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Single Value Decomposition For Stability Analysis of Nonlinear Poiseuille Flows

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Abstract

In nonlinear analysis of fluid mechanics problems, small amplitude oscillations near the Hopf bifurcation point are well-described by the Ginzburg-Landau equation. The coefficients of the Ginzburg-Landau equation can be computed efficiently and conveniently by Singular Value Decomposition (SVD). In this study, the Ginzburg-Landau equation is derived for plane Poiseuille flow problem of a Newtonian fluid and the SVD method is applied in order to show how to find the coefficients of the Ginzburg-Landau equation. The analysis indicates that SVD is easy to implement and straightforward; making it the method of choice for the numerical computations of the coefficients of amplitude equations.

Key Words: Poiseuille flow, Stability, Bifurcation theory, Singular Value Decomposition

Lineer Olmayan Poiseuille Akıştaki Amplitüd Denklemlerinin Katsayılarının Nümerik Hesabı

Özet

Lineer olmayan akışkanlar mekaniği problemlerinin analizinde, Hopf çatallaşma noktası yakınındaki düşük amplitüdlü osilasyonlar Ginzburg-Landau denklemi ile formüle edilebilirler. Ginzburg-Landau denkleminin katsayıları Tekil Değer Ayrışımı (TDA) metodu ile pratik ve kolay olarak hesaplanabilir. Bu çalışmada, Ginzburg-Landau denklemi Newtoniyen bir akışkanın Poiseuille akışı için çıkarılmış ve denklemin katsayılarının TDA metodu uygulanarak nasıl bulunabileceği gösterilmiştir. Bu analizin sonuçları göstermektedir ki TDA, kolay ve direkt olarak uygulanabilir olması özelliğinden dolayı, amplitüd denklemlerinin katsayılarının nümerik olarak hesaplanması arzulanan durumlarda tercih edilebilecek bir metoddur.

Anahtar Sözcükler: Poiseuille akış, Stabilite, Çatallaşma teorisi, Tekil Değer Ayrışımı

Introduction

It is known from the experiments of Davies and White (1928) that in plane Poiseuille flow turbulence of some kind can exist at Reynolds numbers, Re, (based on half channel width and maximum velocity) as low as 1000. However, the linearized theory of instability (Orszag, 1971) gives a value of Re_c, critical Reynolds number, of about 5772. Consequently the linearized theory gives results radically different from those of relevant experiments. A possible explanation for this discrepancy is that even for $Re<Re_c$, nonlinear effects might provide a threshold amplitude above which velocity perturbations could

grow and stimulate instabilities. Therefore, nonlinear theory provides information on the following two important questions that arise in hydrodynamic instability: whether or not a flow which is stable to infinitesimal distrubances might be unstable to disturbances of some finite amplitude, and what the nature of the finite amplitude equilibrium flow which develops as a result of an initial instability would be.

The nonlinear stability analysis presented here is based on the bifurcation theory. This theory is restricted in applications to those cases in which there is a threshold for instability, i.e. stable solutions exist. In these cases, one may go beyond bifurcation into monochromatic waves and derive amplitude equations which allow for slow modulations of wavy flow in space and time. This amplitude equation is called the Ginzburg-Landau equation and is applicable to small-amplitude waves which modulate a monochromatic wave of wavelength $2\pi/\alpha_c$, where α_c is the critical wavenumber at the nose of the neutral curve.

There are various ways to compute amplitude equations of bifurcation problems. The bifurcation parameters and the coefficient for the amplitude equations can be determined by formulas expressing the requirements of the Fredholm alternative. The Fredholm alternative requires that the inhomogeneous terms in he underlying system of differential equations, which contain the unknown coefficients, be orthogonal to the indepented eigenvectors that span the null-space of the adjoint system of differential equations. These formulas involve many unit operations, explicit calculation of the adjoint, and integration over the flow domain of a multiplicative composition of eigenfunctions and adjoint eigenfunctions. These operations can not usually be carried out analytically and require numerical computation. This conventional solution procedure requires too much work especially for complicated problems like those which arise in two-fluid dynamics. Therefore the numerical computation of bifurcation has become increasingly popular in recent years.

