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A Special Quasi-Linear Mapping and its Degree

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Abstract

In this article, for the purpose of expanding to the mappings between Banach manifolds, a degree is determined in for the mappings between Banach spaces, which are from the obvious class.

0. Introduction

This work is related to the topological methods of global analysis and is devoted to the theory of degree mapping of domains of Banach spaces.

In 1912, time the degree concept was introduced by L.E.Brouwer for finite dimensional (continuous) mappings using the basic results of algebraic topology basic results. Later, the degree concep was defined in different ways and with its help many concrete problems were solved.

However, as is well known, it is not possible to introduce a definition of degree for arbitrary continuous mappings of infinite - dimensional domains [1]; e have to restrict the class of mappings.

The first such class has been defined by Leray and Schauder [6]. It consist of mappings of the kind "identical + compact." Other classes have also been introduced ([2],[3],[4],[5], e.t.c.). One of the suitable classes for studying nonlinear (pseudo) differential equations of smooth functions is the class of Fredholm quasi-linear mappings introduced by A. I. Shnirelman [7]. He introduced the definition of degree for such mappings as a limit of değrees of finite dimensional mappings.

However, the definition given by A. I. Shnirelman uses global geometrical constructions in Banach space and so cannot be used for the definition of degree of mappings of Banach manifolds. The definition of degree that does not have the indicated deficiency is given in this article. For this aim the new class of FSL mappings, for which the "local" definition of degree is given naturally, is introduced. But FSQL mapping f is determined as a uniform limit (in each bounded domain) of sequence of FSL mappings $f^{n_k}, k = 1, 2, \ldots$.

Then it is proved that in this case the degrees of FSL mappings are stabilized. So, $d(f)$ is determined as a limit: $d(f) = \lim_{k\to\infty}(f^{n_k})$. The coincidence of classes FQL- and FSQL mappings is proved. In addition the main properties of degree, similar to finite-dimensional mappings, and also equality of degrees of f, considered as a FQL or FSQL mapping, are proved. For the sake of simplicity, constructions are conducted in Hilbert spaces.

In later works the theories of quasi-linear manifolds (for example, the manifolds of smooth mappings of finite- dimensional manifolds) and the degree of FSQL- mappings between such manifolds, as well as the homological theory of degree of such mappings will be presented.

1. FSQL-mapping

Let H_1 and H_2 be real Hilbert spaces, and let $\| \|_1, \| \|_2$ be the corresponding norm in them. Let $\{X^n_\alpha\}, \alpha \in M_n$, be a family of pairs of disjoint closed planes in H_1 of codimension n, continuously depending on $\alpha; M_n$ is manifold of dimension n. Suppose $\{Y^n_\beta\}, \beta \in N_n$, is an analogious family in H_2. Let $\tilde{M}_n \bigcup_\alpha X^n_\alpha, \tilde{N}_n = \bigcup_\beta X^n_\beta$. Let us determine the projections $\pi_n : \tilde{M}_n \to M_n, p_n : \tilde{N}_n \to N_n$ in the following way: $\pi_n : x \mapsto \alpha$, if $x \in X^n_\alpha$; $p_n : y =\mapsto \beta$, if $y \in Y^n_\beta$.

It is obvious that the triples $\xi(\pi_n, \tilde{M}_n, M_n)$ and $\eta = (p_n, \tilde{N}_n, N_n)$ are affine bundles.

Definition 1 *Continuous mapping $f : \tilde{M}_n \to \tilde{N}_n$ is called a Fredholm Special Linear (SFL), if $\forall\alpha \in M_n$ $f^n_\alpha \equiv f\,|_{X^n_\alpha}$ is a affine invertible mapping from X^n_α on some $Y^n_\beta, f^n_\alpha \in Aff(X^n_\alpha, Y^n_\beta)$ and f^n_α depends continuously on α.*

It is obvious that FSL-mapping induces bimorphism between affine bundles ξ and η. The restriction of FSL-mapping on any domain $\Omega, \bar{\Omega} \subset \tilde{M}_n$, is also called FSL-mapping.

Let $\omega, \bar{\Omega} \subset \tilde{M}_n$, be a bounded domain in H_1; let $f : \Omega \to H_2$ be an FSL-mapping, and

$$\Big|||f||\Big|_\Omega = \sup_{x^n_\alpha \cap \Omega \neq \phi} \inf \{C | \left\|f^n_\alpha(x)\right\|_2 \leq C(1 + \|x\|_1), \|x\|_1 \leq C(1 + \left\|f^n_\alpha(x)\right\|_2), x \in X^n_\alpha\}$$

Definition 2 *Continuous mapping $f : \Omega \to H_2$ is called Fredholm Special Quasi-Linear (FSQL), if there exists a sequence of FSL-mappings $f^{n_i} : \Omega \to H_2, i = 1, \ldots,$ unifomly approximating f on Ω and*

$$\Big|||f||\Big|_\Omega \leq C(\Omega), \forall i > i(\Omega). \tag{1}$$

Moreover, $C(\Omega)$ does not depend on i for $i(\Omega)$.

Definition 3 *Continuous mapping $f : H_1 \to H_2$ is called a FSQL mapping, if in any bounded domain $\Omega \subset H_1$ unclear.*

As we have already noted, FL and FQL mappings were defined by A. I. Shnirelman, who also proved their basic properties and gave examples of FQL mappings (see[7]).

