r_{22}\,Sinh\,2\,b - r_{23}\,Cosh\,2b \\ +\,r_{24}\,Sinh\,2b\, + r_{25}\,Cosh\,b - r_{26}\,Sinh\;b - r_{27} Cosh\,b + r_{28}\,Sinh\,b \\ + r_{29}\,Cosh\,b - r_{30}\,Sinh\,b + r_{31} - r_{32} + r_{33} \end{array}\right)$$

$$D_{24} = -(r_{34} + r_{35} + r_{36} + r_{37} + r_{.38})\,; \quad D_{25} = -(r_{29} + r_{19})$$

$$D_{26} = (\,2bK\, r_{20} + K\, r_{23} + \; K\, r_{27} + \; K\, br_{30})$$

$$e_1 = -\begin{pmatrix} l_5\, Cosh\, 2b - l_6 Sinh\;\, b - l_7\, Cosh\, 2b + l_8 Sinh\, b + l_9\, Cosh\, b - l_{10} Sinhb \\ -l_{11}\, Cosh\, b + l_{12} Sinhb + l_{13}\, Cosh 2b - l_{14} Sinh 2b + l_{15}\, Cosh\, b \\ -l_{16} Sinhb + l_{17} \; - l_{18} + l_{19} \; - l_{20} + l_{21} \end{pmatrix}$$

$$e_2 = -(l_{22} + l_{23} + l_{24} + l_{25} + l_{26} + l_{27} + l_{28})\,; \quad e_3 = (l_{13} + l_{15} + l_{21})\,; \quad e_4 = \left(m\, a^2\, b - \, b^3\right)$$

$$e_5 = \left(m\, a^2 \; - 3\; b^2\right); \quad e_6 = -\begin{pmatrix} m\, a^2\, l_7 + \, m\, a^2\, l_{11} + \, 2\, b\, m\, a^2 l_{14} + m\, b\, a^2 l_{16} + m\, a^2 l_{20} - 4l_5 \\ -12\, b\, l_6 - 12\, b^2\, l_7 - 8\, b\, l_{10} - 3\, b^2\, l_{11} - 8\, b^3\, l_{14} - b^3\, l_{16} - 6\, l_{18} \end{pmatrix}$$

$$e_7 = b^2\, Cosh\, b \quad ; \quad e_8 = b^2\, Sinh\, b; \quad e_9 = \left(-b^2\, Cosh\, b - \, 2\, b\, Sinh\, b\right)$$

$$e_{10} = \left(b^2\, Sinh\, b + \, 2\, b\, Cosh\, b\right)$$

$$e_{11} = -\begin{pmatrix} 4\; b^2\, l_5\, Cosh\, 2b + 4\, b\, l_5\, Sinh\, 2b + 4b\, l_5\, Sinh\, 2b - 4\, l_5\, Cosh\, 2b \\ -4\, b^2\, l_6\, Sinh\, 2b - 8b\, l_6\, Cosh\, 2b - 2\, l_6\, Sinh\, 2b - 4\; b^2\, l_7\, Cosh\, 2b \\ -4b\, l_7\, Sinh\, 2b + 4\, b^2\, l_8\, Sinh\, 2b + 4b\, l_8 Cosh\, 2b \; + b^2\, l_9\, Cosh\, b + \\ 4b\, l_9\, Sinh\, b + 2\, l_9\, Cosh\, b - b^2 l_{10}\, Sinh\, b - 4b\, l_{10} Cosh\, b - 2\, l_{10} Sinh\, b - \\ b^2\, l_{11}\, Cosh\, b - 2b\, l_{11}\, Sinh\, b + b^2\, l_{12}\, Sinh\, b + 2b\, l_{12} Cosh\, b + b^2\, l_{15}\, Cosh\, b \\ 4\, b^2 l_{13}\, Cosh\, 2b - 4\, b^2\, l_{14}\, Sinh\, 2b - b^2\, l_{16}\, Sinh\, b + 12\, l_{17} - 6\, l_{18} + 2\, l_{19} \end{pmatrix}$$

$$e_{12} = -\left(4\; b\, l_8 \; + 2\; l_9 + \; 2\, b\, l_{12} + \; 4\, b^2 l_{13} + b^2\, l_{15} + 2\, l_{19}\right)$$

$$e_{13} = \left(b^2\, e_{10} - 2\, b\, e_7\right); \quad e_{14} = \left(b^2\, e_{11} - e_7\, e_{12}\right); \quad e_{15} = (e_8 Coshb - e_7\, Sinh\, b)$$

