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Free vibration analysis of short fiber reinforced laminated plates with first shear deformation theory

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Abstract

Vibration analysis of simply supported square laminated plates containing randomly and unidirectionally aligned short fibers was performed. The effective elastic modulus of composite was expressed by using the Mori-Tanaka mean field approach. The results were compared with the rule of mixture and Hashin bounds. The governing equations were obtained by means of Hamilton's principle and solved by using the Navier type solution and Ritz method. The effects of the fiber orientation, the degree of orthotropy, the fiber aspect ratio, the plate span to thickness ratio, and the fiber volume fraction on the vibration behavior of the laminated plates were studied. The mode frequency results are compared with results of the finite element model. It is observed that for increasing degree of orthotropy, the difference between the frequency parameters increases for increasing aspect ratio. The mode frequency results show that for larger aspect ratios, the frequencies for short fibers approach those of continuous fibers.

Key Words: Free vibration, composite plate, short fibers, effective moduli, Mori-Tanaka theory, mode shapes

1. Introduction

In recent years short fiber reinforced composite applications have increased in many branches of engineering. The shape and orientation of short fibers vary depending on structural requirements and loading conditions. It is difficult to control the orientation of fibers. Therefore, probabilistic studies are used for determining the orientation of fibers in a composite. The effective elastic constants of short fiber reinforced composites are predicted by many approaches in the literature. When the fiber volume fraction of the short fiber reinforced composite is small in the dilute case Eshelby's method estimates the elastic constants reasonably (Eshelby, 1957). When the volume fraction of the short fiber reinforced composite becomes significant, proposed aggregate models are used (Halpin et al., 1971; Christensen and Malls, 1972). The geometric aspect ratios of fibers and

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interactions between the fibers and the surrounding material are ignored in these models. The shear lag method considers the effect of short fiber length but no information can be obtained about the stress and strain fields of short fibers by this method (Aveston and Kelly, 1973). Another important approach is the Mori-Tanaka mean field theory based on Eshelby's solution of an ellipsoidal inclusion (Mori and Tanaka, 1973). It is useful for the calculation of non-dilute concentrations. Several approaches have been proposed in the literature by using the Mori-Tanaka method. Weng obtained the elastic moduli of identical shaped multiphase composites by the Mori-Tanaka method (Weng, 1990). Benveniste (1987) applied the Mori-Tanaka method to find stress and strain tensors of composites. Huang and coworkers used the Mori-Tanaka mean field theory and elucidated the effect of the aspect ratio and orientation effects on the elastic moduli (Chao et al., 1999). The elastic moduli results were used for the vibration and postbuckling analysis of a short fiber reinforced composite for various boundary conditions (Huang, 2000; Chang et al., 2004; Shukla et al., 2004; Huang and Shukla, 2005).

The dynamic behavior of laminated composite plates has received considerable attention in the past. On the basis of classical lamination theory, Jones (1973) obtained a closed form solution for the vibration and buckling analysis of cross-ply laminated plates with simply supported boundary conditions. The first-order shear deformation plate theory, commonly known as Mindlin plate theory, accounts for layer-wise constants states of transverse shear stresses (Mindlin, 1951). A Levy type solution was developed by Reddy and Khedir (1989) on the basis of parabolic shear deformation theory. Approximate methods are developed for the vibration and buckling analysis of laminated plates for various boundary conditions. Some earlier studies employing the Ritz method are based on the classical plate theories with displacement components assumed as double series of trigonometric functions (Leissa and Narita, 1989). Baharlou and Leissa (1987) employed simple algebraic polynomials in applying the Ritz method for various boundary conditions.

In the present study the vibration behavior of short fiber reinforced cross-ply laminated square plates for randomly and unidirectional fiber alignments were studied. The effective elastic moduli of the composite were expressed by using the Mori-Tanaka mean field theory for the unidirectionally aligned case. The governing equations of the vibration were solved by using a Navier type solution and the Ritz method. The effects of the fiber orientation, the ratio of orthotropy, the fiber aspect ratio, and the plate span to thickness ratio on the vibration behavior of the laminated plates were studied. The mode frequency results were determined and compared with the results of the finite element method.

2. The effective elastic moduli of composite

In this section the evaluation of the effective elastic properties for a composite containing unidirectionally aligned fibers is presented. To simulate geometrical configurations ranging from short fiber to continuous fiber spheroidal inclusion is defined by the following equation:

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} + \frac{z^2}{a_3^2} \leq 1 \tag{1}$$

For the calculation of the aspect ratio (for a prolate spheroid aspect ratio is defined as $a = a_3/a_1$) semi-axes of the spheroidal inclusions were taken as ($a_3 > a_1 = a_2$) prolate spheroid form (Mura, 1987).

For non-dilute composites the effective elastic modulus of a 2 phase composite are obtained explicitly as follows (Huang, 2001):

$$C = f_0 C_0 A_0 + \sum_{r=1}^{N} f_r C_r A_r \tag{2}$$

where A_0 and A_r are strain concentration factors of matrix and r_{th} fiber phases, respectively.