We shall prove that the classes of FSQL and FQL mappings coincide. First let us define some notions.

Let X^m, X^n be closed planes in H_1 having codimensions m and $n, m > n$. Let us transfer each of them parallely to itself into the origin in H_1. Let us denote the obtained subspaces by $'X^m$ and $'X^n$.

Definition 4 *The number*

$$\sin(X^m, X^n) = \sup_{x \in' X^m} \{\rho(x,' X^n) \cap B_1(1)\}$$

is called the sinus of the angle between the planes X^m and X^n.

Here, $B_1(1)$ is sphere in H_1 with radius 1 and a centre at zero.

Theorem 5 *FQL mapping $f : H_1 \to H_2$ is uniformly continuous and bounded in each bounded domain $\Omega \subset H_1$.*

Proof. Suppose that FL mapping $f^n : \Omega \to H_2$ uniformly approximates f on Ω to a precision of $\in; o$; that is

$$\forall x \in \Omega \|f(x) - f^n(x)\|_2 < \epsilon. \tag{2}$$

Suppose that $\{X^n_\alpha\}$ is a family of parallely closed planes in H_1 of codimension n, corresponding to the EL mapping f^n according to definition (see Appendix), where $f^n_\alpha \equiv f|_{X^n_\alpha}$ depends continuously on α. We may suppose that $B = \overline{\{\alpha | X^n_\alpha \cap \Omega \neq \phi\}} \subset R^n$. The compactness of B implies that f^n_α is uniformly continuous on $\alpha \in B$. So, for any given Ω

$$\forall \epsilon > o\, \exists \delta_1, \delta_2, \forall \alpha_1, \alpha_2 \in B, \|\alpha_1 - \alpha_2\|_{R^*} < \delta, \forall x_1 \in X^n_{\alpha_1} \cap \Omega, \forall x_2 \in X^n_{\alpha_2} \cap \Omega,$$

$$\|x_1 - x_2\|_1 < \delta_2, \|f^n_{\alpha_1}(x_1) - f^n_{\alpha_2}(x_2)\|_2 < \epsilon.$$

From the parallelism of the planes X^n_α we have

$$\forall d_1 > 0\, \exists d_2 > o,\ \forall x_1 \in X^n_{\alpha_1},\ \forall x_2 X^n_{\alpha_2}, \|x_1 - x_2\|_1 < d_2 \Rightarrow \|\alpha_1 - \alpha_2\|_{R^n} < \delta_1.$$

That is, from the closeness of x_1 and x_2 implies the closeness of parameters α_1 and α_2. So,

$$\forall\epsilon > o\,\exists\delta_2 > o, \forall x_1 \in X^n_{\alpha_1} \cap \Omega,\, \forall x_2 \in X^n_{\alpha_2} \cap \Omega,\, \|x_1 - x_2\|_1 > \delta_2,$$

$$\|f^n_{\alpha_1}(x_1) - f^n_{\alpha_2}\| < \epsilon.$$

Then

$$\forall\epsilon > o\,\exists\delta_2 > o,\, \forall x_1, x_2 \in \Omega,\, \|x_1 - x_2\| < \delta_2,$$
$$\|f^n(x_1) - f^n(x_2)\|_2 = \|f^n_{\alpha_1}(x_1) - f^n_{\alpha_2}(x_2)\|_2 < \epsilon. \tag{3}$$

Here, $x_1 \in X^n_{\alpha_1}, x_2 \in X^n_{\alpha_2}$. So, FL mapping f^n is uniformly continuous on Ω. From (2) and (3) we have

$$\forall\epsilon > o\,\exists\delta > o, \forall x_1, x_2 \in \Omega\|x_1 - x_2\| < \delta,$$

$$\|f(x_1) - f(x_2)\|_2 \le \|f(x_1) - f^n(x_1)\|_2 + \|f^n(x_1) - f^n(x_2)\|_2 + \|f^n(x_2) - f(x_2)\|_2 < 3\epsilon.$$

This means that FQL mapping f is uniformly continuous on Ω.

2) Now let us prove the boundness of FQL mapping f on Ω. Let us take a δ_1- net $\alpha_{1,\ldots,\alpha_k}$ of compact B, such that

$$\forall\epsilon > o\forall\alpha \in B\exists\alpha_i \in \{\alpha_1, \ldots, \alpha_k, \exists\tau_2 > o, \forall x_1 \in X^n_\alpha \cap \Omega, \forall x_2 \in X^n_{\alpha_i} \cap \Omega,$$
$$\|x_1 - x_2\| < \delta_2,\, \left\|f^n_\alpha(x_1) - f^n_{\alpha_i}(x_2)\right\|_2 < \epsilon. \tag{4}$$

According to definition, $\forall\alpha f^n_\alpha$ is bounded. So,

$$\exists M > o, \forall i = 1, \ldots, k\forall x \in X^n_{\alpha_i} \cap \Omega\left\|f^n_{\alpha_i}(x)\right\|_2 \le M. \tag{5}$$

