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On the Asymptotics of Fourier Coefficients for the Potential in Hill's Equation

Haskız Coşkun

Abstract

We consider Hill's equation $y'' + (\lambda - q)y = 0$ where $q \in L^1[0, \pi]$. We show that if l_n−the length of the $n - th$ instability interval− is of order $O(n^{-k})$ then the real Fourier coefficients a_n, b_n of q are of the same order for($k = 1, 2, 3$), which in turn implies that $q^{(k-2)}$, the $(k - 2)th$ derivative of q, is absolutely continuous almost everywhere for $k = 2, 3$.

1. Introduction

We consider the differential equation

$$y''(t) + (\lambda - q(t))y(t) = 0 \tag{1}$$

on $[0, \pi]$ where λ is a real parameter, $q(t)$ is integrable over the interval $[0, \pi]$ which may be extended to the real line by periodicity. We associate three types of boundary conditions with (1) over $[0, \pi]$:

- periodic boundary conditions

$$y(0) = y(\pi),\ y'(0) = y'(\pi), \tag{2}$$

- semi-periodic boundary conditions

$$y(0) = -y(\pi),\ y'(0) = -y'(\pi), \tag{3}$$

- auxiliary boundary conditions

$$y(\tau) = y(\tau + \pi) = 0, \tag{4}$$

where $0 \le \tau < \pi$.

Let $\lambda_n(n = 1, 2, \ldots)$ denote the periodic eigenvalues of the problem (1) and (2), and $\mu_n(n = 1, 2, \ldots)$ the semi-periodic eigenvalues of the problem (1) and (3), and $\Lambda_n(\tau), (n = 1, 2, \ldots)$ the auxiliary eigenvalues of (1) with condition (4). It is well known that , see for example [4],

$$\lambda_0 < \mu_0 \le \mu_1 < \lambda_1 \le \lambda_2 < \mu_2 \le \mu_3 < \ldots$$

We define the instability intervals of (1) as follows:

$I_0 = (-\infty, \lambda_0)$,$I_{2m+1} = (\mu_{2m}, \mu_{2m+1})$, $I_{2m+2} = (\lambda_{2m+1}, \lambda_{2m+2})$, and for $n \ge 1$ their lengths by l_n. It is shown in [10] that (1) with (4) is equivalent to the Dirichlet problem

$$y''(t) + (\lambda - q(t + \tau))y(t) = 0, \tag{5}$$

$$y(0) = y(\pi) = 0. \tag{6}$$

Some asymptotic estimates for the eigenvalues and instability intervals of (1) are provided in [6],[7], respectively. We suppose without loss of generality that

$$\int_0^\pi q(t)dt = 0$$

and let a_n,b_n denote the real Fourier coeeficients of q on $[0, \pi]$, i.e,

$$a_n = \frac{2}{\pi}\int_0^\pi q(t)cos(2nt)dt, b_n = \frac{2}{\pi}\int_0^\pi q(t)sin(2nt)dt. \tag{7}$$

As a result of the needs of modern physics, inverse problems became a hot research area. One of the earliest such problems formulated and solved by Ambarzumian[1]. In 1929, he considered the following question:

$$y''(t) + (\lambda - q(t))y(t) = 0$$

and

$$y'' + \lambda y = 0$$

subject to the boundary conditions

$$y'(0) = y'(\pi) = 0$$

with the same eigenvalues. What can be said about $q(z)$? Ambarzumian's answer was that $q(z) = 0$.

Borg[4] considered the general problem of what can be said about $q(z)$ from a knowledge of spectrum. Similar problems have also been investigated by Hochstadt[10] and Ungar[12]. Now, we state a result proven independently by Hochstadt and Ungar.

Theorem 1.1 *[10] If $q(z)$ is real and integrable, and if all finite instability intervals vanish then $q(z) = 0$ almost everwhere.*

In this paper we assume that all finite instability intervals are $O(n^{-k})$ and show that $a_n, b_n = O(n^{-k})$, from which we deduce that $q^{(k-2)}$ is absolutely continuous a.e.($k = 2, 3$).

Central to our analysis is the following theorem of Hochstadt [10] which involves the auxiliary eigenvalues of (1) considered on the interval $[\tau, \tau + \pi]$ where $0 \leq \tau < \pi$ with boundary conditions

$$y(\tau) = y(\tau + \pi) = 0.$$

Theorem 1.2 *[10] The ranges of $\Lambda_{2m}(\tau)$ and $\Lambda_{2m+1}(\tau)$, as functions of τ are $[\mu_{2m}, \mu_{2m+1}]$ and $[\lambda_{2m+1}, \lambda_{2m+2}]$ respectively.*

Remark: We make use of the Theorem 1.2 in the sense that if all finite instability intervals are $O(n^{-k})$ then $\Lambda_n(\tau_2) - \Lambda_n(\tau_1) = O(n^{-k})$ for any $\tau_1, \tau_2 \in [0, \pi)$.