For non-dilute multiphase composites strain concentration factors are defined by

$$A_0 = \left[f_0 I + \sum_{r=1}^{N} f_r A_r^{dil} \right]^{-1} \tag{3}$$

$$A_r = A_r^{dil} A_0 \tag{4}$$

A_r^{dil} is dilute concentration factor of r_{th} phase and it is given by

$$A_r^{dil} = \left[I + S_r C_0^{-1} (C_r - C_0) \right]^{-1}_{,} \tag{5}$$

where I is the unit tensor and S_r is the Eshelby tensor for spheroidal inclusions in a transversely isotropic medium (Eshelby, 1957).

Variation of the Eshelby tensor according to the aspect ratio can be found in Sanboh et al. (1999).

This formulation is based on the assumption that principal axes of fibers coincide with the directions of the composite matrix. The effects of orientation on elastic properties of composites containing oriented fibers are evaluated in this section. Fiber orientation distribution can be expressed as (Huang, 2001):

$$L_{ij} = \frac{1}{4\theta_0 \sin\phi_0} \int_{-\theta_0}^{\theta_0} \int_{0}^{\pi} C_{ij}(\theta, \phi) \sin\phi \, d\phi \, d\theta \tag{6}$$

For spatially oriented fiber in a generally orthotropic medium, its orientation can be described by 2 Euler angles: θ, ϕ (Figure 1).

$$\rho_r(\theta, \phi) = 4\theta_0 \sin\phi_0 \tag{7}$$

$$-\theta_0 \le \theta \le \theta_0 \quad \frac{\pi}{2} - \phi_0 \le \phi \le \frac{\pi}{2} + \phi_0 \tag{8}$$

$\rho_r(\theta, \phi)$ is the probability density function of the r_{th} phase fibers and the matrix is independent of fiber orientation. $\sin\phi$ in the integrand is to account for the surface area of a sphere. $C(\theta, \phi)$ is the effective stiffness of composite obtained from Eq. (2) by performing tensor transformation.

For the case of the $\theta_0 = \pi$ and $\phi_0 = \frac{\pi}{2}$ fibers are uniformly distributed in all directions (Huang, 2001). The composite behavior is independent of direction. The composite has completely random distribution and is macroscopically isotropic. In this case, elastic modulus and effective Poisson ratio ν are obtained through the relations:

$$E = L_{11} - \frac{2L_{12}^2}{L_{11} + L_{12}} \quad \nu = \frac{L_{12}}{L_{11} + L_{12}} \tag{9}$$

The elastic modulus is obtained for transversely isotropic composites. In addition, the random distribution is expressed with the orientation distribution function. The results are compared with the rule of mixture (Jones, 1998; Calister, 2007) and Hashin bounds (Hashin and Shtrikman, 1963) for random distribution.

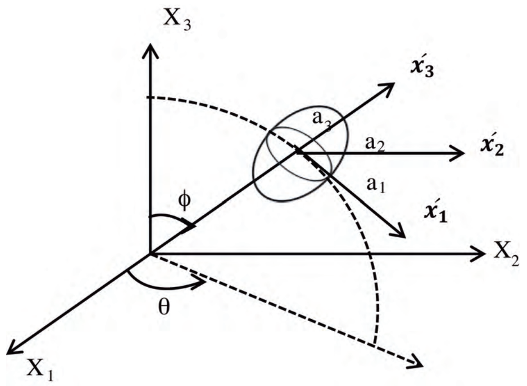


Figure 1. Orientation of short fibers.

3. Governing equations

The displacement field for the plate is assumed on the basis of the general shear deformable shell theory presented in the literature (Soldatos and Timarci, 1993).

$$\begin{aligned} &U(x,y,z;t)=u(x,y;t)-zw_{,x}+\phi_1(z)u_1(x,y;t),\\ &V(x,y,z;t)=v(x,y;t)-zw_{,y}+\phi_2(z)v_1(x,y;t),\\ &W(x,y,z;t)=w(x,y;t) \end{aligned} \tag{10}$$

where u, v, w, u_1, and v_1 are the 5 unknown displacement functions of the middle surface of the plate, while ϕ_1 and ϕ_2 represent shape functions determining the distribution of the transverse shear strains and stresses along the thickness. Upon employing Hamilton's principle, the 5 variationally consistent governing equations of the plate are obtained as:

$$\begin{aligned} &N^c_{x,x}+N^c_{xy,y}=\left(\rho_0 u-\rho_1 w_{,x}+\bar{\rho}_0^{11}u_1\right)_{,tt}\\ &N^c_{y,y}+N^c_{xy,x}=\left(\rho_0 v-\rho_1 w_{,y}+\bar{\rho}_0^{21}v_1\right)_{,tt}\\ &M^c_{x,xx}+M^c_{y,yy}+2M^c_{xy,xy}+q+N^c_x w_{,xx}+N^c_y w_{,yy}+N^c_{xy}w_{,xy}=\\ &\left[\rho_0 w-\rho_1 v_{,y}-\rho_2\left(w_{,yy}+w\right)+\bar{\rho}_1^{11}u_{1,x}+\bar{\rho}_1^{21}v_{1,y}+\rho_1 u_{,x}\right]_{,tt}\\ &M^a_{x,x}+M^a_{xy,y}-Q^a_x=\left(\bar{\rho}_0^{11}u-\bar{\rho}_1^{11}w_{,x}+\bar{\rho}_0^{12}u_1\right)_{,tt}\\ &M^a_{y,y}+M^a_{yx,y}-Q^a_y=\left(\bar{\rho}_0^{21}v-\bar{\rho}_1^{21}w_{,y}+\bar{\rho}_0^{22}v_1\right)_{,tt} \end{aligned} \tag{11}$$

Here q is the transverse load, and N^c_x, N^c_y *and* N^c_{xy} are the constant in-plane edge loads. The inertias ρ_i and $\bar{\rho}_i^{lm}$ are defined by

$$\begin{aligned} &\rho_i=\int_{-h/2}^{h/2}\rho z^i dz, \quad (i=0,1,2),\\ &\bar{\rho}_i^{lm}=\int_{-h/2}^{h/2}\rho z^i\phi_l^m dz, \quad (i=0,1;\quad l=m=1,2) \end{aligned} \tag{12}$$

where ρ is the mass per unit volume.

Here c and a indices denote classical plate theory (CPT) and first shear deformation theory (FSDT), respectively. Although different shape functions are applicable, only the one that converts the present theory to the corresponding FSDT is employed in the present study. This is achieved by choosing the shape functions as follows:

$$FSDT: \quad \phi_1(z)=\phi_2(z)=z \tag{13}$$

and shear correction factor is taken as $k = \sqrt{5/6}$.

By substituting the stress-strain relations into definitions of force and moment resultants the following constitutive equations are obtained:

$$\begin{bmatrix} N^c \\ M^c \\ M^a \end{bmatrix} = \begin{bmatrix} A_{ij} & B_{ij} & B_{ijl} \\ B_{ij} & D_{ij} & D_{ijl} \\ B_{ijl} & D_{ijl} & D_{ijlm} \end{bmatrix} \begin{bmatrix} e^c \\ k^c \\ k^a \end{bmatrix} \tag{14}$$

$$N^c = \begin{bmatrix} N_x^c \\ N_y^c \\ N_{xy}^c \end{bmatrix}, \quad M^c = \begin{bmatrix} M_x^c \\ M_y^c \\ M_{xy}^c \end{bmatrix}, \quad M^a = \begin{bmatrix} M_x^a \\ M_y^a \\ M_{xy}^a \\ M_{yx}^a \end{bmatrix} \tag{15}$$

$$e^c = \begin{bmatrix} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \end{bmatrix}, \quad k^c = \begin{bmatrix} -w_{,xx} \\ -w_{,yy} \\ -2w_{,xy} \end{bmatrix} \quad k^a = \begin{bmatrix} M_x^a \\ M_y^a \\ M_{xy}^a \\ M_{yx}^a \end{bmatrix} \tag{16}$$

$$\begin{bmatrix} Q_y^a \\ Q_x^a \end{bmatrix} = \begin{bmatrix} A_{4422} & 0 \\ 0 & A_{5511} \end{bmatrix} \begin{bmatrix} v_1 \\ u_1 \end{bmatrix} \tag{17}$$

The extensional, coupling, bending, and transverse rigidities are defined as follows:

$$A_{ij} = \int_{-h/2}^{h/2} Q_{ij}^k dz, \quad A_{pqlm} = \int_{-h/2}^{h/2} Q_{pq}^k \phi_l' \phi_m' dz$$

$$B_{ij} = \int_{-h/2}^{h/2} Q_{ij}^k z dz \quad B_{ijl} = \int_{-h/2}^{h/2} Q_{ij}^k \phi_l dz$$

$$D_{ij} = \int_{-h/2}^{h/2} Q_{ij}^k z^2 dz \quad D_{ijl} = \int_{-h/2}^{h/2} Q_{ij}^k \phi_l(z) z dz$$

$$D_{ijlm} = \int_{-h/2}^{h/2} Q_{ij}^k \phi_l \phi_m dz \tag{18}$$

The governing equations are solved by using the Navier type solution and the Ritz method (Aydogdu and Timarci, 2003).