From (4) and (5) we have

$$\forall\alpha \in B\exists\alpha_i \in \{\alpha_1, \ldots, \alpha_k\}, \forall x \in X^n_\alpha \cap \Omega\exists x' \in X^n_{\alpha_i}(x') \cap \Omega,$$
$$\|f^n_\alpha(x)\|_2 \le \|f^n_{\alpha_i}(x')\|_2 + \|f^n_\alpha(x) - f^n_{\alpha_i}(x')\|_2 \le M + \epsilon. \tag{6}$$

That means that FL-mapping f^n is bounded on Ω. From (2) and (6), we have

$$\forall x \in \Omega\|f(x)\|_2 \le \|f^n(x)\|_2 + \|f^n(x) - f(x)\|_2 \le M + 2\epsilon.$$

This means that FQL mapping f is bounded on Ω. □

Theorem 6 *Let $f^{n_0} : \Omega \to H_2$ be an FL mapping. Then there exists a sequence of FSL mappings $f^{n_i} : \Omega \to H_2, i = 1, 2, \ldots,$ uniformly converging to f^{n_0} on Ω.*

Proof. Let $\{X_\alpha^{n_0}\}$ be a family of parallel closed planes, having codimension n_0 corresponding to FL-mapping f^{n_0}. It is obvious that $B\overline{\{\alpha | X_\alpha^{n_0} \cap \Omega \neq \phi\}}$ is compact, as Ω is bounded. Therefore the continuous family $\{f_\alpha^{n_0} | \alpha \in B\}$ of affine mappings is uniformly continuous on B. So, the family of planes $\{Y_\alpha^{n_0} | Y_\alpha^{n_0} = f_\alpha^{n_0}(x_\alpha^{n_0})\}$ is uniformly continuous is α. Hence $\forall \epsilon > 0$ there is a final number of elements $\alpha_1, \ldots, \alpha_k \in B$ such that

$$\forall \alpha \in B \, \exists \alpha_i \in \{\alpha_1, \ldots, \alpha_k\}, \sin(Y_\alpha^{n_0}, Y_{\alpha_i}^{n_0}) < \epsilon. \tag{7}$$

Let us transfer the planes of $Y_{\alpha_i}^{n_0}, i = \overline{1, k}$, parallely to itself to the origin of H_2 and take the intersection of all these subspaces. Let us denote this intersection by $Y^m, m \geq n_0$. Taking into account (7), let us project Y^m on $Y_\alpha^{n_0}, \alpha \in B$ orthogonally and partition each plane $Y_\alpha^{n_0}$ into planes of codimension m that are parallel to the projection of Y^m. As a result we get a continuous family of planes, $\{Y_{\alpha,\beta}^m(\alpha, \beta) \in B \times R^{m-n_0}\}$ having codimension m, and satisfying the following conditions: □

a) $\forall \alpha, \forall \beta_1, \beta_2 \, Y_{\alpha,\beta_1}^m // Y_{\alpha,\beta_2}^m;$

b) $\forall \alpha \, Y_\alpha^{n_0} = \bigcup\limits_\beta Y_{\alpha,\beta}^m;$

c) $\forall a_1, \alpha_2 \in B, \quad \forall \beta_1, \beta_2 \in R^{m-n_0} \quad \sin(Y_{\alpha_{11},\beta_1}, Y_{\alpha_2,\beta_2}^m) < \epsilon.$

Because of affine isomorphism of mappings of $f_\alpha^{n_0}, \alpha \in B$, each plane of $X_\alpha^{n_0}$ can be divided into parallel subplanes $X_{\alpha,\beta}^m = (f_\alpha^{n_0})^{-1}(Y_{\alpha,\beta}^m)$ of codimension m. The obtained family $\{X_{\alpha,\beta}^m\}$ will be continuous is (α, β) as $(f_\alpha^{n_0})^{-1}$ is continuous in α. Let us take the subspace $Y_m \subset H_2$, having dimension m, perpendicular to some Y_{α_0,β_0}^m. Because of the smallness of ϵ, all the planes $Y_{\alpha,\beta}^n$ will be transversal to Y_m. So, $\forall \alpha, \beta \; Y_{\alpha,\beta}^m \cap Y_m$ will consist of one point. Let us draw the plane of $'Y_{\alpha,\beta}^m$ over each such point, parallel to Y_{α_0,β_0}^m. As a result we is get a continuous family $\{'Y_{\alpha,\beta}^m\}$ of parallel planes. According to Theorem 5, the sef $f(\Omega)$ is bounded, so it is contained in some sphere $B_2(R)$. According to the construction

$$\forall \alpha, \beta \; \sin(Y_{\alpha,\beta}^m, 'Y_{\alpha,\beta}^m) < \epsilon.$$

Because of the smallness of ϵ we may state that the planes $Y_{\alpha,\beta}^m$ and $Y_{\alpha,\beta}^m$ are sufficiently close to one another in $B_2(R)$.

Note 7. In what follows we shall make use of this construction several times.