In numerical computation, one works entirely with the matrix formulations generated by the initial discretization. SVD is a natural, powerful and practical method to carry out these numerical computations. Langford (1977) who studied two-point boundary value problems shows how SVD can be applied to the numerical solution of perturbation problems. He proposed an algorithm converting a two-point boundary value problem to an initial value problem plus a least squares problem. He solved the best least squares problem by applying SVD. However, the solvability condition was still enforced by evaluating a complicated integral involving the adjoint eigenvector. A full application of SVD to bifurcation problems was made by Chen and Joseph (1991). They applied the method to compute the coefficients of the Ginzburg-Landau equation for the nonlinear evolution of interfacial waves arising from axisymmetric disturbances of core-annular flow of two fluids in a pipe.

The objective of this study is to present a general review of the methodology of SVD and its application to bifurcation problems. SVD method is applied to amplitude equations resulting from the consideration of small-amplitude oscillations near the Hopf bifurcation point and the numerical computation of the coefficients of amplitude equations for the case of plane Poiseuille flow is considered.

Amplitude Equations

The onset of instabilities due to infinitesimal disturbances can be predicted accurately by linear stability analysis in a fluid flow problem and the critical value of the flow controlling prameter, say Reynolds number, Re_c can be determined (see, for example, Pınarbaşı and Liakopoulos, 1995). If the amplitude of disturbances after Re_c becomes too large, a nonlinear theory is required in order to follow the evolution of such perturbations. Small-amplitude oscillations near the Hopf bifurcation point are generally governed by a simple evolution equation. If such oscillations form a field through diffusion-coupling, the governing equation is a simple partial differential equation called the Ginzburg-Landau equation. A brief description of the amplitude equations will be presented here, but the derivation of the Ginzburg-Landau equation will be given in the next section by considering nonlinear analysis of plane Poiseuille flow. It should be noted here that the Ginzburg-Landau equation is not only related to a few fluid mechanical or optical problems but that it can be deduced from a rather general class of partial differential equations.

Many theories on the nonlinear dynamics of the dissipative systems are based on the first-order ordinary differential equations (Kuramoto, 1984)

$$\frac{dX_i}{dt} = F_i(X_1, X_2, \ldots, X_n; \mu), i = 1, 2, \ldots, n \quad (1)$$

which include some control parameters represented by μ. For some range of μ, the system may remain stable in a time-independent state. In particular, this is usually the case for macroscopic physical systems which lie sufficiently close to the thermal equilibrium. In many systems, such a steady state loses stability at some critical value μ_c of μ, and beyond it (say $\mu > \mu_c$), turns into a periodic motion. In the parameter-amplitude plane, this appears as a branching of some time-periodic solutions from a stationary solution branch, and this phenomenon is generally called *Hopf bifurcation.* As μ increases further, the system may show more and more complicated dynamics through a number of bifurcations. It may show complicated periodic oscillations, quasi-periodic oscillations or a variety of non-periodic behaviors.

Employing a multi-scale method, Eq. (1) can be contracted to a very simple universal equation called the Stuart-Landau equation (Eq. (2) below). In fact, Landau (1994) was first to conjecture the equation form, and Stuart (1960) was the first to derive it through an asymptotic method. Stuart (1960) who studied the evolution of monochromatic waves in parallel shear flows suggested that the evolution of disturbances near criticality can be treated by means of an expansion in powers of (Re-Re$_c$) or of some parameter close to that Reynolds number difference. After some analysis, it was deduced that the time-dependent amplitude A of the leading Fourier mode of the expansion satisfied the nonlinear ordinary differential equation

$$\frac{d}{dt}|A|^2 = k_1|A|^2 + k_2|A|^4 \tag{2}$$

where k_1 and k_2 are constants.

