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Zeros of $\zeta''(s)$ & $\zeta'''(s)$ in $\sigma < \frac{1}{2}$

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Abstract

There is only one pair of non-real zeros of $\zeta''(s)$, and of $\zeta'''(s)$, in the left half-plane. The Riemann Hypothesis implies that $\zeta''(s)$ and $\zeta'''(s)$ have no zeros in the strip $0 \le \Re s < \frac{1}{2}$.
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1. Introduction

The Riemann zeta-function defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (\sigma > 1) \tag{1}$$

(as usual we write $s = \sigma + it$; $\sigma, t \in \mathbb{R}$), can be analytically continued to the whole complex plane, with a simple pole at $s = 1$, and satisfies the functional equation

$$\zeta(1-s) = 2(2\pi)^{-s}(\cos\frac{\pi s}{2})\Gamma(s)\zeta(s)\,. \tag{2}$$

From (2) it is seen that $\zeta(-2n) = 0$, $\forall n \in \mathbb{Z}^+$ (trivial zeros of ζ). From Hadamard's theory of entire functions it follows that $\zeta(s)$ also has infinitely many (nontrivial) zeros in the strip $0 < \sigma < 1$. The nontrivial zeros are situated symmetrically with respect to the real axis and also with respect to the line $\sigma = \frac{1}{2}$. Applying the argument principle, von Mangoldt proved that the number of nontrivial zeros $\rho = \beta + i\gamma$ with $0 < \gamma \le T$ is $\frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$, as $T \to \infty$. Riemann's yet unproved assertion that all of these zeros lie on the critical line $\sigma = \frac{1}{2}$ is known as the Riemann Hypothesis (RH). For the fundamentals of the theory of $\zeta(s)$ we refer the reader to Davenport's book [3].

The origin of our topic is Speiser's proof [6] that the Riemann Hypothesis is equivalent to $\zeta'(s)$ having no zeros in $0 < \sigma < \frac{1}{2}$. In a comprehensive article on the zeros of derivatives of $\zeta(s)$, Levinson and Montgomery [4] gave a different proof of this and that $\zeta'(s)$ has only real zeros in the closed left half-plane, vanishing exactly once in the interval $(-2n-2, -2n)$ for $n \geq 1$ (these are the zeros between the trivial zeros of ζ guaranteed by Rolle's theorem). Moreover they showed that for any $k \geq 1$, $\zeta^{(k)}(s)$ has at most a finite number of nonreal zeros in $\sigma < \frac{1}{2}$ as a consequence of RH. Spira [7] calculated the zeros of ζ' and ζ'' in the rectangle $-1 \leq \sigma \leq 5$, $|t| \leq 100$, and found out that $\zeta''(s) \neq 0$ in $0 \leq \sigma \leq \frac{1}{2}$, $|t| \leq 100$. However, Spira also found that ζ'' has zeros at $-0.355084.. \pm i \cdot 3.59083..$ (to be denoted as b_0 and $\overline{b}_0$ below).

Berndt [2] showed that the number of nonreal zeros of $\zeta^{(k)}(s)$ with imaginary parts in $[0, T]$ is $\dfrac{T}{2\pi}\log T - \dfrac{1+\log 4\pi}{2\pi}T + O(\log T)$. For each $k \geq 0$, the nonreal zeros of $\zeta^{(k)}(s)$ all lie in a strip $\alpha_k < \sigma < \sigma_k$. The existence of α_k was deduced by Spira [8]. That $\zeta(s) \neq 0$ in the region $\sigma \geq 1 - \dfrac{c}{\log T}$, $t \geq 2$ (in fact the very first zero of $\zeta(s)$ is at $\frac{1}{2} + i \cdot 14.134..$, and the first $1.5 \cdot 10^9$ zeros of $\zeta(s)$ have all been verified in [5] to lie on the critical line) implies the prime number theorem. Titchmarsh [9, Theorem 11.5c] proved $\sigma_1 < 3$. Later Spira [7] calculated that $\sigma_2 = 4.98..$, $\sigma_3 = 6.01, \ldots, \sigma_{10} = 13.68$, and in general $\sigma_k = \frac{7}{4} + 2$ for $k \geq 3$ is acceptable. Verma and Kaur [10] have improved the last estimate to $\sigma_k = ak + 2$ for $k \geq 3$ with $a = 1.13...$

In this paper, we shall be concerned with the zeros of $\zeta''(s)$ and $\zeta'''(s)$ lying to the left of the critical line. Our results for the left half-plane are unconditional (i.e. without assuming RH), since here $\zeta(s)$ can be expressed via the functional equation in terms of its values in $\sigma > 1$, but to get results for the strip $0 < \sigma < \frac{1}{2}$ we assume RH. Most of our results appeared in [11] which contained only the proof of Theorem 1 fully.

In our calculations we will repeatedly use the well-known formula

$$\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{m}\frac{a_k}{n+\alpha_k}\Big) = -\sum_{k=1}^{m} a_k \psi(1+\alpha_k), \quad (a_k, \alpha_k \in \mathbb{C})$$

where $\psi = \frac{\Gamma'}{\Gamma}$ is the digamma function.

2. ζ'' to the left of the critical line

Theorem 1. *The Riemann Hypothesis implies that $\zeta''(s)$ has no zeros in the strip $0 \le \sigma < \frac{1}{2}$.*

Proof. Let us denote the real zeros of ζ' as $-a_n$, $n \ge 1$, where $a_n \in (2n, 2n+2)$. A nonreal zero of ζ' will be represented as $\rho_1 = \beta_1 + i\gamma_1$. By what was recounted above, $\frac{1}{2} \le \beta_1 < 3$ for all ρ_1 (the lower-bound is upon RH). Since $\Re\frac{\zeta'}{\zeta}(s) < 0$ on $\sigma = \frac{1}{2}$ except when $\zeta(s) = 0$, one has $\beta_1 = \frac{1}{2}$ only at a possible multiple zero of $\zeta(s)$ (see [4]). We start with the partial fraction representation

$$\frac{\zeta''}{\zeta'}(s) = \frac{\zeta''}{\zeta'}(0) - 2 - \frac{2}{s-1} + \sum_{\rho_1}\Big(\frac{1}{s-\rho_1} + \frac{1}{\rho_1}\Big) + \sum_{n}\Big(\frac{1}{s+a_n} - \frac{1}{a_n}\Big), \tag{3}$$