Let $\forall\alpha,\beta P^m_{\alpha,\beta}: Y^m_{\alpha,\beta} \to' Y^m_{\alpha,\beta}$ be an orthogonal projection. It is continuous in (α,β) and by the construction

$$\left\|P^m_{\alpha,\beta}(y) - y\right\|_2 < \epsilon\,\forall\alpha,\beta,\,\forall y \in B_2(R) \cap Y^m_{\alpha,\beta}.$$

Now let us consider the mapping

$$f^m: \Omega \to H_2, f^m\Big|_{\Omega\cap X^m_{\alpha,\beta}} \equiv f^m_{\alpha,\beta}, f^m_{\alpha,\beta}(x)' = P^m_{\alpha,\beta} \circ f^{n_0}(x)\, x \in \Omega \cap X^m_{\alpha,\beta}. \tag{8}$$

It is obvious from the construction that $f^m: \Omega \to H_2$ is an FSL mapping and

$$\forall x \in \Omega \left\|f^m(x) - f^{n_0}(x)\right\|_2 < \epsilon.$$

Similarly, we can of prove the following theorem is the inverse of Theorem 6.

Theorem 7 *Let $f^{n_0}: \Omega \to H_2$ be an FSL-mapping. Then there exists a sequence of FL-mappings $f^{n_i}: \Omega \to H_2, i = 1,2,\ldots,$ uniformly approximating f^{n_0} on Ω.*

From theorems 6 and 8 the identity of the classes of FSQL and FQL mappings follows.

Note 9. Theorem 5 is valid for FSQL mapping as well.

Making use of the above technique we may also prove the following theorem.

Theorem 8 *Let $f: H_1 \to H_2, g: H_2 \to H_3$ be FSQL mappings. Then $g \circ f: H_1 \to H_3$ will also be an FSQL mappings. Here H_3 is a real Hilbert space.*

Making use of the coincidence of classes FQL and FSQL mappings, as an example of FSQL mappings, we may give the following example (see [7]).

Let $f: [0,2) \times R \to R$ by a smooth function and $grad_x f \neq 0 \forall\tau, x$. Then the operator $\tilde{f}: H_k(S^1) \to H_k(S^1), k \geq 2)$, defined as $\tilde{f}: x(\tau) \mapsto f(\tau, x(\tau)), x(\tau) \in H_k(S^{-1})$, will be FSQL mapping.

Here

$$H_k(S^1) = \left\{x(\tau) \setminus \|x\|^2_k = \int_0^{2\pi} \sum_{l=0}^{k} \left|x^{(l)}\tau\right|^2 d\tau < \infty\right\}$$

is Sobolev space.

2. The degree of an FSQL mapping

Let $f: H_1 \to H_2$ be an FSQL mapping; suppose that the following apriori estimate is satisfied

$$\|x\|_1 \leq \phi\big(\|f(x)\|_2\big), \tag{9}$$

where ϕ is some pozitive monotonous function. Therefore all solutions of the equation

$$f(x) = y_0,\, y_0 \in H_2, \tag{10}$$

will be contained in the ball $\Omega{+}\left\{x|\, \left\|x\right\|_1 \leq \phi\big(\|y_0\|_2\big)\right\}$.

The degree $\deg(f)$ of mapping of f is the number of solutions of equation (10). To give an exact definition of $\deg(f)$, we need to introduce some notions.

Let $f^{n_k}, k = 1, 2, \ldots,$ be the sequence of FSL mappings, uniformly converging of f in the ball Ω^{R_0}, $R_0\phi(\|y_0\|_2 + 2\delta)$, $\delta > 0$. Let us consider the equation

$$f^{n_k}(x) = y_0; \tag{11}$$

we are looking for its solutions, contained in the ball Ω^{R_0}. This problem may reduced to a finite-dimensional one.

Indeed, each FSL mapping f^{n_k} being bimorphism between affine bundles ξ_{n_k} and η_{n_k}, induces continuous mapping $f_{n_k} : M_{n_k} \to N_{n_k}$ between bases M_{n_k} and N_{n_k} of the corresponding bundles. Here, M_{n_k} and N_{n_k} are n_k - dimensional manifolds. S, $f_{n_k}(\alpha) = \beta$, where $f^{n_k}(X\alpha_\alpha^{n_k}) = Y_\beta^{n_k}$.

Let $y_0 \in Y_{\beta_0}^{n_k}$. Then equation (11) reduces us to the finite-dimensional equation

$$f_{n_k}(\alpha) = \beta_0. \tag{12}$$

Let us prove that at sufficiently big k the finding of the solutions of the equation (11) contained in Ω^{R_0} is equivalent to the finding of the solutions of the equation (12), contained in $\pi_{n_k}(\Omega^{R_0}) \subset M_{n_k}$, where π_{n_k} is a projection from $\xi_{n_k} = (\tilde{M}_{n_k}\pi_{n_k}, M_{n_k})$. Indeed, let $x \in \Omega^{R_0}$ and $f^{n_k}(x) = y_0$. Then there exists $\alpha \in \pi_{n_k}(\Omega^{R_0}$ such that $x \in X_\alpha^{n_k}$; therefore $f_{n_k}(\alpha) = \beta_0$.

Conversely, let $f_{n_k}(\alpha) = \beta_0$. This means, that f^{n_k} is a affine isomorphism between the planes $X_\alpha^{n_k}$ and $Y_{\beta_0}^{n_k}$.

Therefore, there exists unique point $x \in X_\alpha^{n_k}$, such that $f^{n_k}(x) = y_0$. However, it may be that point x will lie outside the set of Ω^{R_0}. Let us show that this is impossible.