4. Ritz solution for vibration of cross-ply laminated plates

The vibration analysis of cross-ply laminated plates with various boundary conditions is performed by the Ritz method. It is a variational approach and requires the expansion of the unknown functions of displacement components in terms of infinite series. Trigonometric functions (Leissa and Narita, 1989), algebraic polynomials (Baharlou and Leissa, 1987), and orthogonal polynomials have been employed on the basis of different plate theories in composite plate studies (Bhat, 1985; Messina and Soldatos, 1999). In the present study, after defining non-dimensional coordinates as $\xi = \mathrm{x/L}$ and $\eta = y/L$ the following simple algebraic polynomials are used:

$$\begin{aligned}
u(\xi,\eta;t) &= \sum_{i=0}^{I-1}\sum_{j=0}^{J-1} A_{ij}X_i(\xi)\,Y_j(\eta)\,\sin\omega t,\\
v(\xi,\eta;t) &= \sum_{k=0}^{K-1}\sum_{l=0}^{L-1} B_{kl}X_k(\xi)\,Y_l(\eta)\,\sin\omega t,\\
w(\xi,\eta;t) &= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} C_{mn}X_m(\xi)\,Y_n(\eta)\,\sin\omega t\\
u_1(\xi,\eta;t) &= \sum_{p=0}^{P-1}\sum_{q=0}^{Q-1} D_{pq}X_p(\xi)\,Y_q(\eta)\,\sin\omega t,\\
v_1(\xi,\eta;t) &= \sum_{r=0}^{R-1}\sum_{s=0}^{S-1} E_{rs}X_r(\xi)\,Y_s(\eta)\,\sin\omega t,
\end{aligned} \tag{19}$$

where the polynomials are defined as:

$$X_f = \xi^f(\xi+1)^{B_1}(\xi-1)^{B_3}, \qquad f = i,k,m,p,rY_g(\eta) = \eta^g(\eta+1)^{B_2}(\eta-1)^{B_4}, \qquad g = j,l,n,q,s \tag{20}$$

and A_{ij}, B_{kl}, C_{mn}, D_{pq} $and\, E_{rs}$ are unknown undetermined coefficients. Here B_i can take values that are chosen according to the type of boundary conditions imposed at the edges of the plate; the index i denotes the subsequent edges of the plate in the counterclockwise direction. The edge numbered 1 is the one at $\xi = -1$. The values of $B=$ 0, 1, and 2 correspond to the free, simply supported, and clamped edge, respectively (Narita, 2000). In our work, for simply supported boundary condition B = 1 is taken for numerical analysis.

The dimensionless free vibration frequencies are defined as follows:

$$\lambda^2 = (\rho L_x^4\omega^2/E_2h^3) \tag{21}$$

5. Numerical results

The effective elastic moduli of the composite are expressed by using the Mori-Tanaka mean field approach for unidirectionally aligned inclusions. The effective elastic moduli of composite with randomly oriented fibers are obtained by orientation distribution function. The elastic moduli are obtained for different aspect ratios namely a = 1, 4, 40, and 100. The elastic moduli are taken as $E_0 = 5.35\,GPa$ for matrix and $E_1 = 73\,GPa$ for inclusions.

The elastic modulus results are compared with results in the literature (Huang, 2000) for the unidirectionally aligned case and good agreement is observed. For random orientation of fibers the Mori-Tanaka results are compared with the rule of mixture and Hashin bounds in Figure 2. The elastic modulus results are shown to be close for lower fiber volume fractions. The elastic modulus results are used for the free vibration analysis of cross-ply laminated plates with simply supported boundary conditions. The frequency parameter results of the Navier type solution and the Ritz method for randomly oriented short fiber reinforced cross-ply laminate of

[0/90/90/0] with a = 1, f = 0.3 are presented in Figure 3 as a function of plate span to thickness ratio (L/h). It is observed that for larger L/h ratios the Ritz results converge to Navier solutions.

The degree of orthotropy (E1/E2) on the vibration behavior of the laminated plates is shown in Figure 4. The analysis was performed for (0°) simply supported cross ply laminated composite plates containing unidirectionally aligned inclusions. It is observed that for increasing (E_1/E_2) ratio the difference between the frequency parameters increases for increasing aspect ratio.

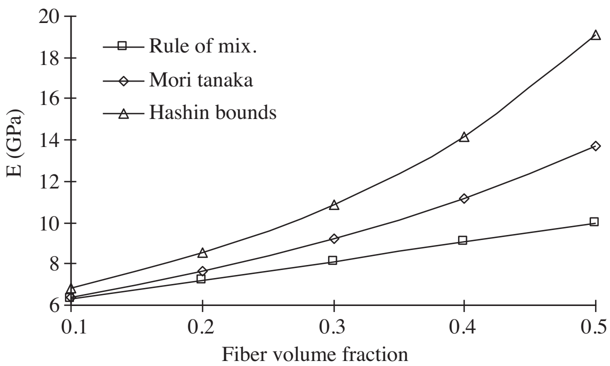


Figure 2. Variation in elastic modulus with fiber volume fraction.