In many physical problems, partial differential equations describing the process in the space-time domain prove to be a more useful mathematical tool. Thus, it is desirable that the Stuart-Landau equation be generalized so as to cover such circumstances. An appropriate mathematicl model is then obtained by simply adding diffusion terms to Eq. (1) as

$$\frac{\partial X}{\partial t} = F(X) + D\nabla^2 X. \tag{3}$$

Eq. (3) is called a reaction-diffusion equation (Kuramoto, 1984), and D is a matrix of diffusion constants. In addition to depending on time scales, Eq. (3) now also has slow space dependence.

Fluid mechanicians have developed theories which proved to be very useful in understanding instabilities arising in systems in one or two dimensions. A typical example in the Stewartson-Stuart theory (1971) on plane Poiseuille flow. They worked with partial differential equations throughout, not transforming them into ordinary differential equations. They generalized the form of the Stuart-Landau equation, thereby admitting slow spatial and temporal modulation of the envelope of the bifurcating flow patterns. The amplitude equation they derived is called the Ginzburg-Landau equation

$$\frac{\partial A}{\partial \tau} - a_2\frac{\partial^2 A}{\partial \xi^2} = \frac{d_1}{d_{1r}}A - \kappa A|A|^2 \tag{4}$$

where A is the amplitude of the waves; ξ and τ are scaled length and time, respectively; a_2 and d_1 are constants that are properties of the flow obtained from linear stability theory, and κ is the Landau constant from which the effect of nonlinear interactions is determined.

In the next section, Eq. (4) will be derived by considering plane Poiseuille flow of a Newtonian fluid and SVD will be applied to find the coefficients that appear in Eq. (4).

Application of Singular Value Decomposition to Find the Coefficients of Amplitude Equations

Consider the plane Poiseuille flow of an incompressible viscous fluid in steady motion at a Reynolds number Re close to critical value Re_c, above which small velocity perturbations may be amplified. The governing equations in a suitably normalized form are

$$\nabla.V = 0 \tag{5}$$

$$\frac{\partial V}{\partial t} + (V.\nabla)V = -\nabla P + \frac{1}{Re}\nabla^2 V \tag{6}$$

The corresponding boundary conditions are that $u = v = 0$ at $y = \mp 1$ (no-slip condition at walls). In the undisturbed motion, the base flow is

$$u_b = 1 - y^2, v_b = 0, dP_b/dx = -2/Re$$

which is the fully developed flow under a uniform pressure gradient. Introducing a streamfunction

$$u = \frac{\partial \Psi}{\partial y}, v = -\frac{\partial \Psi}{\partial x},$$

Eq. (5) is satisfied exactly and eq. (6), after eliminating the pressure terms by cross-differentiation, becomes

$$\frac{\partial}{\partial t}(\nabla^2\psi)+\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2\psi)-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2\psi)=\frac{1}{Re}(\nabla^4\psi) \tag{7}$$

where

$$\nabla^2\psi=\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}.$$

Imposing two-dimensional disturbances on the base flow and denoting perturbation streamfunction by Ψ, the governing equations take the form

$$\frac{\partial}{\partial t}(\nabla^2\Psi)+(\frac{\partial\Psi}{\partial y}+u_b)\frac{\partial}{\partial x}(\nabla^2\Psi) \\ -\frac{\partial\Psi}{\partial x}(u_b^{''}+\frac{\partial}{\partial y}(\nabla^2\Psi))=(\nabla^2\Psi))\frac{1}{Re}(\nabla^4\Psi) \tag{8}$$

where primes denote differentiation with respect to y.

In the linear analysis, the nonlinear terms in Eq. (8) are neglected and Ψ is assumed to have the form, $\Psi=\theta(y)exp[i\alpha(x-ct)]$. Then the well-known Orr-Sommerfeld equation is obtained

$$\mathcal{L}(\phi)=i\alpha Re\{(u_b-c)(\phi''-\alpha^2\phi)-u_b''\phi\} \\ -(\phi^{IV}-2\alpha^2\phi''+\alpha^4\phi)=0 \tag{9}$$

where $\mathcal{L}$ is the linear Orr-Sommerfeld operator and $c=c_r+ic_i$ is the wavespeed. Let $\mathcal{L}_1$ denote the linear Orr-Sommerfeld operator at criticality, i.e. $Re=Re_c, \alpha=\alpha_c$ and $c=c_c=c_r$ since $c_i=0$ at critical conditions. The results of this linear stability analysis give Re_c=5772.2, α_c=1.02 (see Fig. 1).