Lemma 9 *There exists* k_0 *such that if* $k \geq k_0, x \in \tilde{M}_{n_k}, \pi_{n_k}(0) \in \pi_{n_k}(\Omega^{R_0})$ *and* $f^{n_k}(x) = y_0$, *then* $x \in \Omega^{R_0}$.

Proof. According to the condition (1), will be found such $C > 0$, that $\left\| |f^{n_k}| \right\|_{\Omega_{R_0}} < C$ as sufficiently big k. Therefore, preimage of the point y_0, if it is contained in the domain $\pi_{n_k}^{-1}(\pi_{n_k}(\omega^{R_0}))$, is in sphere Ω^{C_1}, where $C_1 = C(1 + \left\|y_0\right\|_2)$. Let k_0 be so big that

$$\left|f^{n_k} - f\right|_{\Omega^{C_1}} = \sup_{x \in \Omega^{C_1}} \left\|f^{n_k}(x) - f(x)\right\|_2 < \delta, \forall k \geq k_0,$$

where δ is the same, as in definition R_0. So, if $x \notin \Omega^{R_0}$, then $R_0 \leq \|x\|_1 < C_1$; but then

$$\left\|f^n(x)\right\| \geq \|f(x)\|_2 - \left|f^{n_k} - f\right|_{\Omega^{C_1}} > \left(\|y_0\|_2 + 2\delta\right) - \delta = \|y_0\|_2 + \delta,$$

so that x cannot be a preimage of the point y_0. Therefore $\|x\|_1 < R_0$.

Thus, equation (11) reduces to the finite-dimensional equation (12). Therefore we can give the following. □

Definition 10

$$\deg(f^{n_k}) = \deg(f_{n_k}).$$

Theorem 11 *Suppose that FSL mappings* $f^{n"}, f^{n''} : \Omega \to H_2$ *are sufficiently close to each other in* Ω. *Then*

$$\deg(f^{n'}) = \deg(f^{n''}).$$

3. Proof of Theorem 13

Let $\{X_{\alpha'}^{n'}\}, \{Y_{\beta'}^{n'}\}$ be continuous families of closed planes of codimension n' corresponding to FSL mapping $f^{n'}$ (see definition 1); let $\{X_{\alpha''}^{n''}\}, \{Y_{\beta''}^{n''}\}$ be analogous families for $f^{n''}$. Similar to the proof of theorem 6, we obtain continuous families $\{'X_{\alpha'\gamma'}^{m'}\}, \{X_{\alpha'',\gamma''}^{m''}\}$, satisfying to the following conditions:

a) $m' \geq n'$, $m'' \geq n''$;

b) $\forall(\alpha_1', \gamma_1', (\alpha_2", \gamma_2')\, 'X_{\alpha_1'\gamma_1'}^{m'} \| 'X_{\alpha_2', \gamma_2'}$;

c) $\forall(\alpha_1'', \gamma_1''), (\alpha_2'', \gamma_2'')\, 'X_{\alpha_1''\gamma_1''}^{m''} \| 'X_{\alpha_2'', \gamma_2''}$;

d) $\forall(\alpha', \gamma'), (\alpha'', \gamma'') \sin(X_{\alpha',\gamma'}^{m'}, 'X_{\alpha',\gamma'}^{m'}) < \epsilon, \sin(X_{\alpha'',\gamma''}^{m''}, 'X_{\alpha'',\gamma''}^{m''}) < \epsilon$;

e) $\forall\alpha', \gamma_1', \gamma_2'\, X_{\alpha',\gamma_1'} \| X^{m'}_{\alpha',\gamma_2'},\ X^{m'}_{\alpha'\gamma_2'}, X^{n'}_{\alpha'} = \bigcup\limits_{\gamma'} X^{m'}_{\alpha',\gamma'},\ {}'X^{n'}_{\alpha'} = \bigcup\limits_{\gamma'}{}' X^{m'}_{\alpha'\gamma'}$ is a plane of codimension n';

f) $\forall\alpha'', \gamma_1'', \gamma_2'',\ X^{m''}_{\alpha'',\gamma''} \| X^{m''}_{\alpha'',\gamma_2''},\ X^{n''}_{\alpha''} = \bigcup\limits_{\gamma'} X^{m''}_{\alpha''\gamma''},\ X^{m''}_{\alpha''} = \bigcup\limits_{\gamma'}{}' X^{m''}_{\alpha'',\gamma''}$ is a plane of codimension n''.

Let $\forall\alpha', \gamma'\ \pi^{m'}_{\alpha',\gamma'} :' X^{m'}_{\alpha',\gamma'} \to X^{m'}_{\alpha',\gamma'}$ and $\forall\alpha'', \gamma''\, \pi^{m''}_{\alpha'',\gamma''} :' X^{m''}_{\alpha'',\gamma''} \to X^{m''}_{\alpha'',\gamma''}$ be ortogonal projections. According to our construction they induce isomorphisms between the corresponding bundles, and

$$\left\|x - \pi^{m'}_{\alpha',\gamma'}(x)\right\|_1 < \delta\, \forall x \in \Omega \cap' X^{m'}_{\alpha',\gamma'}, \forall\alpha', g';$$

$$\left\|x - \pi^{m''}_{\alpha'',\gamma''}(x)\right\|_1 < \delta\, \forall x \in \Omega \cap' X^{m''}_{\alpha'',\gamma''}, \forall\alpha'', \gamma'',$$

where $\delta > 0$ is a small number, corresponding to $\epsilon > 0$ (according to uniform continuity of mappings $f^{n'}$ and $f^{n''}$ in Ω).