$$e_{16} = (\, e_7\, Cosh\, b + e_9 Cosh\, b\,)\,; \quad e_{17} = (\, e_7\, Sinh\, b - e_{10} Cosh\, b\,)\,; \quad e_{18} = (e_1\, e_7 \; - e_{11} Cosh\, b\,)$$

$$e_{19} = \left(b^2\, e_9\, e_{15} - b^2\, e_8 e_{16}\right); \quad e_{20} = \left(e_{13}\, e_{15} + b^2\, e_8 e_{17}\right); \quad e_{21} = \left(e_{14}\, e_{15} + b^2\, e_8 e_{18}\right)$$

$$e_{22} = \left(-b^2\, e_8\right); \quad e_{23} = \left(e_{13} - \frac{b^2\, e_9\, e_{20}}{e_{19}}\right); \quad e_{24} = \left(e_{14} - \frac{b^2\, e_9\, e_{21}}{e_{19}}\right)$$

$$e_{25} = \left(e_{17} + \frac{e_{16}\, e_{20}}{e_{19}}\right); \quad e_{26} = \left(e_{18} + \frac{e_{16}\, e_{21}}{e_{19}}\right); \quad l_1 = \frac{A_1 C_1}{a^2\, \sigma^2}; \quad l_2 = \frac{A_1 C_2}{a^2\, \sigma^2}$$

$$l_3 = \frac{-\,Gr\, C_3}{6\, Re}; \quad l_4 = \frac{-\,Gr\, C_4}{2\, Re}; \quad l_5 = \frac{A_1 r_{21}}{a^2\, \sigma^2\, S_2}; \quad l_6 = \frac{A_1 r_{22}}{a^2\, \sigma^2\, S_1}$$

$$l_7 = \left[\frac{A_1 r_{23}}{a^2\, \sigma^2\, S_2} - \frac{2\, S_3 A_1 r_{22}}{a^2\, \sigma^2\, S_1}\right]; \quad l_8 = \left[\frac{A_1 r_{24}}{a^2\, \sigma^2\, S_2} - \frac{2\, S_3 A_1 r_{21}}{a^2\, \sigma^2\, S_2}\right]$$

$$l_9 = \frac{A_1 r_{25}}{a^2\, \sigma^2\, S_8}; \quad l_{10} = \frac{A_1 r_{26}}{a^2\, \sigma^2\, S_8}; \quad l_{11} = \left[\frac{A_1 r_{27}}{a^2\, \sigma^2\, S_8} - \frac{2\, S_9 A_1 r_{26}}{a^2\, \sigma^2\, S_8}\right]$$

$$l_{12} = \left[\frac{A_1 r_{28}}{a^2\, \sigma^2\, S_8} - \frac{2\, S_9 A_1 r_{25}}{a^2\, \sigma^2\, S_8}\right]; \quad l_{13} = \left[\frac{A_1 r_{19}}{a^2\, \sigma^2\, S_1} - \frac{2\, S_4 A_1 r_{21}}{a^2\, \sigma^2\, S_2} + \frac{2\, S_3^2 A_1 r_{21}}{a^2\, \sigma^2\, S_2} - \frac{S_3 A_1 r_{24}}{a^2\, \sigma^2\, S_2}\right]$$

$$l_{14} = \left[\frac{A_1 r_{20}}{a^2\, \sigma^2\, S_1} - \frac{2\, S_4 A_1 r_{22}}{a^2\, \sigma^2\, S_1} + \frac{2\, S_3^2 A_1 r_{22}}{a^2\, \sigma^2\, S_1} - \frac{S_3 A_1 r_{23}}{a^2\, \sigma^2\, S_2}\right] \; l_{15} = \left[\frac{A_1 r_{29}}{a^2\, \sigma^2\, S_8} - \frac{2\, S_{10} A_1 r_{25}}{a^2\, \sigma^2\, S_8} + \frac{2\, S_9^2 A_1 r_{25}}{a^2\, \sigma^2\, S_8}\right];$$

$$l_{16} = \left[\frac{A_1 r_{30}}{a^2\,\sigma^2\,S_8} - \frac{2\,S_{10} A_1 r_{26}}{a^2\,\sigma^2\,S_8} + \frac{2\,S_9^2 A_1 r_{26}}{a^2\,\sigma^2\,S_8}\right]$$

