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Some Results on Derivation Groups

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Abstract

In this paper we describe a share package XMOD [1] of functions for computing with finite, permutation crossed modules, their morphisms and derivations; cat^1-groups, their morphisms and their sections, written using the GAP [5] group theory programming language. We also give some mathematical results for derivations. These results are suggested by the output produced by the XMOD [1] package.

Key Words: Crossed modules, derivation, whitehead multiplication.

1. Introduction

A starting point for this paper was to consider the possibility of implementing functions for doing calculations with crossed modules, derivations, actor crossed modules, cat1-groups, sections, induced crossed modules and induced cat1-groups in GAP [5].

We should first explain the importance of crossed modules. The general points are:

- crossed modules may be thought of as 2-dimensional groups;
- a number of phenomena in group theory are better seen from a crossed module point of view;
- crossed modules occur geometrically as $\pi_2(X, A) \to \pi_1 A$ when A is a subspace of X or as $\pi_1 F \to \pi_1 E$ where $F \to E \to B$ is a fibration;
- crossed modules are usefully related to forms of double groupoids.

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Particular constructions, such as induced crossed modules, are important for the applications of the 2-dimensional Van-Kampen Theorem of Brown and Higgins [2], and so for the computation of homotopy 2-types.

For all these reasons, the facilitation of the computations with crossed modules should be advantageous. It should help to solve specific problems, and it should make it easier to construct examples and see relations with better known theories.

The powerful computer algebra system GAP provides a high level programming language with several advantages for the coding of new mathematical structures. The GAP system has been developed over the last 15 years at RWTH in Aachen. Some of its most exciting features are:

- it has a highly developed, easy to understand programming language incorporated;
- it is especially powerful for group theory;
- it is portable to a wide variety of operating systems on many hardware platforms.
- it is public domain and it has a lively forum, with open discussion. These make it increasingly used by the mathematical community.

On the other hand, GAP has some disadvantages, too:

- the built in programming language is an interpreted language, which makes GAP programs relatively slow compared to compiled languages such as C or Pascal. GAP source can not be compiled. This will change in version 4 to be released during 1997;
- the demands on system resources are quite high for the serious calculations.

However, the advantages outweigh the disadvantages, and so GAP was chosen.

The term crossed module was introduced by J. H. C. Whitehead in [6]. Most references of crossed modules state the axioms of a crossed module using left actions, but we shall use right actions since this is the convention used by most computational group packages. In section 2 we recall the basic properties of crossed modules and derivation groups. We also give some mathematical results in section 3.

2. Crossed Modules and derivation groups

We recall some standard results on crossed modules, groups of derivations in this paper. Let S and R be groups. If R acts on S then we denote the action of r on s by s^r. If

$S = R$ then R acts on itself by conjugation

$$r_1^{r_2} = r_2^{-1}\, r_1\, r_2 \quad for\ r_1, r_2 \in R$$

Let S and R be groups together with a homomorphism $\partial : S \to R$. Suppose that R acts on S, so that there is a homomorphism $R \to \mathrm{Aut}(S)$. If the following conditions are satisfied then the $\mathcal{X} = (\partial : S \to R)$ is a crossed module.

$$\begin{array}{lcl} \mathrm{CM1}: \quad \partial(s^r) & = & r^{-1}\partial s\, r \quad for\ all\ s \in S, r \in R \\ \mathrm{CM2}: \quad s^{\partial(t)} & = & t^{-1} s\, t \quad for\ all\ s, t \in S \end{array}$$

The homomorphism ∂ is called the boundary map of the crossed module.

Standard examples of crossed modules are :

i) $\mathcal{X} = (\iota : S \to R)$ where ι is the inclusion of a normal subgroup S of a group R, with the action of R on S given by conjugation. If x, y are elements of some group then by the conjugate of x by y we mean element $x^y = y^{-1}xy$;

ii) $\mathcal{X} = (0 : S \to R,\ s \mapsto 1_R)$ where the boundary is the zero morphism and S is a R-module in the usual sense;

iii) $\mathcal{X} = (\chi : M \to \mathrm{Aut}(M))$ with range the automorphism group of M, where $\chi(m)$ is the inner automorphism determined by $m \in M$, together with the standard action of $\mathrm{Aut}(M)$ on M.

Let $\mathcal{X} = (\partial : S \to R)$ and $\mathcal{X}' = (\partial' : S' \to R')$ be crossed modules. A crossed module morphism

$$\langle \sigma, \rho \rangle : \mathcal{X} \to \mathcal{X}'$$

is a pair of homomorphisms $\sigma : S \to S', \rho : R \to R'$

$$\begin{tikzcd} S \arrow[r, "\sigma"] \arrow[d, "\partial"'] & S' \arrow[d, "\partial'"] \\ R \arrow[r, "\rho"'] & R' \end{tikzcd}$$

such that,

i) $\partial'\sigma(s) = \rho\partial(s)$ for all s $\in$ S

ii) $\sigma(s^r) = \sigma(s)^{\rho(r)}$ for all s $\in$ S, r $\in$ R.