$$f^{m'} : \Omega \to H_2; \forall\alpha', \gamma'\, f^{m'}\Big|_{\Omega\cap' X^{m'}_{\alpha',\gamma'}} \equiv f^{m'}_{\alpha',\gamma'},\ f^{m'}_{\alpha',\gamma'}(x) = f^{n'} \circ \pi^{m'}_{\alpha',\gamma'}(x),\ x \in \Omega \cap' X^{m'}_{\alpha',\gamma'};$$

$$f^{m''} : \Omega \to H_2; \forall\alpha'', \gamma'' f^{m''}\Big|_{\Omega\cap'' X^{m''}_{\alpha'',\gamma''}} \equiv f^{m''}_{\alpha'',\gamma''}, f^{m''}_{\alpha'',\gamma''}(x) = f^{n''} \circ \pi^{m''}_{\alpha'',\gamma''}(x), x \in \Omega\cap' X^{m''}_{\alpha'',\gamma''};$$

According to the construction,

$$\forall x \in \Omega \left\|f^{m'}(x), f^{n'}(x)\right\|_2 < \epsilon,\ \left\|f^{m''}(x) - f^{n''}(x)\right\|_2 < \epsilon$$

Moreover, $f^{m'}$ will be FSL mapping between the families $\{{}'X^{n'}_{\alpha'}\}$ and $\{Y^{n'}_{\beta'}\}$. Let us denote it by $f^{n'}_1$. Then

$$\deg(f^{n'}_1) = \deg(f^{n'}), \tag{13}$$

as both mappings induce one and the same finite - dimensional mapping. This is also valid for the mapping $f^{m''}$, in that will be a FSL mapping (let us denote it as $f^{n''}_2$) between the familes $\{{}'X^{n''}_{\alpha''}\}$ and $\{Y^{n''}_{\beta''}\}$; that is

$$\deg(f^{n''}_1) = \deg(f^{n''}). \tag{14}$$

Now led us consider the intersection of families $\{'X^{m'}_{\alpha',\gamma'}\}$ and $\{'X^{m'}_{\alpha',\gamma'}\}$. It will consist of parallel planes, as the familes themselves are parallel planes. Let us denote this intersection by $\{X^m_\lambda\}$, $m \geq \max\{m', m''\} \cdot f_1^{n'}$ and $f_2^{n''}$, being considered in the family $\{X^m_\lambda$, will be FL mappings. Let us denote them accordingly by f_1^m and f_2^m. Let us consider continuous families $\{Y^m_{1,\lambda} \setminus Y^m_{l,\lambda} = f_1^m(X^m_\lambda)\}$ and $\{Y_{2,\lambda} \setminus Y^m_{2,\lambda} = f_2^m(X^m_\lambda)\}$. By analogy with the proof of Theorem 6, let us approximate them correspondingly with families $\{'Y^{m+l'}_{1,\lambda,\lambda'}\}$ and $\{'Yillmi\}$, such that

a) $\forall(\lambda_1, \lambda'_1\ 'Y^{m+l'}_{1,\lambda_1,\lambda'_1} \|' Y^{m+l'}_{1,\lambda_2,\lambda'_2}$;

b) $\forall(\lambda_1, \lambda''_1), (\lambda_2, \lambda''_2)\ 'Y^{m+l''}_{1,\lambda_1,\lambda''_1} \|' Y^{m+l''}_{1,\lambda_2,\lambda''_2}$;

c) $\forall(\lambda_1, \lambda'_1)\ \sin('Y^{m+l'}_{1,\lambda_1,\lambda'_1}, Y^{m+1'}_{1,\lambda_1,\lambda'_1}) < \epsilon$, $\sin('Y^{m+l''}_{1,\lambda_1,\lambda''_1}, Y^{m+1''}_{1,\lambda_1,\lambda''_1}) < \epsilon$,.

Making use of orthogonal projections $P^{m+l'}_{\lambda,\lambda"} : Y^{m+l'}_{1,\lambda,\lambda'} \to' Y^{m+l'}_{1,\lambda,\lambda'}$ and $P^{m+l''}_{\lambda,\lambda''} ::$ $Y^{m+l'}_{1,\lambda,\lambda'} \to' Y^{m+l'}_{1,\lambda,\lambda'}$ we shall get the FSL mappings:

$$f^{m+l'} : \Omega \to H_2; \forall \lambda, \lambda'\, f^{m+l'}\Big|_{X^{m+l'}_{\lambda,\lambda"}} \equiv f^{m+l'}_{\lambda,\lambda"},\ f^{m+l'}_{\lambda,\lambda"}(x) = P_{llmb} \circ f_1^m(x),\ x \in \Omega \cap X^{m+l'}_{\lambda,\lambda"};$$

$$f^{m+l''} : \Omega \to H_2; \forall \lambda, \lambda''\, f^{m+l''}\Big|_{X^{m+l''}_{\lambda,\lambda''}} \equiv f^{m+l''}_{\lambda,\lambda''},\ f^{m+l''}_{\lambda,\lambda''}(x) = P_{llmi} \circ f_1^m(x),\ x \in \Omega \cap X^{m+l''}_{\lambda,\lambda''}.$$