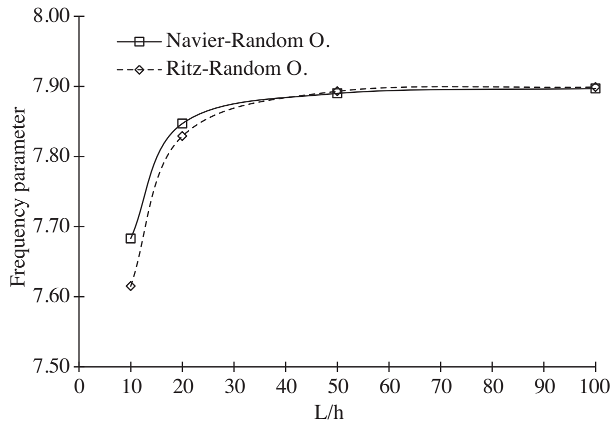


Figure 3. Variation in frequency parameter with (L/h) ratio.

In Figures 5 and 6, effects of the aspect ratio and plate span-to-thickness ratio effect on the free vibration results of a simply supported composite plate with [0/90/90/0] layup are shown.

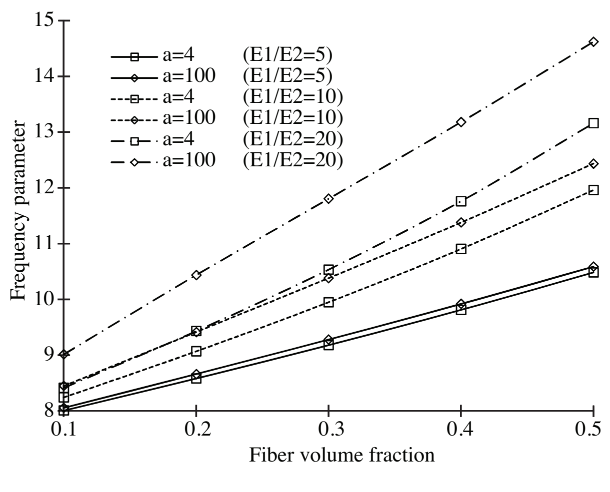


Figure 4. Variation in frequency parameter with (E_1/E_2) ratio.

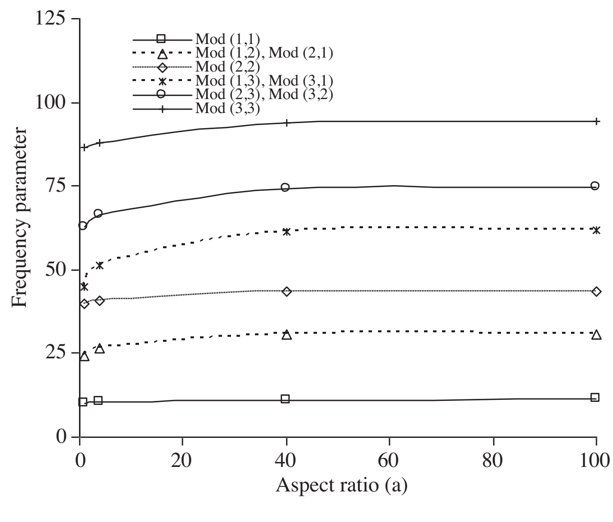


Figure 5. Variation in frequency parameter for [0/90/90/0] layup with aspect ratio and for (f_r = 0.3, L/h = 10).

This study was performed for the first 9 modes of laminated plates. Figures show that the frequency parameter is more sensitive for smaller aspect ratios (1-5) and with increasing aspect ratio they asymptotically converge to a value. It can be said that for larger aspect ratio frequency of the short fibers, the results approach those of continuous fibers (Eruslu and Aydogdu, 2009). It is observed that the frequency parameter increase for larger L/h ratios.

In Figures 7 and 8 the free vibration frequencies of symmetric and anti-symmetric cross-ply plate with a = 40, L/h = 50 are obtained and presented for different volume fractions (f_r). It can be seen that all frequency parameters increase with increasing fiber volume fraction. The frequency parameter result for symmetric cross-ply layups comes out to be higher than the results for anti-symmetric cross-ply layups.

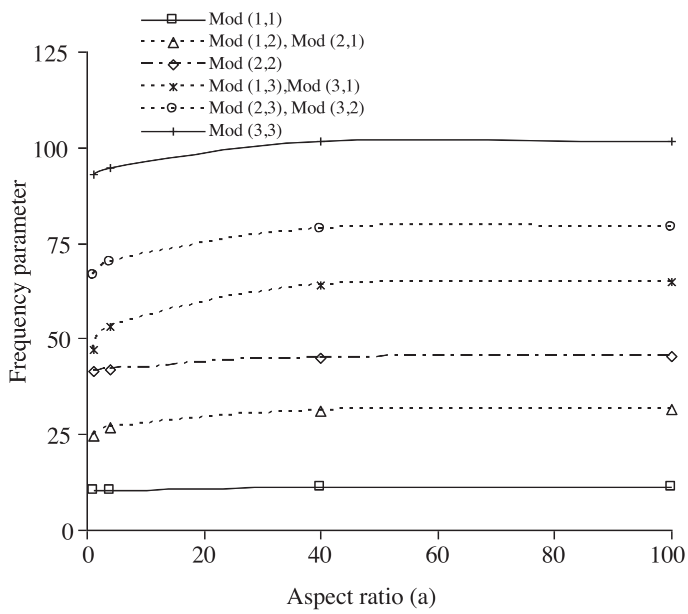


Figure 6. Variation in frequency parameter for [0/90/90/0] layup with aspect ratio and for (f_r = 0.3, L/h = 100).