Given a crossed module $\mathcal{X} = (\partial : S \to R)$ denote by $\mathrm{Der}(\mathcal{X})$ the set of all derivations from R to S, i. e. all maps $\chi : R \to S$ such that for all $q, r \in R$,

$$\chi(qr) = (\chi q)^r \chi(r)$$

It follows that

$$\chi(1) = 1,\ (\chi q^{-1})^q = (\chi q)^{-1}. \tag{1}$$

Each derivation χ defines endomorphisms $\rho(= \rho_\chi)$ and $\sigma(= \sigma_\chi)$ of R, S

$$\begin{tikzcd}
S \arrow[r, "\sigma"] \arrow[d, "\partial"] & S \arrow[d, "\partial"] \\
R \arrow[r, "\rho"'] \arrow[u, bend left=60, "\chi"] & R \arrow[u, bend right=60, "\chi"']
\end{tikzcd}$$

where

$$\rho_\chi(r) = r(\partial\chi r),\ \sigma_\chi(s) = s(\chi\partial s) \tag{2}$$

$$\rho\partial(s) = \partial\sigma(s) = (\partial s)(\partial\chi\partial s), \tag{3}$$

$$\sigma\chi(r) = \chi\rho(r) = (\chi r)(\chi\partial\chi r), \tag{4}$$

$$\sigma(s^r) = (\sigma s)^{\rho(r)}. \tag{5}$$

The Whitehead multiplication in $\mathrm{Der}(\mathcal{X})$ is defined by the following

$$(\chi_1 \circ \chi_2)(x) = (\chi_1 x)(\chi_2 x)(\chi_1\partial\chi_2 x). \tag{6}$$

This multiplication is associative and the composite of three derivations is given by:

$$\begin{aligned}(\chi_1 \circ \chi_2 \circ \chi_3)(x) = {} & (\chi_1 x)(\chi_2 x)(\chi_1\partial\chi_2 x)(\chi_3 x)(\chi_1\partial\chi_3 x) \\ & (\chi_2\partial\chi_3 x)(\chi_1\partial\chi_2\partial\chi_3 x)\end{aligned} \tag{7}$$

Alternatively, the composite derivation $\chi = \chi_1 \circ \chi_2$ may be defined using σ or ρ by:

$$\chi(x) = \chi_2(x)\chi_1\rho_2(x), \tag{8}$$

$$\chi(x) = \chi_1(x)\sigma_1\chi_2(x). \tag{9}$$

This multiplication turns $\mathrm{Der}(\mathcal{X})$ into a monoid, with identity element the derivation which maps each element of R into the identity of S. There are homomorphisms of monoids

$$\mathrm{Der}(\mathcal{X}) \to \mathrm{End}(S), \chi \mapsto \sigma_\chi, \tag{10}$$

$$\mathrm{Der}(\mathcal{X}) \to \mathrm{End}(R), \chi \mapsto \rho_\chi. \tag{11}$$

The Whitehead group $\mathcal{W}(\mathcal{X})$ is defined to be group of units of $\mathrm{Der}(\mathcal{X})$. The elements of $\mathcal{W}(\mathcal{X})$ will be called regular derivations. Now we can give some examples of Whitehead groups [3]

i) If S is a R - module, then the trivial homomorphism $S \to R$ is a crossed module and $\mathrm{Der}(\mathcal{X}) = \mathcal{W}(\mathcal{X})$ is the usual abelian group of derivations.

ii) Together with the conjugation action of a group R on itself, the identity map $\mathcal{X} = (\mathrm{id} : R \to R)$ is a crossed module. An automorphism α of R determines its displacement derivation $\delta_\alpha \in \mathcal{W}(\mathcal{X})$ given by $\delta_\alpha(r) = \alpha(r)r^{-1}$, and the correspondence $\alpha \mapsto \delta_\alpha$ is an isomorphism $\delta : \mathrm{Aut}R \to \mathcal{W}(\mathcal{X})$.

iii) Generalising (ii) we have the inclusion $S \to R$ of a normal subgroup S of a group R, with R acting by conjugation. Then $\mathcal{W}(\mathcal{X})$ is isomorphic to the subgroup of $\mathrm{Aut}R$ consisting of all those α whose displacement derivations take values in S,

$$\mathcal{W}(\mathcal{X}) \cong \{\alpha \in \mathrm{Aut}R \mid \ \textit{for all } r \in R,\ \alpha(r)r^{-1} \in S\}.$$

In particular, if S is a characteristic subgroup of R, then $\mathcal{W}(\mathcal{X})$ is the kernel of the canonical map from $\mathrm{Aut}R$ to $\mathrm{Aut}(R/S)$.

iv) A crossed module $\mathcal{X} = (\partial : S \to R)$ with surjective boundary map amounts to a central extension of $\ker\partial$ by R; so let E be a group and K a central subgroup of E. Let $\mathrm{Aut}_K E$ be the subgroup of $\mathrm{Aut}E$ consisting of those automorphism of E that act trivially on K. The natural map $\mathcal{N} = (\nu : E \to E/K)$ is a crossed module and $\mathcal{W}(\mathcal{N})$ is isomorphic to $\mathrm{Aut}_K E$.