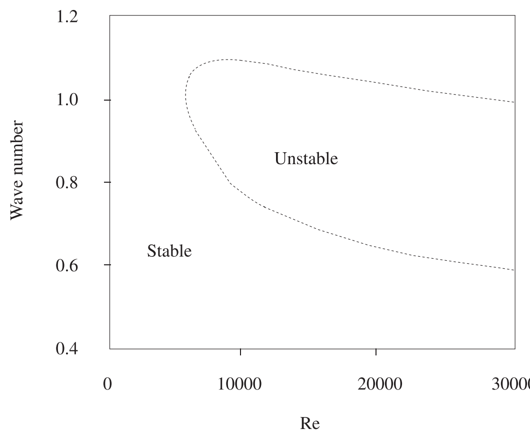

Figure 1. Neutral stability curve for Poiseuille flow (Re_c=5772.2, α_c=1.02).

In order to perform nonlinear stability analysis and to derive amplitude equations, the multiple-scales method is used near critical conditions. Here, the methodology used by Chen and Joseph (1991) will be followed. In this method, first introduce a small perturbation parameter ϵ, defined by

$$\epsilon^2=|d_{1r}(Re-Re_c)| \tag{10}$$

where

$$d_{1r}=Real\{d_1\}, d_1=-i\{\frac{\partial(\alpha c)}{\partial Re}\}_{(\alpha_c,Re_c)} \tag{11}$$

Here, $(-i\alpha c)$ is the linear complex growth rate for the linear instability of the base flow and (α_c, Re_c) is the point at the nose of the neutral curve.

Next, introduce the slow spatial and time scales

$$\xi=\epsilon(x-c_gt), \tau=\epsilon^2 t \tag{12}$$

where c_g is the group velocity at criticality. The scales are appropriate for a wave packet centered at the nose of the neutral curve and the long-time behavior of this wavetrain is examined in the frame moving with its group velocity. The perturbation streamfunction, Ψ, is assumed to be slowly varying functions of ξ and τ;

$$\Psi\rightarrow\Psi(\xi,\tau;x,y,t)$$

then,

$$\frac{\partial}{\partial t}\rightarrow\frac{\partial}{\partial t}-\epsilon c_g\frac{\partial}{\partial\xi}+\epsilon^2\frac{\partial}{\partial\tau} \\ \frac{\partial}{\partial x}\rightarrow\frac{\partial}{\partial x}+\epsilon\frac{\partial}{\partial\xi}. \tag{13}$$

Then define the travelling wave factor of the amplitude

$$E=exp[i\alpha_c(x-c_rt)] \tag{14}$$

where c_r is the phase speed at criticality. For a wave packet centered around the critical state, ψ can be assumed to be of the following form:

$$\Psi=\psi_0(y,\xi,\tau)+\{\psi_1(y,\xi,\tau)E+c.c.\}+ \\ \{\psi_2(y,\xi,\tau)E^2+c.c.\}+h.h. \tag{15}$$

where $c.c.$ stands for complex conjugate and $h.h.$ for higher harmonics. Assume that the fundamental wave $\psi_1(y,\xi,\tau)E$ is of order ϵ and expansions in ϵ yield

$$\begin{aligned}\psi_1 &= \epsilon\psi_{11}(y,\xi,\tau)+\epsilon^2\psi_{12}(y,\xi,\tau)\\ &\quad+\epsilon^3\psi_{13}(y,\xi,\tau)+O(\epsilon^4)\\ \psi_2 &= \epsilon^2\psi_{22}(y,\xi,\tau)+O(\epsilon^4)\\ \psi_0 &= \epsilon^2\psi_{02}(y,\xi,\tau)+O(\epsilon^4)\end{aligned} \tag{16}$$