According to the construction,

$$\forall x \in \omega \left\| f^{m+l'}(x) - f_1^m(x) \right\|_2 < \epsilon,\ \left\| f^{m+l''}(x) - f_2^m(x) \right\|_2 < \epsilon.$$

Now let us consider the intersection of the families $\{Y^{m+l'}_{1,\lambda,\lambda'}\}$ and $\{'Y^{m+l''}_{2,\lambda,\lambda''}\}$. It will consist of parallel planes, as both families themselves consist of parallel planes. Let us denote it by $\{Y^k_\mu\} \cdot f^{m+l'}$ and $f^{m+l''}$, being regarded correspondingly as continuous families $X^k_{1,\mu} \setminus X^k_{1,\mu} = (f^{m+l'})(Y^k_\mu)\}$ and $\{X^k_{2,\mu} \setminus X^k_{2\mu} = (f^{m+l'})^{-1}(Y^k_\mu)\}$, will be FSL mappings. Let us denote them correspondingly by f_1^k and f_2^k. It is clear from the constructions that

$$\forall x \in \Omega \left\| f_1^k(x) - f_2^k(x) \right\|_2 < \epsilon.$$

Lemma 12 $\quad \deg(f_1^k) = \deg(f_1^{n'}, \deg(f_2^k) = \deg(f_2^{n''}). \qquad (15)$

Proof. Let us prove that first equality from 515). FSL mapping f_1^k induces a finite - dimensional mapping

$$f_{1,k} : M_{1,k} \to N_{1,k}.$$

and the FSL mapping $f_1^{n'}$ induces a finite-dimensional mapping

$$f_{l,n'} : M_{n'} \to N_{n'}.$$

According to the construction the triplet $(M_{1,k}, \pi_{1,k}, M_{n'})$ and $N_{1,k}, p_{1,k}, N_{n'}$ will be affine bundles with the fibres isomorphic to the finite - dimensional space of $R^{k-n'}$. Here,

$$\pi_{1,k} : M_{1,k} \to M_{n'},\ \pi_{1,k} : (\alpha', \tau') \mapsto \alpha';$$

$$p_{1,k} : N_{1,k} \to N_{n'},\ p_{1,k} : (\beta', \nu') \mapsto \beta'.$$

In this case (again according to the construction) $f_{1,k}$ will be-bimorphism between the indicated bundles. So,

$$\deg(f_{1,k}) = \deg(f_{1,n'},$$

thas is

$$\deg(f_1^k) = \deg(f_1^{n'}).$$

Hence (acc. to the Definition 12) we get the first equality from (15).

Now let us prove that Hausdorff distance, $dist(X^k_{l,\mu} \cap \Omega, X^k_{2,\pi} \cap \Omega)$, between the planes $X^k_{1,\mu}$ and $X^k_{2,\mu}$, is small. Here,

$$dist(X^k_{1,\mu} \cap \Omega, X^k_{2,\mu} \cap \Omega) = \max(\sup_{x_1} \inf_{x_1} \rho(x_1, x_2), \sup_{x_1} \inf_{x_2} \rho(x_1, x_2)\}, x_1 \in X^k_{1,\mu} \cap \Omega, x_2 \in X^k_{2,\mu} \cap \Omega.$$

□

Lemma 13 *Let X, Y be Banach spaces; let X_1, X_2 be closed planes in X and suppose that $f_1, f_2 : X \to Y$ are continuous mappings satisfying the following conditions:*

a) $f_1 \to Y$ is an isomorphism from X at $Y_1 = f_1(X)$; moreover, $\|f_1\|_X \leq C, \|f_1^{-1}\|_{Y_1} \leq C$;

b) $\forall x \in X \cap B_x(r) \|f_1(x) - f_2(x)\|_2 < \epsilon$;

c) $dist(f_1(X_1) \cap B_y(R), f_2(X_2) \cap B_y(R))\epsilon$.

Then

$$dist(X_1 \cap B_x(r), X_2 \cap B_x(r)) \le 2C\epsilon.$$

Here $B_x(r)$ and $B_y(R)$ are balls in X and Y radius r and R respectively.

Proof. Let $x_1 \in X_1 \cap B_x(r),, x_2 \in X_2 \cap B_x(r)$. From isomorphism $f_1 : X \to Y_1$ we have

$$\|x_1 - x_2\| \le C\|f_1(x_1 - f_1(x_2)\| \Rightarrow \|f_1(x_1) - f_1(x_2)\| \ge C^{-1}\|x_1 - x_2\|.$$

Hence it follows that

$$\|f_1(x_1) - f_2 x_2)\| = \|f_1(x_1) - f_1(x_2)) + f_1(x_2) - f_2(x_2))\| \ge$$

$$\ge \|f_1(x_1) - f_1(x_2)\| - \|f_1(x_2) - f_2(x_2)\| \ge C^{-1}\|x_1 - x_2\| - \epsilon.$$