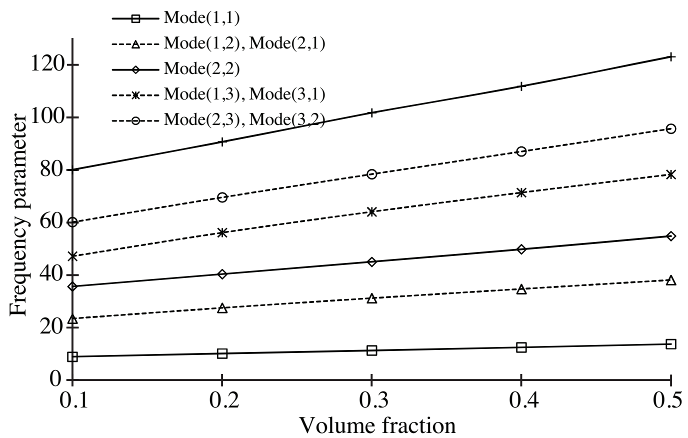


Figure 7. Variation in frequency parameter for [0/90/90/0] layup with fiber volume fraction for (L/h = 50, a = 40).

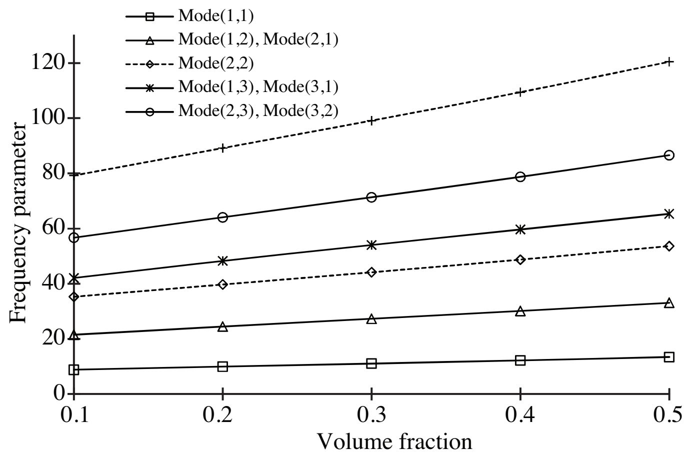


Figure 8. Variation in frequency parameter for [0/90/0/90] layup with fiber volume fraction for (L/h = 50, a = 40).

The frequency parameter results of the Navier type exact solution and ANSYS (finite element code) are compared for a = 1, 40, f = 0.1, L/h = 50 in Tables 1 and 2, respectively. The analysis is performed for simply supported antisymetric cross-ply laminated plates.

Table 1. Frequency parameter results of composite plate with [0/90/0/90] layup for the first 9 modes (a = 1).

Mode	M	n	Frequency parameter (Navier)	Frequency parameter (ANSYS)
1	1	1	8.5	8.3
2	1	2	20.0	20.3
3	2	1	20.0	20.3
4	2	2	33.9	33.1
5	1	3	38.0	39.9
6	3	1	38.0	39.9
7	2	3	53.8	53.3
8	3	2	53.8	53.3
9	3	3	76.2	67.2

Table 2. Frequency parameter results of composite plate with [0/90/0/90] layup for the first 9 modes (a = 40).

Mode	M	n	Frequency parameter (Navier)	Frequency parameter (ANSYS)
1	1	1	8.8	8.8
2	1	2	21.5	22.0
3	2	1	21.5	22.0
4	2	2	35.3	35.0
5	1	3	42.0	44.0
6	3	1	42.0	44.0
7	2	3	56.7	56.8
8	3	2	56.7	56.8
9	3	3	79.2	74.9

According to these tables, the largest difference between solutions is obtained as 11% for a = 1 and 5% for a = 40. The mode frequency results show that both results are close to each other, especially for larger aspect ratios. This means that for larger aspect ratios the frequencies of the short fibers are close to those of continuous fibers.

The mode shape results show that the in plane displacements (u,v) are negligible and the displacements including transverse shear effects (u_1, v_1) are insignificant in contrast to out-of-plane displacements (w). The mode shapes of symmetric modes are similar. In Figure 10, the effect of L/h ratio on transverse shear effects is shown for 2 different modes. It is found that transverse shear effects are effective for small L/h ratios as expected.

In Figure 9a and b the first 9 mode shapes of plate with [0/90/0/90] layup are presented.