Substituting the above expressions into the nonlinear systems of equations and identifying different orders $(k,n) \Leftrightarrow (E^k, \epsilon^n)$ results in a sequence of differential equations. At order (1, 1) one obtains

$$\begin{aligned}&i\alpha_c Re_c\{(u_b-c_c)(\psi_{11}''-\alpha^2\psi_{11})-u_b''\psi_{11}\}\\ &(\psi_{11}^{IV}-2\alpha^2\psi_{11}''+\alpha^4\psi_{11})=0\end{aligned}$$

which is in the same form as the linear eigenvalue problem at criticality. Denoting the eigenfunction at criticality by ϕ,

$$\psi_{11}(y,\xi,\tau)=A(\xi,\tau)\phi(y) \tag{17}$$

where $A(\xi,\tau)$ is the slowly varying amplitude of the fundamental wave to be found. The equations that arise at orders (0, 2), (2, 2) and (1, 2) support separated product solutions of the following type

$$\begin{aligned}\psi_{02}(y,\xi,\tau)&=|A(\xi,\tau)|^2F(y)\\ \psi_{22}(y,\xi,\tau)&=A^2(\xi,\tau)G(y)\\ \psi_{12}(y,\xi,\tau)&=\frac{\partial A(\xi,\tau)}{\partial\xi}H(y)+A_2(\xi,\tau)\phi(y)\end{aligned} \tag{18}$$

Then at orders (1, 2) and (1, 3) one finds

$$\mathcal{L}_1(\Psi_{12})=Z(\phi(y),c_g)$$

$$\begin{aligned}\mathcal{L}_1(\Psi_{13})&=J_1\frac{\partial A}{\partial\tau}+J_2\frac{\partial^2A}{\partial\xi^2}+J_3A+\\ &J_4A|A|^2+J_5\frac{\partial A_2}{\partial\xi}\end{aligned} \tag{19}$$

where $\mathcal{L}_1$ is the linear Orr-Sommerfeld operator at criticality and J_i, i=1, 2,. . .,5 are the functions of $\phi(y), F(y), G(y)$ and $H(y)$. At order (1, 3), the application of the Fredholm alternative yields the Ginzburg-Landau equation governing the amplitude $A(\xi,\tau)$ of the fundamental wave

$$\frac{\partial A}{\partial\tau}-a_2\frac{\partial^2A}{\partial\xi^2}=\frac{d_1}{d_{1r}}A-\kappa A|A|^2. \tag{20}$$

The coefficients a_2, d_1 and κ are complex in general and can be computed using the Fredholm alternative.

The problem at order (1, 1) is spectral (linear stability equations at criticality) while the problems at orders (0, 2), (2, 2) are invertible and at orders (1, 2), (1, 3 are singular. Therefore, SVD can be used to solve the problems at orders (1, 2) and (1, 3). At all orders, a system of algebraic equations of the form

$$(A-cB)x=0 \tag{21}$$

or,

$$(A-cB)x=f \tag{22}$$

arises after discretization. In Eqs. (21) and (22), **A** and **B** are both square, N×N complex matrices. Assume that c is an eigenvalue with multiplicity K. Then applying SVD to the matrix *(*$\mathbf{A}$-$c\mathbf{B}$*)* one gets

$$\mathbf{A}-c\mathbf{B}=\mathbf{U}diag[\sigma_1,\sigma_2,\ldots,\sigma_{N-K},0,0\ldots,0]\ \mathbf{V}^H \tag{23}$$

where $\sigma_1 \geq \sigma_2 \geq \ldots\ldots \geq \sigma_{N-K} \geq 0$ are real constants,

$$\begin{aligned}\mathbf{U}&=[\mathbf{u_1},\mathbf{u_2},\ldots,\mathbf{u_{N-K}},\mathbf{u_{N-K+1}},\ldots,\mathbf{u_N}],\\ \mathbf{V}&=[\mathbf{v_1},\mathbf{v_2},\ldots,\mathbf{v_{N-K}},\mathbf{v_{N-K+1}},\ldots,\mathbf{v_N}],\end{aligned}$$