$$l_{17} = \frac{A_1 r_{31}}{a^2\,\sigma^2}; \quad l_{18} = \frac{A_1 r_{32}}{a^2\,\sigma^2}; \quad l_{19} = \left[\frac{12\,A_1\,m\,r_{31}}{a^2\,\sigma^4} + \frac{A_1 r_{33}}{a^2\,\sigma^2}\right]$$

$$l_{20} = \left[\frac{6\,A_1\,m\,r_{32}}{a^2\,\sigma^4} + \frac{A_1 C_{11}}{a^2\,\sigma^2}\right]; \quad l_{21} = \left[\frac{24\,A_1\,r_{31}\,m^2}{a^2\,\sigma^6} - \frac{24 A_1 r_{31}}{a^4\,\sigma^4} + \frac{2\,A_1 r_{33}\,m}{a^2\,\sigma^4} + \frac{A_1 C_{12}}{a^2\,\sigma^2}\right]$$

$$l_{22} = \frac{-\,Gr\,r_{34}}{56\,Re}; \quad l_{23} = \frac{-\,Gr\,r_{35}}{42\,Re}; \quad l_{24} = \frac{-\,Gr\,r_{36}}{30\,Re}; \quad l_{25} = \frac{-\,Gr\,r_{37}}{20\,Re}$$

$$l_{26} = \frac{-\,Gr\,r_{38}}{12\,Re}; \quad l_{27} = \frac{-\,Gr\,C_{13}}{6\,Re}; \quad l_{28} = \frac{-\,Gr\,C_{14}}{2\,Re}$$

$$r_1 = \left(A_2\,b^2\,C_5^2 + A_2\,C_8^2 + 2\,A_2 b\,C_5\,C_8 + A_3\,C_6^2\right)$$

$$r_2 = \left(A_2\,b^2\,C_6^2 + A_2\,C_7^2 + 2\,A_2 b\,C_6\,C_7 + A_3\,C_5^2\right)$$

$$r_3 = \left(A_2\,b^2\,C_7^2 + A_3\,C_8^2\right); \quad r_4 = \left(A_2\,b^2\,C_8^2 + A_3\,C_7^2\right)$$

$$r_5 = \left(A_2\,b^2\,C_5 C_6 + A_2 b\,C_5\,C_7 + A_2 b\,C_6\,C_8 + A_2\,C_7\,C_8 + A_3\,C_5\,C_6\right)$$

$$r_6 = 2\,\left(A_2\,b^2\,C_5 C_7 + A_2 b\,C_8\,C_7 + A_3\,C_6\,C_8\right)$$

$$r_7 = 2\,\left(A_2\,b^2\,C_6 C_8 + A_2 b\,C_8\,C_7 + A_3\,C_5\,C_7\right)$$

$$r_8 = \left(A_2\,b^2\,C_7 C_6 + A_2 b\,C_7^2 + A_2 b^2\,C_5\,C_8 + A_2 b\,C_8^2 + A_3\,C_5\,C_8 + A_3\,C_6\,C_7\right)$$

$$r_9 = 2\,(A_2\,b\,C_5 l_1 + A_2\,C_8\,l_1 + A_3\,C_6\,l_2);$$

$$r_{10} = 2\,(A_2\,b\,C_6 l_1 + A_2\,C_7\,l_1 + A_3\,C_5\,l_2)$$

$$r_{11} = \left(A_2\,b^2\,C_7 C_8 + A_3\,C_8\,C_7\right); \quad r_{12} = 2\,(A_2\,b\,C_7 l_1 + A_3 C_6 l_1 + A_3 C_8 l_2)$$

$$r_{13} = 2\,(A_2\,b\,C_8 l_1 + A_3 C_5 l_1 + A_3 C_7 l_2); \quad r_{14} = 2\,A_3 C_7 l_1; \quad r_{15} = 2\,A_3 C_8 l_1$$

$$r_{16} = A_3\,l_1^2; \quad r_{17} = 2\,A_3 l_2 l_1; \quad r_{18} = \left(A_2 l_1^2 + A_3 l_2^2\right)$$

$$r_{19} = \left(\frac{r_1}{8\,b^2} + \frac{r_2}{8\,b^2} + \frac{3\,r_3}{16\,b^4} + \frac{3\,r_4}{16\,b^4} - \frac{r_8}{4\,b^3}\right); \quad r_{20} = \left(\frac{r_5}{4\,b^2} - \frac{r_6}{8\,b^3} - \frac{r_7}{8\,b^3} + \frac{3\,r_{11}}{8\,b^4}\right)$$