which follows from Hadamard factorization. Taking real parts in (3), we have

$$\Re\frac{\zeta''}{\zeta'}(s) = \frac{\zeta''}{\zeta'}(0) - 2 + \frac{2(1-\sigma)}{|s-1|^2} + \sum_{\rho_1}\Re\frac{1}{s-\rho_1} + \sum_{\rho_1}\frac{1}{\rho_1}$$
$$+ \sum_{n}\Big(\frac{\sigma + a_n}{|s+a_n|^2} - \frac{1}{a_n}\Big), \tag{4}$$

since $\zeta'(\overline{\rho_1}) = 0$ as well. We should first like to put a bound on $\sum \frac{1}{\rho_1}$ (in this series it is understood that the terms from ρ_1 and $\overline{\rho_1}$ are grouped together). At $s = 6$, Eq. (4) reads

$$\frac{\zeta''}{\zeta'}(6) = \frac{\zeta''}{\zeta'}(0) - \frac{12}{5} + \sum_{\rho_1}\frac{6-\beta_1}{(6-\beta_1)^2 + \gamma_1^2} + \sum_{\rho_1}\frac{\beta_1}{\beta_1^2+\gamma_1^2}$$
$$- \sum_{n}\frac{6}{a_n(a_n+6)}. \tag{5}$$

It is known that $\frac{\zeta''}{\zeta'}(0) = 2.183..$(see [1]), and $\frac{\zeta''}{\zeta'}(6) = -0.773...$ Also

$$\sum_{n}\frac{6}{a_n(a_n+6)} < \sum_{n=1}^{\infty}\frac{6}{2n(2n+6)} = \frac{11}{12}\ .$$

Since the least $|\gamma_1|$ is 23.3.. (see [7]), for all ρ_1 we have

$$\frac{6-\beta_1}{(6-\beta_1)^2+\gamma_1^2} > \frac{\beta_1}{\beta_1^2+\gamma_1^2}.$$

Plugging all these in (5), it follows that

$$\sum_{\rho_1} \frac{1}{\rho_1} < 0.185 \tag{6}$$

(from Spira's list of ρ_1 with $|\gamma_1| < 100$ one calculates $\sum \frac{1}{\rho_1} > 0.0249$).

We now examine the value of $\Re\frac{\zeta''}{\zeta'}(s)$ in the region $0 \le \sigma \le \frac{1}{2}$, $|t| \ge 100$. If ever a zero of ζ' exists on the critical line, this region is to be modified by deleting an arbitrarily small neighbourhood around such a zero. For any s in our region, $\frac{2(1-\sigma)}{|s-1|^2} < \frac{1}{5000}$ and $\Re\frac{1}{s-\rho_1} < 0$ for all ρ_1 (on RH), and also

$$\begin{aligned}
\sum_n \Big(\frac{\sigma+a_n}{|s+a_n|^2} - \frac{1}{a_n}\Big) &\le \sum_n \frac{-10^4}{a_n((a_n+\frac{1}{2})^2+10^4)} \\
&< \sum_{n=2}^{\infty} \frac{-10^4}{2n((2n+\frac{1}{2})^2+10^4)} \\
&< \frac{1}{2}\Big(1+\psi(1)-\Re\psi(\frac{5}{4}+50i)\Big) + \frac{\Im\psi(\frac{5}{4}+50i)}{400} \\
&< -1.74.
\end{aligned} \tag{7}$$

Together with (6), these estimates used in (4) give $\Re\frac{\zeta''}{\zeta'}(s) < -1.37$ at all points of our region. □

Notice that $\zeta''(s)$ can be zero on the critical line only at a multiple zero (of at least third order) of $\zeta(s)$ if ever this exists.

Theorem 2. (unconditional) *There is only one pair of nonreal zeros of $\zeta''(s)$ in the left half-plane.*

To prove Theorem 2 we shall consider the change in the argument of $\frac{\zeta''}{\zeta'}(s)$ as s goes around the rectangle R with corners at $\pm iN$, $\sigma_N \pm iN$, where $\sigma_N = -2N - 2$ with an arbitrarily large $N \in \mathbb{N}$. The reason behind this choice of σ_N will be clear after the following lemma.

Lemma 1. $-a_n = -2n - 2 + \frac{1}{\log n} + O(\frac{1}{\log^2 n})$, *as* $n \to \infty$.

Proof. Differentiating the functional equation (2) we have

$$\zeta'(1-s) = \zeta(1-s)\Big[\log 2\pi + \frac{\pi}{2}\tan\frac{\pi s}{2} - \psi(s) - \frac{\zeta'}{\zeta}(s)\Big]\,, \tag{8}$$

so we see that $\zeta'(1-\sigma) = 0$ with $\sigma > 1$ if

$$\log 2\pi + \frac{\pi}{2}\tan\frac{\pi\sigma}{2} - \psi(\sigma) - \frac{\zeta'}{\zeta}(\sigma) = 0\,. \tag{9}$$

We are interested in the situation when $\sigma \to \infty$, in which case we use

$$\psi(\sigma) = \log\sigma - \frac{1}{2\sigma} + O(\frac{1}{\sigma^2})\,, \tag{10}$$

$$\frac{\zeta'}{\zeta}(\sigma) = -(\frac{\log 2}{2^\sigma} + \frac{\log 3}{3^\sigma} + \frac{\log 2}{4^\sigma} + \ldots) = O(\frac{1}{\sigma^2})\,. \tag{11}$$

Thus as $\sigma \to \infty$, (9) becomes

$$\frac{\pi}{2}\tan\frac{\pi\sigma}{2} = \log\frac{\sigma}{2\pi} - \frac{1}{2\sigma} + O(\frac{1}{\sigma^2})\,.$$

Since the right-hand side tends to ∞, to maintain equality we must have σ tend to ∞ through values close to and to the left of odd integers. So the negative zeros of ζ' lie slightly to the right of negative even integers, i.e.

$$-a_n = -2n - 2 + \epsilon(n)\,, \quad (\epsilon(n) > 0)\,.$$

Carrying this out in more detail, taking $\sigma = 2n + 3 - \epsilon(n)$, we have

$$\frac{\pi}{2}\tan\frac{\pi\sigma}{2} = \frac{1}{\epsilon(n)} - \frac{\pi^2\epsilon(n)}{12} + O(\epsilon^3(n))\,.$$