where $\mathbf{u}_j$ and $\mathbf{v}_j$ $j=1,2,\ldots N$ are the column vectors of orthonormal matrices **U** and **V**. Note that $\mathbf{U}\mathbf{U}^H$=$\mathbf{U}^H\mathbf{U}$=$\mathbf{V}\mathbf{V}^H$=$\mathbf{V}^H\mathbf{V}$ =**I** where superscript H denotes Hermitian. A matrix **A** is called Hermitian if it equals the complex conjugate of its transpose.

In order to find the solution of the homogenous problem in Eq. (21), multiply Eq. (23) with **x** to get

$$(\mathbf{A}-c\mathbf{B})\mathbf{x}=\mathbf{U}\ diag[\sigma_1,\sigma_2,\ldots\sigma_{N-K},0,0,\ldots0]\ \mathbf{V}^H\mathbf{x}=\mathbf{0} \tag{24}$$

and then multiply Eq. (24) with $\mathbf{U}^H$ from left and let $\mathbf{y}$=$\mathbf{V}^H\mathbf{x}$, to find

$$diag[\sigma_1,\sigma_2,\ldots,\sigma_{N-K},0,0,\ldots,0]\mathbf{y}=\mathbf{0} \tag{25}$$

since $\mathbf{U}^H\mathbf{U}$=**I**. Eq. (25) indicates that the elements of the vector **y** up to (N-K) should be zero, i.e.

$$\mathbf{y}=[0,0,\ldots,0,y_{N-K+1},\ldots,y_N] \tag{26}$$

where $y_{N-K+1}, \ldots, y_N$ are K arbitrary constants. Since $\mathbf{y}=\mathbf{V}^H\mathbf{x}$, one finds $\mathbf{x}=\mathbf{V}\mathbf{y}$ by multiplying both sides with $\mathbf{V}$. Recalling that $\mathbf{x}$ is an eigenvector of ($\mathbf{A}$-c$\mathbf{B}$), one finds that the column vectors $\mathbf{v}_j$ j = N-K+1,. . . , N are K independent eigenvectors corresponding to c.

Now consider the adjoint problem $(\mathbf{A}\text{-c}\mathbf{B})^H\mathbf{x}=\mathbf{0}$ of Eq. (21). Following the same steps as above,

$$(\mathbf{A} - c\mathbf{B})^H\mathbf{x} = \{\mathbf{U}\}, diag[\sigma_1, \sigma_2, \ldots, \sigma_{N-K}, 0, 0, \ldots, 0]$$
$$\{\mathbf{V}\}^H\mathbf{x} = \mathbf{0} \tag{27}$$

$$\mathbf{V}\ diag[\sigma_1, \sigma_2, \ldots, \sigma_{N-K}, 0, 0, \ldots, 0]\mathbf{U}^H\mathbf{x} = \mathbf{0}. \tag{28}$$

Multiplying Eq. (28) with $\mathbf{V}^H$,

$$diag[\sigma_1, \sigma_2, \ldots, \sigma_{N-K}, 0, 0, \ldots, 0]\mathbf{y} = \mathbf{0}$$

where $\mathbf{y}=\mathbf{U}^H\mathbf{x}$, or $\mathbf{x}=\mathbf{U}\mathbf{y}$. Since $\mathbf{x}$ is an eigenvector of Eq. (27), the columns of $\mathbf{u}_j$, j=N-K+1,. . ., N are the K independent eigenvectors corresponding to c in Eq. (27).