$$r_{21} = \left(\frac{r_3}{8\,b^2} + \frac{r_4}{8\,b^2}\right); \quad r_{22} = \left(\frac{r_{11}}{4\,b^2}\right); \quad r_{23} = \left(\frac{r_6}{8\,b^2} + \frac{r_7}{8\,b^2} - \frac{r_{11}}{2\,b^3}\right)$$

$$r_{24} = \left(-\frac{r_3}{4\,b^3} - \frac{r_4}{4\,b^3} + \frac{r_8}{4\,b^2}\right); \quad r_{25} = \left(\frac{r_{14}}{b^2}\right); \quad r_{26} = \left(\frac{r_{15}}{b^2}\right); \quad r_{27} = \left(\frac{r_{13}}{b^2} - \frac{4\,r_{15}}{b^3}\right)$$

$$r_{28} = \left(\frac{r_{12}}{b^2} - \frac{4\,r_{14}}{b^3}\right); \quad r_{29} = \left(\frac{r_{10}}{b^2} - \frac{2\,r_{12}}{b^3} + \frac{6\,r_{14}}{b^4}\right); \quad r_{30} = \left(\frac{r_9}{b^2} - \frac{2\,r_{13}}{b^3} + \frac{6\,r_{15}}{b^4}\right)$$

$$r_{31} = \left(\frac{r_4}{24} + \frac{r_{16}}{12} - \frac{r_3}{24}\right); \quad r_{32} = \left(\frac{-r_6}{12} + \frac{r_7}{12} + \frac{r_{17}}{6}\right); \quad r_{33} = \left(\frac{-r_1}{4} + \frac{r_2}{4} + \frac{r_{18}}{2}\right)$$

$$r_{34} = \frac{-9\,l_3^2}{30};\quad r_{35} = \frac{-12\,l_3\,l_4}{20};\quad r_{36} = \frac{-\left(4\,l_4^2+6\,C_9\,l_3\right)}{12}$$

$$r_{37} = \frac{-4\,l_4\,C_9}{6};\quad r_{38} = \frac{-C_9^2}{2};\quad S_1 = \left(\frac{a^2\sigma^2-4\,m\,b^2+16\,b^4}{a^2\sigma^2}\right)$$

$$S_2 = \left(\frac{a^2\sigma^2-4\,m\,b^2+16\,b^4}{a^2\sigma^2}\right);\quad S_3 = \frac{1}{S_2}\left(\frac{32\,b^3-4\,m\,b\,a^2}{a^2\sigma^2}\right)$$

$$S_4 = \frac{1}{S_2}\left(\frac{24\,b^2}{a^2\,\sigma^2}-\frac{m}{\sigma^2}\right);\quad S_5 = \frac{8\,b}{a^2\,\sigma^2 S_2};\quad S_6 = \frac{1}{a^2\,\sigma^2 S_2}$$

Case 3. $\left(1-\frac{4\,\sigma^2}{m^2\,a^2}\right) \quad < 0$

$$\gamma = \frac{1}{\sqrt{2}}\sqrt{\frac{2\,a^2\,\sigma^2+m\,a^2}{2}}\qquad \delta = \frac{1}{\sqrt{2}}\sqrt{\frac{2\,a^2\,\sigma^2-m\,a^2}{2}}$$

$$A_1 = \frac{Gr\,a^2}{Re};\quad A_2 = \frac{-m}{K};\quad A_3 = \frac{-\sigma^2}{K}$$

$$A_5 = -m\,a^2;\quad A_6 = a^2\,\sigma^2;\quad B_1 = \frac{D_9-D_8B_4}{D_7}$$

$$B_2 = \frac{D_4B_1-m_2^2\,Sinh\,m_2B_4}{m_1^2\,Sinh\,m_1};\quad B_3 = \frac{l_1-l_2+Sinh\,m_2B_4+Sinh\,m_1B_2-Cosh\,m_1B_1}{Cosh\,m_2}$$

$$B_4 = \frac{D_9\,D_{10}-D_7D_{12}}{D_8\,D_{10}-D_7D_{11}};\quad B_5 = -\left(l_3+l_4+B_6\right);\quad B_6 = \left(B_1+B_3+l_2\right)$$