Thus we find

$$\epsilon(n) = \frac{1}{\log n} + O(\frac{1}{\log^2 n}),$$

and $\epsilon(n) < 1$ for $n \geq 3$. □

Note that differentiation of (2) also gives

$$\zeta'(-2k) = (-1)^k \pi (2\pi)^{-(2k+1)} (2k)! \zeta(2k+1). \tag{12}$$

Next we observe that $\frac{\zeta''}{\zeta'}(-\sigma_N) < 0$ for all sufficiently large N. For, differentiating the functional equation twice, we get for $k \geq 1$,

$$\zeta''(-2k) = (-1)^k \frac{(2k)!}{(2\pi)^{2k}} \Big[\zeta(2k+1)(\log 2\pi - \psi(2k+1)) - \zeta'(2k+1)\Big] \tag{13}$$

and so we have

$$\frac{\zeta''}{\zeta'}(-2k) = 2\Big(\log 2\pi - \psi(2k+1) - \frac{\zeta'}{\zeta}(2k+1)\Big) < 0, (k \geq 3). \tag{14}$$

Proof of Theorem 2. Inside R there are exactly N zeros of ζ' (all real), so by Rolle's theorem there must be at least $N-1$ real zeros of ζ''. We also know that there exist 2κ, $\kappa \geq 1$, nonreal zeros of ζ'' inside R. Call the number of zeros of $\zeta^{(i)}$ in R as Z_i. By the argument principle we have

$$\frac{1}{2\pi} \Delta_R \arg \frac{\zeta''}{\zeta'}(s) = Z_2 - Z_1 \geq N - 1 + 2\kappa - N = 2\kappa - 1\,.$$

If it is shown that $\arg \frac{\zeta''}{\zeta'}(s)$ changes by 2π as s makes one counterclockwise tour of R, then Theorem 2 is proved. It would also follow that between consecutive negative zeros of ζ', ζ'' vanishes exactly once.

Equation (4) may be rewritten as

$$\Re \frac{\zeta''}{\zeta'}(\sigma + it) = K + \frac{2(1-\sigma)}{(1-\sigma)^2 + t^2} + \sum_n \Big(\frac{(\sigma + a_n)}{(\sigma + a_n)^2 + t^2} - \frac{1}{a_n}\Big)$$
$$+ \sum_{\rho_1} \frac{\sigma - \beta_1}{(\sigma - \beta_1)^2 + (\gamma_1 - t)^2}, \tag{15}$$

where $K = \frac{\zeta''}{\zeta'}(0) - 2 + \sum_{\rho_1} \frac{1}{\rho_1}$ and $0.185 < K < 0.368$.

First consider the left edge of R where $\sigma = \sigma_N = -2N-2$, $|t| \le N$. Here

$$\frac{2(1-\sigma)}{(1-\sigma)^2+t^2} = O(\frac{1}{N}), \tag{16}$$

and $-2N-5 \le \sigma_N - \beta_1 \le -2N-2$, so that (writing $\sum_{\rho_1}$ for the last term of (15))

$$-(2N+5)\sum_{\gamma_1}\frac{1}{(2N+2)^2+(\gamma_1-t)^2} < \sum_{\rho_1} < -(2N+2)\sum_{\gamma_1}\frac{1}{(2N+5)^2+(\gamma_1-t)^2}$$

$$\begin{aligned}&-(2N+5)\Big(\sum_{|\gamma_1-t|<2N+2}\frac{1}{(2N+2)^2}+\sum_{|\gamma_1-t|\ge 2N+2}\frac{1}{(\gamma_1-t)^2}\Big)\\ &<\sum_{\rho_1}<-(2N+2)\Big(\sum_{|\gamma_1-t|<2N+5}\frac{1}{2(2N+5)^2}+\sum_{|\gamma_1-t|\ge 2N+5}\frac{1}{2(\gamma_1-t)^2}\Big).\end{aligned}$$

The sums over γ_1 are evaluated in a standard way using the result of Berndt mentioned in the introduction, giving

$$-\frac{2}{\pi}\log N \lesssim \sum_{\rho_1}\frac{\sigma-\beta_1}{(\sigma-\beta_1)^2+(\gamma_1-t)^2} \lesssim -\frac{1}{\pi}\log N. \tag{17}$$

Now consider the sum over n in (15) for $\sigma_N \le \sigma < 0$, splitting it into two parts: $\sigma + a_n \le 0$ (the finite part) and $\sigma + a_n > 0$ (the infinite part). The finite part is negative and attains its maximum at $|t| = N$. We have

$$\begin{aligned}\sum_{a_n\le-\sigma}\Big(\frac{(\sigma+a_n)}{(\sigma+a_n)^2+N^2}-\frac{1}{a_n}\Big) &\le -\sum_{a_n\le-\sigma}\frac{1}{a_n}+O(1)\\ &\le -\sum_{n<\lceil\frac{-\sigma}{2}\rceil}\frac{1}{2n+2}+O(1)\\ &= -\frac{1}{2}\log(1-\frac{\sigma}{2})+O(1)\end{aligned} \tag{18}$$

(In (18) the sums over a_n are void if $-\sigma < a_1$, and the sum over n is void if $\sigma > -2$. In these cases the O(1)-term takes care of things). Thus on the left edge of R the finite part is always less than $-\frac{1}{2}\log N + O(1)$. On the left edge of R the infinite part is maximum when $t = 0$, and then by Lemma 1

$$\begin{aligned}\sum_{n=N+1}^{\infty}\Big(\frac{1}{\sigma_N + a_n} - \frac{1}{a_n}\Big) &< \sum_{n=N+1}^{\infty}\Big(\frac{1}{2(n-N) - \frac{2}{\log N}} - \frac{1}{2n+2}\Big)\\ &= \frac{1}{2}\sum_{m=1}^{\infty}\Big(\frac{1}{m - \frac{1}{\log N}} - \frac{1}{m+N+1}\Big)\\ &= \frac{1}{2}\Big(\psi(N+2) - \psi(1 - \frac{1}{\log N})\Big)\\ &\leq \frac{1}{2}\log N + O(1)\,. \qquad (19)\end{aligned}$$

Adding up the results of (16)-(19) in (15) we have on the left edge of R

$$\Re\frac{\zeta''}{\zeta'}(\sigma_N + it) \lesssim -\frac{1}{\pi}\log N \quad (|t| \leq N). \qquad (20)$$