For inhomogeneous systems, Eq. (22), the Fredholm alternative can be used to find the solvability condition. The alternative requires that the inhomogeneous terms in the underlying system of differential equations be orthogonal to the independent eigenvectors that span the null-space of the adjoint system of differential equations. Therefore, the solvability condition becomes $\mathbf{U}^H\mathbf{f}$=0, or

$$\mathbf{u}^\star_{ji}\mathbf{f}_j = \mathbf{0},\ i = N - K + 1, \ldots\ldots\ldots, N \tag{29}$$

where $\star$ denotes complex conjugate and there is summation over index j, i.e. j=1, 2,. . .,N. The solution to Eq. (22) can be found as follows: Use SVD to decompose ($\mathbf{A}$-c$\mathbf{B}$) and then substitute it into Eq. (22),

$$\mathbf{U}\ diag[\sigma_1, \sigma_2, \ldots, \sigma_{N-K}, 0, 0, \ldots, 0]$$
$$\mathbf{V}^H\mathbf{x} = \mathbf{f}. \tag{30}$$

Multiplying Eq. (30) first with $\mathbf{U}^H$, then with the inverse of $diag[\sigma_1, \sigma_2, \ldots, \sigma_{N-K}, 0, 0, \ldots, 0]$ and finally with $\mathbf{V}$ gives

$$\mathbf{x} = \mathbf{V} diag[\sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_{N-K}^{-1}, 0, 0, \ldots, 0]$$
$$\mathbf{U}^H\mathbf{f}. \tag{31}$$

Since K rows of Eq. (31) are in the null-space, the solution vector consists of a particular solution $\mathbf{x}_p$ added to any linear combination of K vectors. Therefore,

$$\mathbf{x} = \mathbf{x_p} + \sum_{j=N-K+1}^{N} \gamma_j \mathbf{v}_j \tag{32}$$

where,

$\mathbf{x_p} = \mathbf{V_p}h$ and $\mathbf{v_p} = [\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v}_{N-K}]$ with $\mathbf{v_1}, \ldots, \mathbf{v}_{N-K}$ being the column vectors,

$$h = [\sigma_1^{-1}u^\star_{j1}f_j, \sigma_2^{-1}u^\star_{j2}f_j, \ldots, \sigma_{N-K}^{-1}u^\star_{jN-K}f_j].$$

In Eq. (32), the γ'_j are constants and $\mathbf{v}_j$, j=N-K+1,. . ., N are the column vectors of matrix $\mathbf{V}$.

For the spectral problem, i.e. at order (1, 1), the matrix eigenvalue problem is

$$(\mathbf{A} - c_r\mathbf{B})\psi_{11} = 0 \tag{33}$$

where the matrix ($\mathbf{A}$-$c_r\mathbf{B}$) and the vector ψ_{11} result from the discretization of the Orr-Sommerfeld operator at criticality and the eigenfunction $\psi(y)$, respectively. At orders (1, 2) and (1, 3) following singular algebraic equations exist

$$(\mathbf{A} - c_r\mathbf{B})\psi_{12} = \mathbf{f}(\psi_{11}, c_g) \tag{34}$$

$$(\mathbf{A} - c_r\mathbf{B})\psi_{12} = \frac{\partial A}{\partial \tau}\mathbf{f}_1 + \frac{\partial^2 A}{\partial \xi^2}\mathbf{f}_2 + \frac{A}{d_{1r}}\mathbf{f}_3 + A|A|^2\mathbf{f}_4. \tag{35}$$

The problems at orders (0, 2) and (2, 2) are invertible and ψ_{02} and ψ_{22} can be computed by Gaussian elimination. The problem at criticality, Eq. (33) can be solved easily by standard matrix eigenvalue routines, QZ algorithm for example, to find the critical speed c_r and the eigenvector ψ_{11}.

Assume that at criticality, c_r is an eigenvalue with multiplicity K=1. Then applying SVD algorithm to ($\mathbf{A}$-$c_r\mathbf{B}$), it is easy to compute $\sigma_1, \sigma_2, \ldots, \sigma_{N-1}$, $\mathbf{U}$ and $\mathbf{V}$ with standard SVD codes. Since the left-hand sides of Eqs. (33) and (34) are the same, the right-hand side of Eq. (34) must satisfy the solvability condition. Hence, applying the Fredholm alternative solvability condition, Eq. (29), to Eq. (34), the group velocity c_g can be found. Once c_g is found, ψ_{12} can be computed with Eq. (32).