We also state a sequence of Lemmas which will be used in proving our results. Let

$$c_n = \frac{1}{\pi} \int_0^\pi q(z) e^{-2inz} dz \tag{8}$$

be the Fourier coefficient of $q(z)$ over $[0, \pi]$.

Lemma 1.1 *[10] Let $q(z)$ be periodic with period π, integrable over $[0, \pi]$ and such that*

$$c_n = O(\frac{1}{n^2})$$

as $n \to \infty$. Then $q(z)$ is absolutely continuous almost everywhere.

Lemma 1.2 *Let $q(z)$ be periodic with period π, integrable over $[0, \pi]$ and such that*

$$c_n = O(\frac{1}{n^3})$$

as $n \to \infty$. Then $q'(z)$ is absolutely continuous almost everywhere.

Proof. The proof of Lemma 1.1 goes through.

Lemma 1.3 *[9] For $k = 1, 2, 3, \ldots$, $\tau \leq t \leq \tau + \pi$*

$$\Theta(t, \tau) - \Theta_k(t, \tau) = o(\Lambda^{-k/2})$$

as $\Lambda \to \infty$.

Lemma 1.4 *[2] For q integrable and for any x_1, x_2 such that $\tau \leq x_1 < x_2 \leq \tau + \pi$*

$$\int_{x_1}^{x_2} q(t) sin(2\Lambda^{1/2} t) dt = o(1)$$

as $\Lambda \to \infty$.

Now, we introduce the function $\Theta(t, \Lambda, \tau)$, the so-called modified Prüfer transformation of [2], which is defined for any given solution of (1) as

$$tan\Theta(t, \Lambda, \tau) = \frac{\Lambda^{1/2} y(t, \tau)}{y'(t, \tau)},$$

for $\tau \leq t \leq \tau + \pi$. This fixes Θ to within additive multiples of π. For definiteness we assume that $0 \leq \Theta(t,\tau) \leq \pi$ and observe that the boundary conditions (4) correspond to

$$\Theta(t,\tau) = 0,\ \Theta(t,\tau+\pi) = (n+1)\pi, \tag{9}$$

similarly the boundary conditions (6) correspond to

$$\Theta(t,0) = 0,\ \Theta(t,\pi) = (n+1)\pi. \tag{10}$$

¿From now on, we supress the dependence of Θ on Λ and write $\Theta(t,\tau)$ instead of $\Theta(t,\Lambda,\tau)$. Under the Prüfer transformation the differential equation corresponding to (1) can be written as

$$\Theta'(t,\tau) = \Lambda^{1/2} - \frac{1}{2}\Lambda^{-1/2}q(t) + \frac{1}{2}\Lambda^{-1/2}q(t)cos(2\Theta(t,\tau)), \tag{11}$$

and from (11)

$$\Theta(t,\tau) = (\Lambda^{1/2})(t-\tau) - \frac{1}{2}\Lambda^{-1/2}\int_\tau^t q(s)ds + \frac{1}{2}\Lambda^{-1/2}q(t)\int_\tau^t q(s)cos(2\Theta(s,\tau))ds. \tag{12}$$

We define a sequence of approximating functions for (12) as follows:

$$\Theta_1(t,\tau) := (\Lambda^{1/2})(t-\tau) - \frac{1}{2}\Lambda^{-1/2}\int_\tau^t q(s)ds,$$

$$\Theta_{k+1}(t,\tau) := \Theta_1(t,\tau) + \frac{1}{2}\Lambda^{-1/2}\int_\tau^t q(s)cos(2\Theta_k(s,\tau))ds \tag{13}$$

for $k = 1, 2, \ldots,$ and $\tau \leq t \leq \tau + \pi$.

2. The Results

Theorem 2.1 *For any integer k, the auxiliary eigenvalues of (1), as functions of τ, satisfy*

$$(n+1)\pi = \Lambda_n^{1/2}(\tau)\pi + \frac{1}{2}\Lambda^{-1/2}\int_0^\pi q(t+\tau)cos(2\Theta_k(t,\tau))dt + O(n^{-(k+1)}) \tag{14}$$

as $n \to \infty$.