$$B_7 = \frac{e_{12}+Sinh\,m_1B_8+Sinh\,m_2\,B_{10}}{e_{11}};\quad B_8 = \frac{e_{21}-e_{20}B_{10}}{e_{19}}$$

$$B_9 = \frac{e_1-Cosh\,m_1B_7+Sinh\,m_1B_8+Sinh\,m_2\,B_{10}}{Cosh\,m_2};\quad B_{10} = \frac{e_{18}\,e_{19}-e_{16}\,e_{21}}{e_{17}\,e_{19}-e_{16}\,e_{20}}$$

$$B_{11} = e_2-B_{12};\quad B_{12} = B_7+B_9-e_3;\quad C_1 = \frac{-1}{1+K};\quad C_2 = \frac{K}{1+K}$$

$$C_3 = \frac{-K}{1+K};\quad C_4 = \frac{K}{1+K}\quad C_5 = \frac{\left(l_1-l_2+Cosh\gamma\,Sin\delta\,C_6+Sinh\gamma\,Cos\delta\,C_7-Sinh\gamma\,Sin\delta\,C_8\right)}{Cosh\gamma\,Cos\delta}$$

$$C_6 = \frac{D_{14}-D_5\,C_7-D_{15}C_8}{D_4};\quad C_7 = \frac{D_{20}-D_{19}\,C_8}{D_{18}};\quad C_8 = \frac{D_1\,D_8D_{18}-D_{16}\,D_{20}}{D_{17}\,D_{18}-D_{19}\,D_{16}}$$

$$C_9 = -\left(l_3+l_4+C_{10}\right);\quad C_{10} = \left(l_2+C_5\right);\quad C_{11} = \frac{e_1+e_2+e_4-e_3}{1+K}$$

$$C_{12} = \left(C_{11}-e_1\right);\quad C_{13} = \left(e_2-C_{14}\right);\quad C_{14} = \left(e_3+C_{12}\right)$$

$$C_{15} = \frac{\left(D_{21}+Cosh\gamma\,Sin\delta\,C_{16}+Sinh\gamma\,Cos\delta\,C_{17}-Sinh\gamma\,Sin\delta\,C_{18}\right)}{Cosh\gamma\,Cos\delta}$$

$$C_{16} = \frac{D_{38}-D_{37}\,C_{18}-D_{29}C_{17}}{D_{28}};\quad C_{17} = \frac{D_{49}-D_{48}\,C_{18}}{D_{47}};\quad C_{18} = \frac{D_{46}D_{47}-D_{44}\,D_{49}}{D_{47}\,D_{45}-D_{44}\,D_{48}}$$

$$C_{19} = D_{22} - C_{20}; \quad C_{20} = C_{15} - D_{23}; \quad D_1 = (l_1 - l_2)$$

$$D_2 = -(l_3 + l_4); \quad D_3 = -l_2; \quad D_4 = \left(m\, a^2\delta + \delta^3 - 3\,\gamma^2\delta\right)$$

$$D_5 = \left(m\, a^2 - 3\,b^2\right); \quad D_6 = -m\,l_1\,a^2;$$

$$D_7 = \left(-\delta^2\,Cosh\gamma\,Cos\delta + \gamma^2\,Cosh\gamma\,Cos\delta - 2\,\gamma\,\delta\,Sinh\gamma\,Sin\delta\right)$$

$$D_8 = \left(\delta^2\,Cosh\gamma\,Sin\delta - \gamma^2\,Cosh\gamma\,Sin\delta - 2\,\gamma\,\delta\,Sinh\gamma\,Cos\delta\right)$$

$$D_9 = \left(\delta^2\,Sinh\gamma\,Cos\delta - \gamma^2\,Sinh\gamma\,Cos\delta + 2\,\gamma\,\delta\,Cosh\gamma\,Sin\delta\right)$$

$$D_{10} = \left(-\delta^2\,Sinh\gamma\,Sin\delta + \gamma^2\,Sinh\gamma\,Sin\delta + 2\,\gamma\,\delta\,Cosh\gamma\,Cos\delta\right)$$

$$D_{11} = \left(\gamma^2 - \delta^2\right); \quad D_{12} = D_{10} - \frac{2\gamma\,\delta\,D_7}{D_{11}}; \quad D_{13} = Sinh\gamma\,Sin\delta - \frac{2\gamma\,\delta\,Cosh\gamma\,Cos\delta}{D_{11}}$$

$$D_{14} = D_6 + a^2\,D_2 + a^2\,D_3; \quad D_{15} = -\frac{2\gamma\,\delta\,a^2}{D_{11}}$$