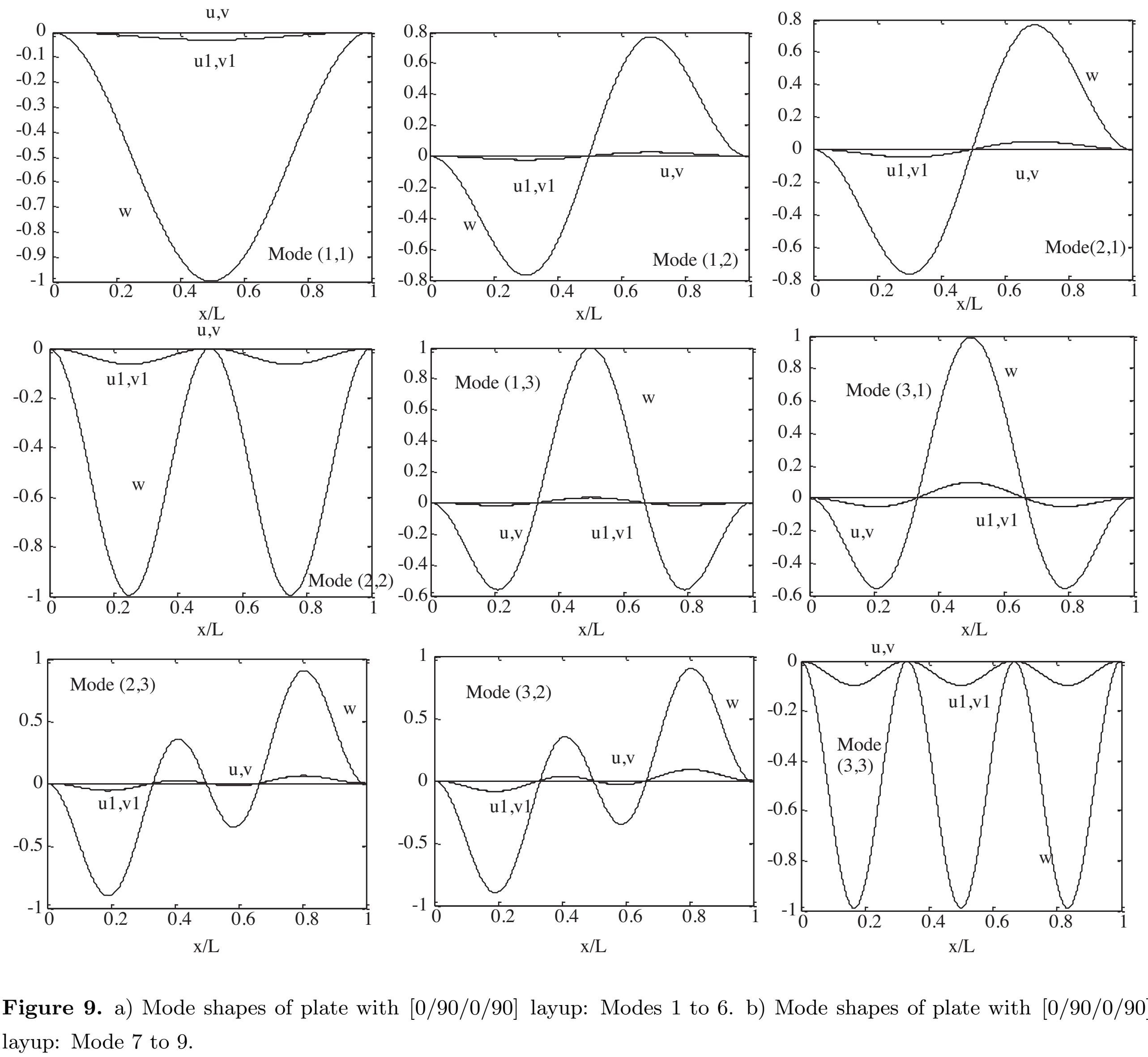
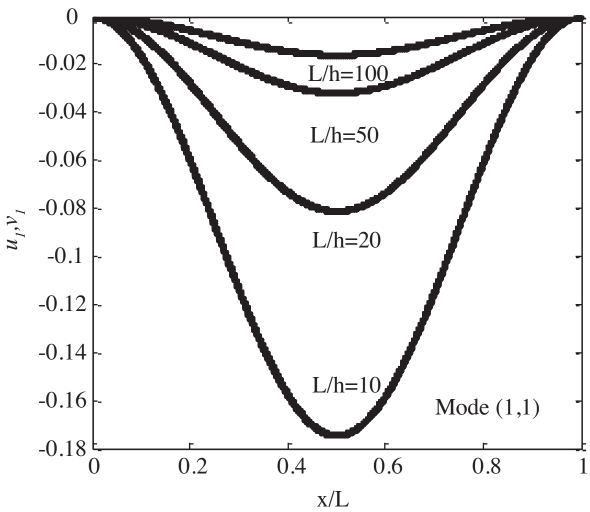


Figure 9. a) Mode shapes of plate with [0/90/0/90] layup: Modes 1 to 6. b) Mode shapes of plate with [0/90/0/90] layup: Mode 7 to 9.