Let us denote by d the distance between the X_1 and X_2 planes. As $d < \|x_1 - x_2\|$, then from the last inequality we have

$$\epsilon \ge C^{-1}d - \epsilon \Rightarrow d \le 2C\epsilon.$$

Let $\pi^k_{1,\mu} : X^k_{1,\mu} \to X^k_{2,\mu}$ be the orthogonal projection. It is obvious that $\mu^k\big|_{X^k_{1,\mu}} \equiv \pi^k_\mu$ $\forall\mu$ induces isomorphism between the families $\{X^k_{1,\mu}\}$ and $\{X^k_{2,\mu}\}$. From the Lemma 15 it follows that π^k is close on Ω to identical mapping. Then the mappings $f^k_1(x)$ and $f^k_2 \circ \pi^k(x), x \in \{X^k_{1,\mu}\}$, will also be close on Ω. So,

$$\deg(f^k_1) = \deg(f^k_2 \circ \pi^k) = \deg(f^k_2) \cdot \deg(\pi^k) = \deg(f^k_2. \tag{16}$$

From equalities (13), (14), (15) and (16) we get:

$$\deg(f^{n'} = \deg(f^{n''}).$$

Theorem 13 is proved.

Now we may give the following. □

Definition 14 *where $f^{n_k}, k = 1, 2, \ldots,$ is sequence FSL mappings, uniformly approximating FSQL mapping f on Ω.*

4. The main property of the degree

Theorem 15 *Suppose that FSQL mapping* $f : H_1 \to H_2$ *satisfies condition (9), and* $\deg(f) \neq 0$. *Then equation (10) has a solution for any* $y_0 \in H_2$.

Theorem 16 *Let* $\{f_t\}$ *be the family of FSQL mappings which depend continuously (but in each sphere* $B_1(r)$ *is uniformly continuous) on parameter* $t \in [0, 1]$, *and for all* $t \in [0, 1]$ *the apriori estimate (9) with the function* ϕ, *independent on* t, *is tasified. Then*

$$\deg(f_0) = \deg(f_1).$$

Theorem 17 *Sappose that the FSQL mapping* $f : H_1 \to H_2$ *satisfies condition (9). Then*

$$\deg(f, y_1) = \deg(f, y_2), y_{1,2} \in H_2.$$

Theorem 18 *Let* f *be an FSQL (FQL) mapping, satisfying condition (9) and* $\deg_1(f)$ *is degree of* f *as FQL mapping, defined by A. I. Shnirelman (see[7]). Then*

$$\deg(f) = \deg_1(f).$$

Theorem 17 is proved analogous to the theorem given in [7] and the proofs of theorems, 18, 19 and 20 are not difficult.

5. Appendix

Let X, Y be real Banach spaces, let Ω be abound domain in X, and suppose that $\pi_\mu : X \to X_\mu$ is a linear mapping from X to a μ - dimensional space X_μ, and $X^\mu_\alpha = \pi^{-1}(\alpha), \alpha \in X_\mu$.

Definition 19 *Continuous mapping* $f^n : \omega \Rightarrow Y$ *is called a Fredholm Linear (FL), if*

a) some linear mapping $\pi_\mu : X \to X_\mu$ is fixed;

b) on each plane $X^\mu_\alpha(\alpha \in X_\mu)$, passing through Ω, $f^\mu_\alpha \equiv f^\mu\big|_{X^\mu_\alpha}$ is an affine invertible mapping from X^μ_α onto its image $Y^\mu_\alpha = f(X^\mu_\alpha)$ that is, closed in Y and has codimension μ and f^μ_α depends continuously on α.

Definition 20 *Continuous mapping* $f : X \to Y$ *is called Fredholm Quasi - Linear (FQL), if there exists a sequence* f^{μ_k} *of FL mappings,* $k = 1, 2, \ldots$, *uniformly approximating* f *on each bounded domain* $\Omega \subset X$ *such that*

$$\left\| f_a^{\mu_k} \right\| < C(\Omega)\,,\ \left\| (f_a^{\mu_k}) \right\| < C(\Omega),$$

with $k > k_0(\Omega)$*, if* $\alpha \in \pi_{\mu_k}(\Omega)$ *and* $C(\Omega)$ *does not depend on* k*, if* $k_0(\Omega)$*.*

References

[1] C.Bessaga, Everi infinite dimensional Hilbert space is diffeomorphic with its unit sphere. *Bull. Acad. Polon. Sci.Ser. Sci. Math. Astronom. Phus.,* **14** (1966), 27-31.

[2] F.E.Browder, Topology and non-linear functional equations. *Studia Math.,* **31,** 2(1968),189-204.

[3] F.E.Browder and R.D.Nussbaum, The topological degree for noncompact nonlinear mappings in Banach spaces, *Bull. Amer. Math. Soc.* **74**, (1967), 671-676.

[4] K.D.Elworthy and A.J.Tromba, Degree theory on Banach manifolds, Proc. Sympos, Pure Math., 18,A.M.S (1970), 86-94.

[5] R.L.Frum-Ketkov, On mappings in Hilbert space, Dokl. Akad. Nauk SSSR, 192 (1970), 6, 1231-1234 (in Russian)

[6] J.Leray and J.Schauder, Topoloqie of equations fonctionnelles, *Ann. Ec. Sup. Paris* **13,** (1934), 45-78.

[7] A.İ.Shnirelman, The degree of quasi-linear mapping and nonlinear problem of Hilbert, Mat. Sb., (1972),**89**, (131),3, 366-389 (in Russian).

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