On $\sigma + iN$, $\sigma_N \leq \sigma < 0$ we rewrite (15) as

$$\Re\frac{\zeta''}{\zeta'}(\sigma + iN) = K + \sum_{a_n > -\sigma} + \sum_{a_n \leq -\sigma} + \sum_{\rho_1} + O(\frac{1}{N})\,,$$

where the sum over ρ_1 takes negative values, and the finite sum was estimated in (18). Now observe that for $\sigma < 0$

$$\begin{aligned}&\sum_{a_n > -\sigma}\Big(\frac{\sigma + a_n}{(\sigma + a_n)^2 + N^2} - \frac{1}{a_n}\Big)\\ &= -(\sigma^2 + N^2)\sum_{a_n > -\sigma}\frac{1}{a_n((\sigma + a_n)^2 + N^2)} - \sigma\sum_{a_n > -\sigma}\frac{1}{(\sigma + a_n)^2 + N^2}\\ &< -(\sigma^2 + N^2)\sum_{n=\lceil\frac{-\sigma}{2}\rceil}^{\infty}\frac{1}{(2n+2)[(\sigma + 2n + 2)^2 + N^2]} + O(1) - \sigma\sum_{n=0}^{\infty}\frac{1}{(2n)^2 + N^2}.\end{aligned}$$

For $\sigma_N \leq \sigma < 0$,

$$-\sigma\sum_{n=0}^{\infty}\frac{1}{(2n)^2 + N^2} = O(1),$$

and we calculate

$$\begin{aligned}
&-(\sigma^2+N^2)\sum_{n=\lceil\frac{-\sigma}{2}\rceil}^{\infty}\frac{1}{(2n+2)[(\sigma+2n+2)^2+N^2]}\\
&\le -\frac{(\sigma^2+N^2)}{8}\sum_{m=1}^{\infty}\frac{1}{(m-\frac{\sigma}{2})(m^2+(\frac{N}{2})^2)}\\
&=\sum_{m=1}^{\infty}\Big(-\frac{1}{2}\frac{1}{m-\frac{\sigma}{2}}-\frac{\sigma-iN}{4Ni}\frac{1}{m+\frac{iN}{2}}+\frac{\sigma+iN}{4Ni}\frac{1}{m-\frac{iN}{2}}\Big)\\
&=\frac{1}{2}\psi(1-\frac{\sigma}{2})+\frac{\sigma}{2N}\Im\psi(1+\frac{iN}{2})-\frac{1}{2}\Re\psi(1+\frac{iN}{2}).
\end{aligned}$$

So for $\sigma_N \le \sigma < 0$

$$\begin{aligned}
\sum_{a_n>-\sigma}\Big(\frac{\sigma+a_n}{(\sigma+a_n)^2+N^2}-\frac{1}{a_n}\Big) &< \frac{1}{2}\psi(1-\frac{\sigma}{2})-\frac{1}{2}\Re\psi(1+\frac{iN}{2})+O(1)\\
&=\frac{1}{2}\log(1-\frac{\sigma}{2})-\frac{\log N}{2}+O(1). \qquad (21)
\end{aligned}$$

Hence on $\sigma \pm iN$, $\sigma_N \le \sigma < 0$ we have

$$\Re\frac{\zeta''}{\zeta'}(\sigma+iN) < -\frac{1}{2}\log N + O(1). \qquad (22)$$

It remains to consider the edge on the imaginary axis, $[-iN, iN]$. Here,

$$\begin{aligned}
\Re\frac{\zeta''}{\zeta'}(it) &= \frac{\zeta''}{\zeta'}(0)-2+\frac{2}{1+t^2}+\sum_n\frac{-t^2}{a_n(a_n^2+t^2)}+\sum_{\rho_1}\frac{1}{\rho_1}\\
&\quad+\sum_{\gamma_1>0}\Big(\frac{-\beta_1}{(\beta_1^2+(\gamma_1-t)^2)}+\frac{-\beta_1}{(\beta_1^2+(\gamma_1+t)^2)}\Big), \qquad (23)
\end{aligned}$$

$$\Im\frac{\zeta''}{\zeta'}(it)=\frac{2t}{1+t^2}-\sum_n\frac{t}{a_n^2+t^2}+\sum_{\gamma_1>0}\frac{2t(\gamma_1^2-\beta_1^2-t^2)}{(\beta_1^2+(\gamma_1-t)^2)(\beta_1^2+(\gamma_1+t)^2)}. \qquad (24)$$

The sums over a_n can be bounded in a similar way to (7), but keeping in mind that $2.6 < a_1 < 2.8$, $4.8 < a_2 < 5$, and $2n+1 < a_n < 2n+2$ for $n \ge 3$ (this can be verified from (8)) in order to get sharper inequalities that will allow us below to determine the signs of $\Re\frac{\zeta''}{\zeta'}(it)$ and $\Im\frac{\zeta''}{\zeta'}(it)$ at certain points. Writing

$$A(t) = -\frac{t^2}{2.8(2.8^2+t^2)} - \frac{t^2}{5(5^2+t^2)} + \sum_{n=1}^{3}\frac{t^2}{2n(2n^2+t^2)}$$

$$B(t) = -\frac{t^2}{2.6(2.6^2+t^2)} - \frac{t^2}{4.8(4.8^2+t^2)} + \frac{t^2}{3(3^2+t^2)} + \frac{t^2}{5(5^2+t^2)}$$

we have

$$\begin{aligned} &B(t) + \frac{1}{2}(\psi(\frac{3}{2}) - \Re\psi(\frac{3}{2}+\frac{it}{2})) \\ &< \sum_n \frac{-t^2}{a_n(a_n^2+t^2)} < A(t) + \frac{1}{2}(\psi(1) - \Re\psi(1+\frac{it}{2})). \end{aligned} \tag{25}$$

Similarly, writing

$$C(t) = -\frac{t}{2.6^2+t^2} - \frac{t}{4.8^2+t^2} + \frac{t}{3^2+t^2} + \frac{t}{5^2+t^2}$$

$$D(t) = -\frac{t}{2.8^2+t^2} - \frac{t}{5^2+t^2} + \frac{t}{2^2+t^2} + \frac{t}{4^2+t^2} + \frac{t}{6^2+t^2}$$

we have

$$C(t) - \frac{1}{2}\Im\psi(\frac{3}{2}+\frac{it}{2}) < \sum_n \frac{-t}{a_n^2+t^2} < D(t) - \frac{1}{2}\Im\psi(1+\frac{it}{2}). \tag{26}$$