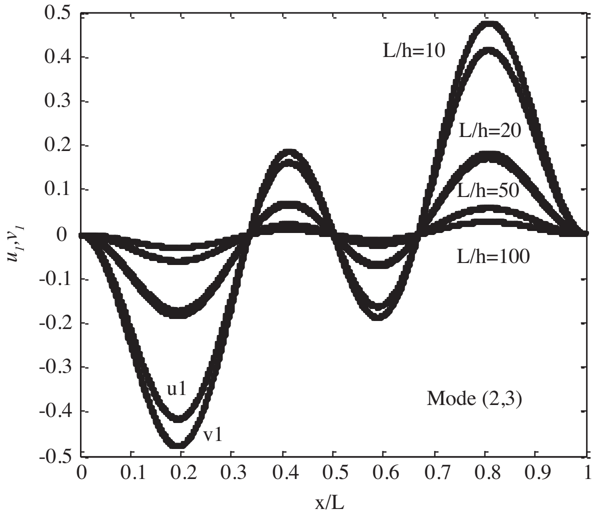


Figure 10. Effect of L/h ratio on mode shapes of plate with [0/90/0/90] layup.

6. Conclusions

In this work, free vibration characteristics of shear deformable, cross-ply laminated short fiber reinforced composite square plates are studied. The effective elastic moduli of composites are expressed using the Mori-Tanaka mean field approach for unidirectionally aligned inclusions. The random orientation is introduced by using the fiber orientation function. The plate with simply supported boundary conditions is solved by Navier's method and the Ritz method. The dimensionless frequency results are given for different aspect ratios, degree of orthotropy, fiber volume fraction, and the length-to-thickness ratio. The mode shape results are given for frequency results.

It is found that for increasing (E_1/E_2) ratio the difference between the frequency parameters increases for increasing aspect ratio. The mode frequency results show that the frequency parameter is more sensitive for smaller aspect ratio (1-5) and with increasing aspect ratio they asymptotically converge to a value. The mode shape results show that the transverse shear effects (u_1, v_1) are insignificant for larger L/h ratios in contrast to out-of-plane displacements. The present study may be extended to higher order shear deformation theories for various boundary conditions.

Nomenclature

a_1, a_2, a_3	semi-axes of the spheroid
a	aspect ratio
C_0, C_r	elastic moduli of matrix and fiber phases
C	effective elastic modulus of 2-phase composite
f_0, f_r	volume fractions of matrix and fiber phases
ε_a	uniform far-field applied load
$\langle\rangle$	volume averaging
$\langle\varepsilon_r^{pt}\rangle, \langle\varepsilon_0^{pt}\rangle$	average perturbed strains in fiber and matrix phases
$\langle\varepsilon_0\rangle, \langle\varepsilon_r\rangle$	average strains in matrix and fiber phases
ε^T	transformation strain
S_r	Eshelby tensor
A_r^{dil}	dilute concentration factor of r_{th} phase

A_0	non-dilute strain concentration factor of matrix phase
A_r	non-dilute strain concentration factor of fiber phase
L_x, L_y	plate dimensions in x, y directions
N	number of layers in the laminated composite plate
E_1, E_2	elastic moduli for a composite layer
G_{12}, G_{13}, G_{23}	shear moduli for a composite layer
ν_{12}, ν_{13}	Poisson's ratios
h	plate thickness
U, V, W	displacements in x-, y-, and z-directions, respectively
u, v, w	displacement components in the mid-plane
u_1, v_1	unknown functions representing the effect of transverse shear strains for the mid-plane
Q_{ij} (i, j = 1, 2, 6)	reduced stiffness of composite
ε_x, ε_y, γ_{xy}, γ_{xz}, γ_{yz}	strain components
x,y,z	Cartesian coordinates
t	time
N_x^c, N_y^c, N_{xy}^c, Q_x^a, Q_y^a	force resultants
M_x^c, M_y^c, M_{xy}^c, M_x^a, M_y^a, M_{xy}^a, M_{yx}^a	moment resultants
A_{ij}, B_{ij}, D_{ij} (i,j=1,2,6)	stiffness matrices
B_{ijl}, D_{ijl}, D_{ijlm}, A_{pqlm} (i, j, l, m = 1, 2; p, q = 4, 5)	shear deformation stiffnesses
q	transverse load
N_x^e, N_y^e, N_{xy}^e	external in-plane loads
ρ_i, $\bar{\rho}_j^{lm}$ $(i = 0, 1, 2; j = 0, 1; l, m = 1, 2)$	inertias
ρ	mass per unit volume
ω	circular frequency
$[K]$	stiffness matrix
$[M]$	mass matrix in free vibration
$\{\Delta\}$	column vector of undetermined coefficients
λ	non-dimensional frequency parameter

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