Proof. We consider the differential equation (5) with the boundary conditions (6). From (11) we get

$$\Theta'(t,\tau) = \Lambda^{1/2} - \frac{1}{2}\Lambda^{-1/2}q(t+\tau) + \frac{1}{2}\Lambda^{-1/2}q(t+\tau)cos(2\Theta(t,\tau)). \tag{15}$$

We also know from Lemma 1.3 that

$$\Theta(t,\tau) - \Theta_k(t,\tau) = o(\Lambda^{-k/2}), \tag{16}$$

so that

$$cos(2\Theta(t,\tau)) = cos(2\Theta_k(t,\tau)) + O(\Lambda^{-k/2}). \tag{17}$$

Substituting (17) into (15) we obtain

$$\Theta'(t,\tau) = \Lambda^{1/2} - \frac{1}{2}\Lambda^{-1/2}q(t+\tau) + \frac{1}{2}\Lambda^{-1/2}q(t+\tau)cos(2\Theta_k(t,\tau)) + O(\Lambda^{-\frac{(k+1)}{2}}). \tag{18}$$

Integrating (18) with respect to τ on $[0,\pi]$ and using (10) we complete the proof.

Corollary 2.1 *As $n \to \infty$ the auxiliary eigenvalues of (1), as functions of τ, satisfy*

$$\Lambda_n^{1/2}(\tau) = (n+1) + \frac{1}{(n+1)}F_1(n,\tau) + O(n^{-2}) \tag{19}$$

where

$$F_1(n,\tau) = -\frac{1}{2\pi}\int_0^\pi q(t+\tau)cos(2(n+1)t)dt. \tag{20}$$

Proof. From (13), we see that

$$cos(2\theta_1(t,\tau)) = cos(2\Lambda^{1/2}t) + O(\Lambda^{-1/2}). \tag{21}$$

Substituting (21) into (14) and using reversion we complete the proof.

Theorem 2.2 *Let $q(t)$ be real-valued integrable function on $[0,\pi]$. If $l_n = O(n^{-1})$, then $a_n, b_n = O(n^{-1})$ as $n \to \infty$.*

Proof. It is easily seen that

$$F_1(n,\tau) = -\frac{1}{4}[cos(2(n+1)\tau)a_{n+1} + sin(2(n+1)\tau)b_{n+1}], \tag{22}$$

where a_{n+1} and b_{n+1} are defined in (7), and $F_1(n,\tau)$ is given by (20). From Corollary 2.1, for any $\tau_1, \tau_2 \in [0,\pi)$

$$\begin{aligned}
\Lambda_n^{1/2}(\tau_2) - \Lambda_n^{1/2}(\tau_1) &= \frac{1}{4(n+1)}\{[cos(2(n+1)\tau_1) - cos(2(n+1)\tau_2)]\, a_{n+1} \\
&+ [sin(2(n+1)\tau_1) - sin(2(n+1)\tau_2)]b_{n+1}\} + O(n^{-2}) \\
&= \frac{1}{2(n+1)}\{sin((n+1)(\tau_1+\tau_2))\, sin((n+1)(\tau_2-\tau_1))a_{n+1} \\
&+ cos((n+1)(\tau_1+\tau_2))sin((n+1)(\tau_1-\tau_2))b_{n+1}\} + O(n^{-2}) \\
&= \frac{1}{2(n+1)}sin((n+1)(\tau_2-\tau_1))\{sin((n+1)(\tau_1+\tau_2))a_{n+1} \\
&- cos((n+1)(\tau_1+\tau_2))b_{n+1}\} + O(n^{-2}).
\end{aligned} \tag{23}$$

On the other hand, from the assumption that $l_n = O(n^{-1})$ and Lemma 1.4.

$$\Lambda_n(\tau_2) - \Lambda_n(\tau_1) = O(n^{-1})$$

and hence of

$$\Lambda_n^{1/2}(\tau_2) - \Lambda_n^{1/2}(\tau_1) = \frac{\Lambda_n(\tau_2) - \Lambda_n(\tau_1)}{\Lambda_n^{1/2}(\tau_2) + \Lambda_n^{1/2}(\tau_1)} = O(n^{-2}). \tag{24}$$

The result follows from (23) and (24).

Corollary 2.2 *As $n \to \infty$ the auxiliary eigenvalues of (1), as functions of τ, satisfy*

$$\Lambda_n^{1/2}(\tau) = (n+1) + \frac{1}{(n+1)}F_1(n,\tau) + \frac{1}{(n+1)^2}F_2(n,\tau) + O(n^{-3}), \tag{25}$$

where $F_1(n,\tau)$ is given by (20) and

$$F_2(n,\tau) = -\frac{1}{2\pi}\int_0^\pi q(t+\tau)(\int_0^t q(s+\tau)ds)sin(2(n+1)t)dt$$

$$+\frac{1}{2\pi}\int_0^\pi q(t+\tau)(\int_0^t q(s+\tau)cos(2(n+1)s)ds)sin(2(n+1)t)dt. \tag{26}$$

Proof. From (13) we observe that for $k=1$

$$cos(2\Theta_2(t,\tau)) = cos(2\Lambda^{1/2}t) + \Lambda^{-1/2}sin(2\Lambda^{1/2}t)(\int_0^t q(s+\tau)ds)$$
$$- \Lambda^{-1/2}sin(2\Lambda^{1/2}t)(\int_0^t q(s+\tau)cos(2\Lambda^{1/2}s)ds) + O(\Lambda^{-1}). \tag{27}$$

Substituting (27) into (14) and using reversion we complete the proof.