Similarly, the left-hand sides of Eqs. (35) and (33) are the same. Therefore, the right-hand side of Eq. (35) has to satisfy the solvability condition. Since K=1, Eq. (29) becomes

$$u^{\star}_{jN}\mathbf{f}_j = 0, \;\; j = 1, 2, \ldots, N$$

or,

$$\frac{\partial A}{\partial \tau}\mathbf{u}^{\star}_N\mathbf{f}_1 + \frac{\partial^2 A}{\partial \xi^2}\mathbf{u}^{\star}_N\mathbf{f}_2 + \frac{A}{d_{1r}}\mathbf{u}^{\star}_N\mathbf{f}_3 + A|A|^2\mathbf{u}^{\star}_N\mathbf{f}_4 = 0 \quad (36)$$

where $\mathbf{u}_N$ is a row vector of length N. Dividing Eq. (36) by $\mathbf{u}^{\star}_N\mathbf{f}_1$

$$\frac{\partial A}{\partial \tau} + \frac{\mathbf{u}^{\star}_N\mathbf{f}_2}{\mathbf{u}^{\star}_N\mathbf{f}_1}\frac{\partial^2 A}{\partial \xi^2} + \frac{\mathbf{u}^{\star}_N\mathbf{f}_3}{\mathbf{u}^{\star}_N\mathbf{f}_1}\frac{A}{d_{1r}} + \frac{\mathbf{u}^{\star}_N\mathbf{f}_4}{\mathbf{u}^{\star}_N\mathbf{f}_1}A|A^2| = 0 \quad (37)$$

Equating the coefficients of Eq. (37) and Eq. (20) gives the coefficients of amplitude equations

$$a_2 = -\frac{\mathbf{u}^{\star}_N\mathbf{f}_2}{\mathbf{u}^{\star}_N\mathbf{f}_1}$$

$$d_1 = -\frac{\mathbf{u}^{\star}_N\mathbf{f}_3}{\mathbf{u}^{\star}_N\mathbf{f}_1}$$

$$\kappa = -\frac{\mathbf{u}^{\star}_N\mathbf{f}_4}{\mathbf{u}^{\star}_N\mathbf{f}_1}$$

The nature of bifurcation is determined by the real part of the Landau constant κ in Eq. (20). If $\kappa_r > 0$, a finite amplitude equilibrium solution exists. On the other hand, if $\kappa_r < 0$, the bifurcation solution of Eq. (20) will burst in finite time and a higher order theory is needed.

For critical point in the Poiseuille flow (α_c=1.02 and Re=5772.2), the coefficients are: $c_g = 0.383$, d_1=(0.168+i0.811)10^{-5}, a_2=0.187+i0.0275 and κ=-30.85+i172.85. A comparison of these results with the result of Chen and Joseph (1991) show that the agreement is excellent.

Conclusions

In this study, SVD is applied to plane Poiseuille flow after a brief review of amplitude equations in order to find the coefficients of amplitude equations resulting from nonlinear stability analysis of the flow. The analysis indicates that SVD appears to be the method of choice for the numerical of the coefficients of amplitude equations. In addition to being straightforward and easily implemented, SVD does not contain too many numerical approximations. As a result, roundoff errors are minimized.

Nomenclature

$a2, d_1$	coefficients of the Ginzburg-Landau equation
c	wavespeed
c_g	group velocity at criticality
p	pressure
Re	Reynolds number
t	time
u, v	x and y components of velocity

Greek Letters

κ	wavenumber
ϵ	perturbation parameter
κ	Landauu constant
ϕ	amplitude of perturbation streamfunction
σ	singular values of a matrix
	τ scaled time
ξ	scaled length
ψ	streamfunction
Ψ	perturbation streamfunction

Subscripts

b	base flow
c	critical condition

Superscripts

H	Hermitian
T	transpose
$\star$	complex conjugate

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