$$D_{16} = \left(D_9\,Cosh\gamma\,Sin\delta - D_8\,Sinh\gamma\,Cos\delta\right); \quad D_{17} = \left(D_{12}\,Cosh\gamma\,Sin\delta + D_8 D_{13}\right)$$
$$D_{18} = \left(D_4\,D_9 - D_8 D_5\right);\ D_{19} = \left(D_4\,D_{12} - D_8 D_{15}\right);\ D_{20} = \left(-D_8\,D_{14}\right)$$

$$D_{21} = -\left(\begin{array}{l} l_5\,Cosh\,2\gamma - l_6\,Sinh\,2\gamma + l_7\,Cos\,2\delta - l_8\,Sin\,2\delta + l_{50}\,Cosh\,2\gamma\,Cos\,2\delta + \\ l_{51}\,Sinh\,2\gamma\,Sin\,2\delta - l_{52}\,Cosh\,2\gamma\,Sin\,2\delta - l_{53}\,Sinh\,2\gamma\,Cos\,2\delta - l_{54}\,Cosh\,\gamma\,Sin\delta \\ -l_{55}\,Sinh\,\gamma\,Cos\,\delta + l_{56}\,Cosh\,\gamma\,Cos\,\delta + l_{57}\,Sinh\,\gamma\,Sin\,\delta + l_{58}\,Cosh\,\gamma\,Sin\,\delta \\ +l_{59}\,Sinh\,\gamma\,Cos\,\delta - l_{60}\,Cosh\,\gamma\,Cos\,\delta - l_{61}\,Sinh\,\gamma\,Sin\,\delta + l_{62} - l_{63} + l_{64} - l_{65} + l_{66} \end{array}\right)$$

$$D_{22} = -\left(l_{67} + l_{68} + l_{69} + l_{70} + l_{71} + l_{72} + l_{73}\right); \quad D_{23} = -\left(l_5 + l_7 + l_{50} + l_{56} + l_{66}\right)$$

$$D_{24} = \left(m\,\delta + K\,\delta^2 - 3\,\gamma^2\delta\,K\right); \quad D_{25} = \left(m\,\gamma - K\,\gamma^3 + 3\,\gamma\,\delta^2\,K\right)$$

$$D_{26} = -\left(\begin{array}{l} 2\,m\,\gamma\,l_6 + 2\,m\,\delta\,l_8 + 2\,m\,\delta\,l_{52} + 2\,m\,\gamma\,l_{53} + m\,\delta\,l_{54} + m\,\gamma\,l_{55} + m\,l_{60} + m\,l_{65} \\ -8K\,\gamma^3\,l_6 + 8K\,\delta^3\,l_8 + 8K\,\delta^3\,l_{52} - 12K\,\gamma^2\delta\,l_{52} + 8K\,\gamma\delta^2\,l_{53} + 24K\,\gamma\delta^2\,l_{53} \\ -8K\,\gamma^3\,l_{53} + K\,\delta^3\,l_{54} - 3\,K\,\gamma^2\delta\,l_{54} + 3\,K\,\gamma\delta^2\,l_{55} - K\,\gamma^3\,l_{55} + K\,\delta^2\,l_{60} \\ -3K\,\gamma^2 l_{60} + 2K\,\delta^2\,l_{60} - 6K\,\gamma\,\delta\,l_{61} - 6\,K\,l_{63} \end{array}\right)$$

$$D_{27} = \left(-\delta^2\,Cosh\gamma\,Cos\delta + \gamma^2\,Cosh\gamma\,Cos\delta - 2\,\gamma\,\delta\,Sinh\gamma\,Sin\delta\right)$$

$$D_{28} = \left(\delta^2\,Cosh\gamma\,Sin\delta - \gamma^2\,Cosh\gamma\,Sin\delta - 2\,\gamma\,\delta\,Sinh\gamma\,Cos\delta\right)$$

$$D_{29} = \left(\delta^2\,Sinh\gamma\,Cos\delta - \gamma^2\,Sinh\gamma\,Cos\delta + 2\,\gamma\,\delta\,Cosh\gamma\,Sin\delta\right)$$

$$D_{30} = \left(\delta^2\,Sinh\gamma\,Sin\delta + \gamma^2\,Sinh\gamma\,Sin\delta + 2\,\gamma\,\delta\,Cosh\gamma\,Cos\delta\right)$$