Using (25) and (6) in (23), where taking 0 as an upper bound for the sum over $\gamma_1 > 0$, it is seen that for $t > 23$, $\Re\frac{\zeta''}{\zeta'}(it) < 0$. In (23) we combine the sums over ρ_1 and $\gamma_1 > 0$ as

$$\sum_{\gamma_1>0} \frac{2t^2\beta_1(\beta_1^2 - 3\gamma_1^2 + t^2)}{(\beta_1^2+\gamma_1^2)(\beta_1^2+(\gamma_1-t)^2)(\beta_1^2+(\gamma_1+t)^2)},$$

and we see that for $|t| < 40$ each term is negative (since $\gamma_1 > 23.298$ and $\beta_1 < 3$ ([7])). Also, the derivative of a term of the sum over γ_1 in (23) is

$$\frac{-4t\beta_1[-\beta^4 + (\gamma^2 - t^2)(2\beta^2 + t^2 + 3\gamma^2)]}{[\beta_1^2+(\gamma_1-t)^2]^2[\beta_1^2+(\gamma_1+t)^2]^2},$$

which is negative for $t \in [0,23]$. So for $t \in [0,23]$ all the terms in the right-hand side of (23) are decreasing functions of t. Hence $\Re\frac{\zeta''}{\zeta'}(it) = 0$ at only one pair of conjugate points on the imaginary axis. As for $\Im\frac{\zeta''}{\zeta'}(it)$, the sum over ρ_1 in (24) is positive for $0 < t \leq 23$. When $t \to 0^{\pm}$ we have $\Im\frac{\zeta''}{\zeta'}(it) \to 0^{\pm}$ ($\lim_{t\to 0} \frac{\Im\psi(1+\frac{it}{2})}{t} = \frac{\pi^2}{12}$ and $\lim_{t\to 0} \frac{\Im\psi(\frac{3}{2}+\frac{it}{2})}{t} = \frac{\pi^2}{2} - 4$). From eqs. (23)-(26) we see that

$$\begin{aligned}\Re\frac{\zeta''}{\zeta'}(i) &> K + 1 + B(1) + \frac{1}{2}[\psi(\frac{3}{2}) - \Re\psi(\frac{3}{2}+\frac{i}{2})] + \sum_{\gamma_1>0} \frac{-4\beta_1}{(\beta_1^2+\gamma_1^2)} \\ &= \frac{\zeta''}{\zeta'}(0) - 1 + B(1) + \frac{1}{2}[\psi(\frac{3}{2}) - \Re\psi(\frac{3}{2}+\frac{i}{2})] - \sum_{\rho_1} \frac{1}{\rho_1} > 0,\end{aligned}$$

$$\Im\frac{\zeta''}{\zeta'}(i) > 1 + C(1) - \frac{1}{2}\Im\psi(\frac{3}{2}+\frac{i}{2}) > 0,$$

$$\Re\frac{\zeta''}{\zeta'}(3.5i) < \frac{\zeta''}{\zeta'}(0) - 2 + \frac{2}{13.25} + A(3.5) + \frac{1}{2}[\psi(1) - \Re\psi(1+1.75\,i)] < 0,$$

$$\Im\frac{\zeta''}{\zeta'}(3.5i) > \frac{7}{13.25} + C(3.5) - \frac{1}{2}\Im\psi(\frac{3}{2}+1.75\,i) + 0.0197 > 0,$$

where 0.0197 is a lower bound for the first two terms of the sum over $\gamma_1 > 0$ in (24) coming from the first two zeros of ζ' at approximately $2.46.. + i\cdot 23.298..$ and $1.29 + i\cdot 31.71..$ ([7]). As t increases from 3.5, $\Im\frac{\zeta''}{\zeta'}(it)$ may change sign, but $\Re\frac{\zeta''}{\zeta'}(it)$ will always be negative. Thus as t moves up on the imaginary axis, the image curve of $\frac{\zeta''}{\zeta'}(it)$ includes just one counterclockwise loop around the origin and the change in $\arg\frac{\zeta''}{\zeta'}$ is roughly 2π. This completes the proof of Theorem 2. □

The graphs of $\frac{\zeta^{(k+1)}}{\zeta^{(k)}}(it)$ for $|t| \leq 40$ and $k = 0,1,2,3$ are included at the end of this paper. These graphs were plotted by M. Özkan in his senior project, using the expressions from the Euler-Maclaurin sum formula

$$(-1)^k\zeta^{(k)}(s)=\sum_{n=1}^{N-1}\frac{\log^k n}{n^s}+\frac{\log^k N}{2N^s}+N^{1-s}\sum_{j=0}^{k}C_{kj}\frac{\log^{k-j}N}{(s-1)^{j+1}}$$
$$+\sum_{\nu=1}^{m}\Big[\sum_{j=0}^{k}\binom{k}{j}\Pi_\nu^{(k-j)}(s)(-1)^{k-j}\log^j N\Big]N^{1-s-2\nu}+R_k, \qquad (27)$$

where

$$C_{kj}=\frac{k!}{(k-j)!},\qquad \Pi_\nu^{(t)}(s)=\frac{B_{2\nu}}{(2\nu)!}\frac{d^t}{ds^t}\prod_{j=0}^{2\nu-2}(s+j),$$

and the error term R_k is neglected in the computations.

In the proof of Theorem 2 one can also notice that if one starts from a point on the negative real axis where $\Re\frac{\zeta''}{\zeta'}(s)>0$ and moves vertically away from the real axis, soon one hits a point where $\Re\frac{\zeta''}{\zeta'}(s)=0$ and further away from the axis $\Re\frac{\zeta''}{\zeta'}(s)<0$.

3. Zeros of $\zeta''(s)$ on the negative real axis

In order to proceed to the investigation of $\zeta'''(s)$ some information on the negative zeros of $\zeta''(s)$ - which will be denoted by $-b_n$, $n\geq 1$ - is needed. The zeros of $\zeta''(s)$ in the right half-plane will be denoted by $\rho_2=\beta_2+i\gamma_2$. From Eq. (13) we see that $\zeta''(-2)<0$, $\zeta''(-4)>0$, and as of $k=3$ the quantity in brackets in (13) will always be negative so that $\operatorname{sgn}[\zeta''(-2k)]=(-1)^{k+1}$, $(k\geq 3)$. Similar to (13) we have for $k\geq 1$

$$\zeta''(1-2k)=(-1)^k\frac{2\Gamma(2k)\zeta(2k)}{(2\pi)^{2k}}[\log^2 2\pi-(\frac{\pi}{2})^2+\psi'(2k)+(\psi(2k))^2+\frac{\zeta''}{\zeta}(2k)$$
$$+2\psi(2k)(\frac{\zeta'}{\zeta}(2k)-\log 2\pi)-2\frac{\zeta'}{\zeta}(2k)\log 2\pi]. \qquad (28)$$