Theorem 2.3 *Let q(t) be real-valued integrable function on* $[0,\pi]$. *If* $l_n = O(n^{-2})$ *then* $a_n, b_n = O(n^{-2})$ *as* $n\to\infty$.

Proof. Since $l_n = O(n^{-2})$ by assumption, it is $O(n^{-1})$ as well. Hence $a_n, b_n = O(n^{-1})$ by Theorem 2.2. From this and Lemma 1.4 we observe that

$$F_2(n,\tau) = O(n^{-1}).$$

Therefore, (25) reduces to

$$\Lambda_n^{1/2}(\tau) = (n+1) + \frac{1}{(n+1)}F_1(n,\tau) + O(n^{-3}).$$

By a similar argument in Theorem 2.2, we complete the proof.

Corollary 2.3 *Let q(t) be a real-valued integrable function on* $[0,\pi]$. *If* $l_n = O(n^{-2})$ *as* $n\to\infty$ *then* $q(t)$ *is absolutely continuous almost everywhere.*

Proof. It follows from Theorem 2.3 and Lemma 1.1

Corollary 2.4 *As* $n\to\infty$ *the auxiliary eigenvalues of (1), as functions of* τ, *satisfy*

$$\Lambda_n^{1/2}(\tau) = (n+1) + \frac{1}{(n+1)}F_1(n,\tau) + \frac{1}{(n+1)^2}F_2(n,\tau) + \frac{1}{(n+1)^3}F_3(n,\tau) + O(n^{-4}), \tag{28}$$

where $F_1(n,\tau)$ *is given by (20),* $F_2(n,\tau)$ *is given by (26) and*

$$F_3(n,\tau) = -\frac{1}{4(n+1)^3\pi}\int_0^\pi q(t+\tau)[\int_0^t q(s+\tau)ds - \int_0^t q(s+\tau)cos(2(n+1)s)ds]^2$$
$$\times cos(2(n+1)t)dt$$
$$+\frac{1}{2(n+1)^3\pi}\int_0^\pi q(t+\tau)(\int_0^t q(s+\tau)(\int_0^s q(l+\tau)dl)sin(2(n+1)s)ds)$$
$$\times sin(2(n+1)t)dt. \tag{29}$$

Proof. From (13) for $k=2$ we have

$$cos(2\Theta_3(t,\tau)) = cos(2\Lambda^{1/2}t) + \Lambda^{-1/2}sin(2\Lambda^{1/2}t)(\int_0^t q(s+\tau)ds)$$
$$-\Lambda^{-1/2}sin(2\Lambda^{1/2}t)(\int_0^t q(s+\tau)cos(2\Lambda^{1/2}s)ds)$$
$$-\frac{1}{2}\Lambda^{-1}\left(\int_0^t q(s+\tau)ds - \int_0^t q(s+\tau)cos(2\Lambda^{1/2}s)ds\right)^2 cos(2\Lambda^{1/2}t)$$
$$-\Lambda^{-1}(\int_0^t q(s+\tau)(\int_0^s q(l+\tau)dl)sin(2\Lambda^{1/2}s)ds)sin(2\Lambda^{1/2}t)$$
$$+O(\Lambda^{-3/2}). \tag{30}$$

Substituting (30) into (14),and using reversion we complete the proof.

Theorem 2.4 *Let q(t) be real-valued integrable function on* $[0,\pi]$. *If* $l_n = O(n^{-3})$ *then* $a_n, b_n = O(n^{-3})$ *as* $n \to \infty$.

Proof. Since $l_n = O(n^{-3})$ by assumption, it is $O(n^{-2})$ as well. Hence $a_n, b_n = O(n^{-2})$ by Theorem 2.3. Therefore, the terms involving $F_2(n,\tau)$ and $F_3(n,\tau)$ in (28) are included in the error term by Lemma 1.4. By a similar argument in Theorem 2.2 we complete the proof.

Corollary 2.5 *Let* $q(t)$ *be a real-valued function on* $[0,\pi]$. *If* $l_n = O(n^{-3})$ *then* $q'(t)$ *is absolutely continuous almost everywhere.*

Proof. It follows from Theorem 2.4 and Lemma 1.2.

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Haskız COŞKUN
Department of Mathematics
Karadeniz Technical University,
Department of Mathematics
61080, Trabzon-TURKEY

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