Proposition 2.1 *[6, 4]* The following conditions on $\chi \in \mathrm{Der}(\mathcal{X})$ are equivalent:

(i) $\chi \in \mathcal{W}(\mathcal{X})$,
(ii) $\rho : R \to R$ is an automorphism,
(iii) $\sigma : S \to S$ is an automorphism.

Consider the group $\mathrm{Aut}(\mathcal{X})$ of automorphisms of the crossed module $\mathcal{X} = (\partial : S \to R)$. Specifically, these consist of pairs $\langle \alpha, \phi \rangle$ with $\alpha \in \mathrm{Aut}(S), \phi \in \mathrm{Aut}(R)$ which satisfy

$$\phi\partial = \partial\alpha$$
$$\alpha(s^r) = (\alpha s)^{\phi(r)}$$

An action of $\mathrm{Aut}(\mathcal{X})$ on the group $\mathcal{W}(\mathcal{X})$ is defined by

$$\chi^{\langle \alpha, \phi \rangle} = \alpha^{-1}\chi\phi. \tag{12}$$

3. Some results on derivation groups

Let $\mathcal{X} = \mathcal{C}_{m,n} = (\iota : C_m \to C_n)$, where ι is the inclusion of a subgroup, C_m, C_n finite permutational groups, and let $d = n/m$.

Theorem 3.1 *Der*$(\mathcal{X})$ comprises derivations χ_k, $(1 \le k \le m)$ where $\chi_k(x) = x^{(k-1)d}$.

Proof: Since C_n is abelian and the action is trivial, every derivation is a homomorphism so χ is determined by the image χx of a generator of x of C_n. There are m possible images for x, namely the m powers of the generator x^d of C_m. □

Corollary 3.2 When $\mathcal{X} = (\iota : C_m \to C_n)$ formulae for $\sigma_k = \sigma_{\chi_k}$ and $\rho_k = \rho_{\chi_k}$ are given by:

$$\sigma_k(x^d) = (x^d)^{1+(k-1)d}$$

and

$$\rho_k x = x^{1+(k-1)d}.$$

When $m = n$ and $d = 1$, $\sigma_k x = \rho_k x = x^k$.

Theorem 3.3 When $\mathcal{X} = (\iota : C_m \to C_n)$ then the composite of two derivations is given by

$$(\chi_k \circ \chi_j)(x) = \chi_{kjd-(j+k-1)(d-1)}(x)$$

Proof:

$$\begin{aligned}(\chi_k \circ \chi_j)(x) &= \chi_k(x)\chi_j(x)\chi_k\iota\chi_j(x) \\ &= x^{(k-1)d}x^{(j-1)d}(x^{(j-1)d})^{(k-1)d} \\ &= x^{(p-1)d}\end{aligned}$$

where

$$\begin{aligned}p &= [k-1+j-1+(k-1)(j_1)d]d+1 \\ &= kjd-(j+k-1)(d-1)\end{aligned}$$

□

Theorem 3.4 When $\gcd(d,m) = 1$, there is an isomorphism between the monoid $\mathrm{Der}(\mathcal{X} = (\iota : C_m \to C_n))$ and the multiplicative monoid $(\mathbb{Z}_m, \times)$.

Proof: Note that

$$(\chi_k \circ \chi_j)(x) = \chi_{kjd-(j+k-1)(d-1)}(x).$$

If we define

$$\begin{array}{cccc}\theta : & \mathcal{W}(\mathcal{X}) & \to & \mathbb{Z}_m \\ & \chi_k & \mapsto & (k-1)d+1.\end{array}$$

then

$$\begin{aligned}\theta(\chi_k \circ \chi_j) &= \theta(\chi_{kjd-(j+k-1)(d-1)}) \\ &= [kjd-(j+k-1)(d-1)-1]d+1 \\ &= [kjd-(jd-j+kd-k-d+1)-1]d+1 \\ &= (kjd-jd+j-kd+k+d-1-1)d+1 \\ &= kjd^2-jd^2+jd-kd^2+kd+d^2-d-d+1 \\ &= kd(jd-d+1)-d(jd-d+1)+jd-d+1 \\ &= (kd-d+1)(jd-d+1) \\ &= ((k-1)d+1)((j-1)d+1) \\ &= \theta(\chi_k)\theta(\chi_j)\end{aligned}$$

and $\theta(\chi_1) = 1$. Hence θ is a homomorphism. Since $\gcd(d, m) = 1$, θ has inverse given by $\theta^{-1}(i) = \chi_k$ where $k \equiv 1 + (i-1)/d$ (modm). $\square$

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