In (28), as k increases, the term $(\psi(2k))^2$ will eventually dominate and the quantity in brackets will always be positive. This happens for $k\geq 16$. We find that

$$b_n\in\begin{cases}[3,4] & n=1\\ [2n+2,2n+3] & 2\leq n\leq 13\\ [2n+3,2n+4] & n\geq 14.\end{cases} \qquad (29)$$

Using MAPLE-V the first few negative zeros of ζ'' are found to be

$$b_1 = 3.595.., \quad b_2 = 6.028.., \quad b_3 = 8.278.., \quad b_4 = 10.446.., \tag{30}$$
$$b_5 = 12.568.., \quad b_6 = 14.662.., \quad b_7 = 16.736.., \quad b_8 = 18.798..,$$
$$b_9 = 20.849.., \quad b_{10} = 22.893.., \quad b_{11} = 24.931..$$

Lemma 2. $-b_n = -2n - 4 + \dfrac{2}{\log n} + O(\dfrac{1}{\log^2 n}),$ *as* $n \to \infty$.

Proof. Differentiating (8) gives

$$-\frac{\zeta''}{\zeta}(1-s) + (\frac{\zeta'}{\zeta}(1-s))^2 = (\frac{\pi}{2})^2(1+\tan^2\frac{\pi s}{2}) - \psi'(s) - (\frac{\zeta'}{\zeta}(s))'. \tag{31}$$

We put $\zeta''(1-\sigma) = 0,\ \sigma > 1$ and use (8) to write

$$\Big(\log 2\pi + \frac{\pi}{2}\tan\frac{\pi\sigma}{2} - \psi(\sigma) - \frac{\zeta'}{\zeta}(\sigma)\Big)^2 = (\frac{\pi}{2})^2(1+\tan^2\frac{\pi\sigma}{2}) - \psi'(\sigma) - (\frac{\zeta'}{\zeta}(\sigma))'.$$

For large σ, using (10), (11) and

$$\psi'(\sigma) = \frac{1}{\sigma} + O(\frac{1}{\sigma^2}), \qquad \frac{\zeta''}{\zeta}(\sigma) = O(\frac{1}{\sigma^2}) \tag{32}$$

this is simplified to

$$[\log\frac{\sigma}{2\pi} - (\pi + O(\frac{1}{\sigma\log\sigma})\tan\frac{\pi\sigma}{2} + O(\frac{1}{\sigma})]\log\frac{\sigma}{2\pi} = (\frac{\pi}{2})^2. \tag{33}$$

It follows that $\log\dfrac{\sigma}{2\pi} \approx \pi\tan\dfrac{\pi\sigma}{2}$, and σ must be close to and to the left of an odd integer. So we plug $\sigma = 2n+5-\delta(n),\ (\delta(n) > 0)$ in (33) and solving for $\delta(n)$ we obtain the result. □

4. Nonreal zeros of $\zeta'''(s)$ in $\sigma < \frac{1}{2}$

Similar to (3) we have

$$\frac{\zeta'''}{\zeta''}(s) = \frac{\zeta'''}{\zeta''}(0) - 3 - \frac{3}{s-1} + \sum_{n=1}^{\infty}\Big(\frac{1}{s+b_n} - \frac{1}{b_n}\Big) + \sum_{\rho_2}\Big(\frac{1}{s-\rho_2} + \frac{1}{\rho_2}\Big)$$
$$+ \Big(\frac{1}{s-b_0} + \frac{1}{b_0} + \frac{1}{s-\overline{b}_0} + \frac{1}{\overline{b}_0}\Big). \tag{34}$$

From Spira [7] we know that $\beta_2 < 5$ for all ρ_2, and analogous to (4) (by using (34) at $s = 10$) we find

$$\sum_{\rho_2} \frac{1}{\rho_2} < 0.12 \tag{35}$$

(from Spira's list of ρ_2 with $|\gamma_2| < 100$ one calculates $\sum \frac{1}{\rho_2} > 0.037$).

Theorem 3. (unconditional) *There is only one pair of nonreal zeros of $\zeta'''(s)$ in the left half-plane.*

Proof. Consider $\Delta_R \arg \frac{\zeta'''}{\zeta''}(s)$ where R is as in the proof of Theorem 2, but with $\sigma_N = -2N - 4$. From our results above, inside R there are N real zeros and two nonreal zeros of ζ''. By Rolle's Theorem there must be at least $N - 1$ real zeros of ζ''' here. Let 2κ be the number of nonreal zeros of $\zeta'''(s)$. Then

$$\frac{1}{2\pi} \Delta_R \arg \frac{\zeta'''}{\zeta''}(s) = Z_3 - Z_2 \geq (N - 1 + 2\kappa) - (N + 2) = 2\kappa - 3\,.$$

We will show that $\Delta_R \arg \frac{\zeta'''}{\zeta''}(s) = -2\pi$ in one tour of the rectangle, implying $\kappa \leq 1$. M. Özkan computed that $\zeta'''(s)$ has zeros at $-2.1101.. \pm i \cdot 2.5842..$, so $\kappa = 1$. This computation was based upon evaluating $\oint \frac{\zeta^{(\mathrm{iv})}}{\zeta'''}(s)\, ds$ around various rectangles. The Euler-Maclaurin formula (27) was used with $N = 10$ and $m = 6$ for the integrand. The line integrations were then done by employing MATHEMATICA.

On the three sides of R in the left half-plane the situation is the same as for $\Re \frac{\zeta''}{\zeta'}$, and there is no need to repeat the arguments in the proof of Theorem 2. On the imaginary axis we have, by (34),

$$\begin{aligned}\Re \frac{\zeta'''}{\zeta''}(it) = {} & \frac{\zeta'''}{\zeta''}(0) - 3 + \frac{3}{1+t^2} + \sum_{n=1}^{\infty} \Big(\frac{b_n}{b_n^2 + t^2} - \frac{1}{b_n} \Big) + \sum_{\rho_2} \frac{1}{\rho_2} \\ & + \sum_{\gamma_2 > 0} \Big(\frac{-\beta_2}{\beta_2^2 + (\gamma_2 - t)^2} + \frac{-\beta_2}{\beta_2^2 + (\gamma_2 + t)^2} \Big) \\ & + \frac{2\Re b_0}{|b_0|^2} - \Re b_0 \Big(\frac{1}{(\Re b_0)^2 + (t - \Im b_0)^2} + \frac{1}{(\Re b_0)^2 + (t + \Im b_0)^2} \Big)\end{aligned} \tag{36}$$

$$\Im\frac{\zeta'''}{\zeta''}(it) = \frac{3t}{1+t^2} - \sum_{n=1}^{\infty}\frac{t}{b_n^2+t^2} + \sum_{\rho_2}\frac{\gamma_2 - t}{\beta_2^2+(\gamma_2-t)^2} \tag{37}$$
$$-\frac{t-\Im b_0}{(\Re b_0)^2+(t-\Im b_0)^2} - \frac{t+\Im b_0}{(\Re b_0)^2+(t+\Im b_0)^2}.$$

In (36), bounding the sum over γ_2 trivially by 0, and using (35), the value of b_0 and $\frac{\zeta'''}{\zeta''}(0) = 2.993..$ ([1]), we have

$$\Re\frac{\zeta'''}{\zeta''}(it) < 0.0595 + \frac{3}{1+t^2} + \sum_{n=1}^{\infty}\Big(\frac{b_n}{b_n^2+t^2} - \frac{1}{b_n}\Big) \tag{38}$$
$$-\Re b_0\Big(\frac{1}{(\Re b_0)^2+(t-\Im b_0)^2} + \frac{1}{(\Re b_0)^2+(t+\Im b_0)^2}\Big).$$

We see that for $t \geq \Im b_0$ the right-hand side is a strictly decreasing function of t. So, if we find a value $t_0 > \Im b_0$ making the right-hand side of (38) negative, then we know that for $t \geq t_0$, $\Re\frac{\zeta'''}{\zeta''}(it) < 0$. To bound the sums over b_n's, using (29) and (30) we take $\hat{b}_n$ and $\tilde{b}_n$ for $1 \leq n \leq 4$ satisfying $\hat{b}_n < b_n < \tilde{b}_n$ and define

$$a(t) = \sum_{n=1}^{4}\frac{-t^2}{\tilde{b}_n(\tilde{b}_n^2+t^2)} + \sum_{n=1}^{6}\frac{t^2}{2n((2n)^2+t^2)}$$
$$b(t) = \sum_{n=1}^{4}\frac{-t^2}{\hat{b}_n(\hat{b}_n^2+t^2)} + \sum_{n=1}^{5}\frac{t^2}{2n((2n)^2+t^2)}$$
$$c(t) = \sum_{n=1}^{4}\frac{-t}{\tilde{b}_n^2+t^2} + \sum_{n=1}^{6}\frac{t}{(2n)^2+t^2}$$
$$d(t) = \sum_{n=1}^{4}\frac{-t}{\hat{b}_n^2+t^2} + \sum_{n=1}^{5}\frac{t}{(2n)^2+t^2}.$$

Then, similar to (25) and (26) we have

$$b(t) < \sum_{n=1}^{\infty}\frac{-t^2}{b_n(b_n^2+t^2)} - \frac{1}{2}\Big(\psi(1) - \Re\psi(1+\frac{it}{2})\Big) < a(t), \tag{39}$$

$$d(t) < \sum_{n=1}^{\infty}\frac{-t}{b_n^2+t^2} + \frac{1}{2}\Im\psi(1+\frac{it}{2}) < c(t). \tag{40}$$

Now by sheer calculation we find that $t_0 = 5.2$ is admissible, and we only need to consider $0 \le t \le 5.2$ to determine $\Delta_R \arg \frac{\zeta'''}{\zeta''}(s)$. The quadrants where $\frac{\zeta'''}{\zeta''}(it)$ lies for various t's can be found from the foregoing expressions (e.g. $\frac{\zeta'''}{\zeta''}(i\Im b_0)$ is in the first quadrant). Also note that as $t \to 0^+$, $\frac{\zeta'''}{\zeta''}(it) \to \frac{\zeta'''}{\zeta''}(0)$ from the first quadrant.

When using (36) to obtain a lower bound for $\Re\frac{\zeta'''}{\zeta''}(it)$ observe that

$$\frac{-\beta_2}{\beta_2^2 + (\gamma_2 - t)^2} \ge \frac{-2\beta_2}{\beta_2^2 + \gamma_2^2} \qquad (0 \le t \le \min|\gamma_2|(1 - \frac{1}{\sqrt{2}}))\,,$$

and since ([7]) the least $|\gamma_2|$ is 23.27.., for $t \le 6.8$ the sum over γ_2 in (36) is $> -\frac{3}{2}\sum_{\rho_2} \frac{1}{\rho_2} > -0.18$. The sum over ρ_2 in (37) is equal to

$$2t \sum_{\gamma_2 > 0} \frac{\gamma_2^2 - \beta_2^2 - t^2}{[\beta_2^2 + (\gamma_2 - t)^2][\beta_2^2 + (\gamma_2 + t)^2]}, \tag{41}$$

all of the terms in this sum being positive for

$$0 < t < \sqrt{(\min \gamma_2)^2 - (\max \beta_2)^2},$$

i.e. certainly for $0 < t < 22.7$. A trivial lower bound for (41) is 0, and one can do better by including the terms corresponding to known values of ρ_2. When using (37) to obtain an upper bound for $\Im\frac{\zeta'''}{\zeta''}(it)$, the sum over ρ_2 presents some difficulty. The quantity in (41) is less than

$$2t \sum_{\gamma_2 > 0} \frac{\gamma_2^2 - t^2}{(\gamma_2 - t)^2(\gamma_2 + t)^2} = 2t \sum_{\gamma_2 > 0} \frac{1}{\gamma_2^2 - t^2} < 2.12t \sum_{\gamma_2 > 0} \frac{1}{\gamma_2^2},$$

where the last inequality holds for $0 \le t \le 5.2$. However we do not know the value of the last sum. If we cheat and assume RH to the effect that $\beta_2 \ge \frac{1}{2}$, then for $0 < t \le 5.2$ we have

$$\sum_{\gamma_2>0}\frac{2t(\gamma_2^2-\beta_2^2-t^2)}{[\beta_2^2+(\gamma_2-t)^2][\beta_2^2+(\gamma_2+t)^2]}<2t\sum_{\gamma_2>0}\frac{1}{\beta_2^2+(\gamma_2-t)^2}$$
$$\leq 2t\sum_{\gamma_2>0}\frac{2\beta_2}{\beta_2^2+\gamma_2^2}\frac{\beta_2^2+\gamma_2^2}{\beta_2^2+(\gamma_2-t)^2}$$
$$<2t\frac{(\max\beta_2)^2+(\min\gamma_2)^2}{(\min\gamma_2-t)^2}\sum_{\rho_2}\frac{1}{\rho_2},$$

and we may use (35) to get an upper bound. We do not assume RH, and instead appeal to the graph of $\frac{\zeta'''}{\zeta''}(it)$. We see that as t moves up on the imaginary axis on our contour $\Delta\arg\frac{\zeta'''}{\zeta''}$ is roughly -2π. This completes the proof. □

Next consider the values of $\Re\frac{\zeta'''}{\zeta''}(s)$, in the region $0\leq\sigma<\frac{1}{2}$, $|t|>T$. If we assume RH, then by Theorem 1, $\Re\dfrac{1}{s-\rho_2}<0$ for all ρ_2. So from (34), when $T\geq|\Re b_0|+|\Im b_0|$, (bounding the sums over b_n as in (39))

$$\Re\frac{\zeta'''}{\zeta''}(s)<\frac{\zeta'''}{\zeta''}(0)-3+\frac{3}{1+T^2}+\sum_{\rho_2}\frac{1}{\rho_2}+g(T)+\frac{2\Re b_0}{|b_0|^2}+$$
$$+(\frac{1}{2}-\Re b_0)\Big(\frac{1}{(T-\Im b_0)^2}+\frac{1}{(T+\Im b_0)^2}\Big)+$$
$$+\frac{2T^2}{1+4T^2}\Big(\psi(1)-\Re\psi(\frac{5}{4}+\frac{iT}{2})+\frac{1}{4T}\Im\psi(\frac{5}{4}+\frac{iT}{2})\Big),$$

where

$$g(t)=\sum_{n=1}^{4}\frac{-t^2}{\tilde{b}_n[(\frac{1}{2}+\tilde{b}_n)^2+t^2]}+\sum_{n=1}^{4}\frac{t^2}{(2n+4)[(2n+\frac{9}{2})^2+t^2]}$$
$$+\frac{t^2}{8[(\frac{5}{4})^2+t^2]}+\frac{t^2}{16[(\frac{9}{4})^2+t^2]}.$$

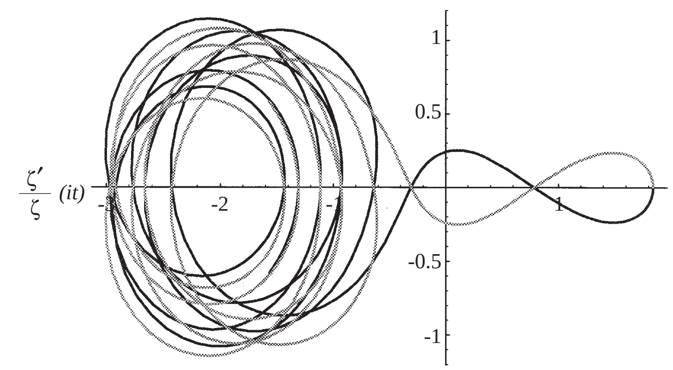
Thus it is seen that $\Re\frac{\zeta'''(s)}{\zeta''(s)}<0$ in $0\leq\sigma<\frac{1}{2},|t|\geq 10$. The region with $|t|<10$ may be swept by integrating $\frac{\zeta^{(\mathrm{iv})}}{\zeta'''}(s)$ around the rectangle, employing the Euler-Maclaurin formulae (27) for the integrand. This computation was carried out by H.E. Yıldırım and no zeros of $\zeta'''(s)$ were found. Hence we have

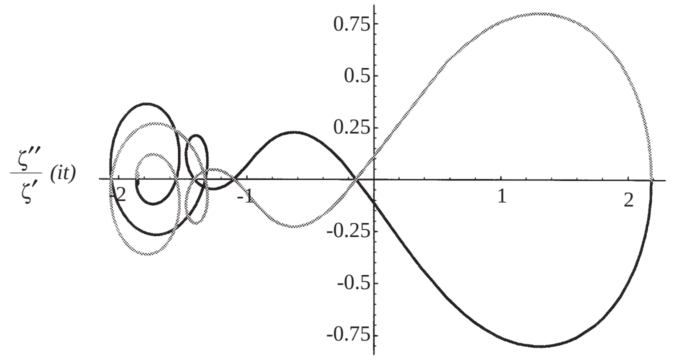
Theorem 4. *The Riemann Hypothesis implies that* $\zeta'''(s)$ *has no zeros in the strip* $0 \le \sigma < \frac{1}{2}$.

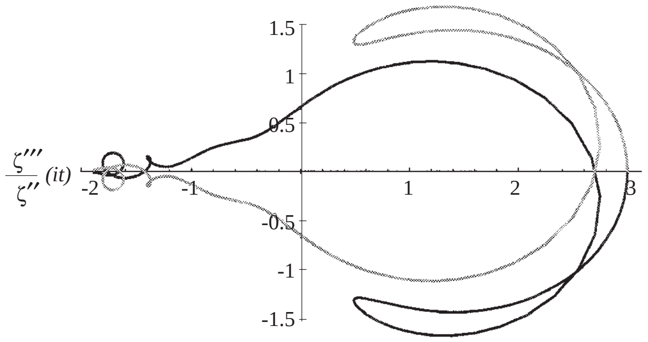
Armed with the methods and results of this paper one may proceed to the investigation of $\zeta^{(\mathrm{iv})}(s)$.

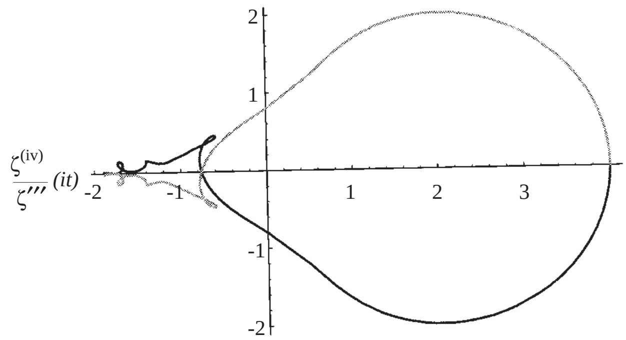
Graphs

The graphs of $\frac{\zeta^{(k+1)}}{\zeta^{(k)}}(it)$, $(k = 0, 1, 2, 3)$ are plotted below for $|t| \le 40$. The darker parts are for $-40 \le t \le 0$.









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