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Multipliers between Orlicz Sequence Spaces *

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Abstract

Let M, N be Orlicz functions, and let $D(\ell_M, \ell_N)$ be the space of all diagonal operators (that is multipliers) acting between the Orlicz sequence spaces ℓ_M and ℓ_N. We prove that the space of multipliers $D(\ell_M, \ell_N)$ coincides with (and is isomorphic to) the Orlicz sequence space $\ell_{M_N^*}$, where M_N^* is the Orlicz function defined by $M_N^*(\lambda) = \sup\{N(\lambda x) - M(x),\ x \in (0,1)\}$.

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Let $M(t)$, $t \geq 0$, be an Orlicz function, that is a non-decreasing convex function such that $M(0) = 0$ and $M(t) \to \infty$ as $t \to \infty$. Orlicz sequence space ℓ_M defined by the function $M(t)$ is the linear space of all sequences of scalars $x = (x_i)_1^\infty$ such that $\sum_i M(x_i) < \infty$. Equipped with the norm

$$\|x\|_M = \inf\{\rho : \sum_i M(|x_i|/\rho) \leq 1\}$$

it is a Banach space.

An Orlicz function $M(t)$ is said to satisfy the Δ_2-condition near 0 if $M(2t) \leq CM(t),\ t \in (0,1)$ for some constant $C > 0$. The following facts are known:

Proposition 1 *Let M be an Orlicz function. Then the subspace*

$$h_M = \{x = (x_i) : \sum M(|x_i|/\rho) < \infty \quad \forall \rho > 0\}$$

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is a closed subspace of ℓ_M*, and the vectors* $(e_n)_1^\infty$ *(where* $e_n = (e_{ni})$*,* $e_{ni} = 0$ *if* $i \neq n$*, and* $e_{nn} = 1$*) form a basis in it.*

Moreover, $h_m = \ell_M$ *if and only if* M *satisfies the* Δ_2*-condition.*

We refer to [1] for a proof of this proposition and the basic theory of Orlicz sequence spaces.

Let $M(t)$ and $N(t)$ be two Orlicz functions. A sequence of scalars $\lambda = (\lambda_i)$ is called a *multiplier* between the Orlicz spaces ℓ_M and ℓ_N if for each $x = (x_i) \in \ell_M$ we have $\lambda x := (\lambda_i x_i) \in \ell_N$. It is easy to see by the Closed Graph Theorem that each multiplier λ defines a continuous *diagonal* operator

$$T_\lambda : \ \ell_M \to \ell_N.$$

Therefore we identify multipliers with diagonal operators and denote by $D(\ell_M, \ell_N)$ the space of all multipliers between ℓ_M and ℓ_N. Regarded with the usual operator norm it is a Banach space.

Consider the function

$$M_N^*(s) = \max(0, \sup_{t\in[0,1]} \{N(st) - M(t)\}), \qquad s \geq 0. \tag{1}$$

Evidently it is an Orlicz function, and by its definition we have

$$N(ts) \leq M(t) + M_N^*(s), \tag{2}$$

which generalizes the classical Young inequality.

Two Orlicz functions $M(t)$ and $\bar{M}(t)$ are *equivalent*, if

$$\exists c > 0,\, t_0 > 0 \ : \quad c^{-1}M(c^{-1}t) \leq \bar{M}(t) \leq cM(ct), \quad \forall\, t \in [0, t_0].$$

Equivalent Orlicz functions generate one and the same Orlicz sequence space and define equivalent norms on it. It is easy to see that if one replaces the functions M and N with equivalent Orlicz functions $\bar{M}$ and $\bar{N}$, then the functions M_N^* and $\bar{M}_{\bar{N}}^*$ will be equivalent.

Proposition 2 *If* $\lambda \in \ell_{M_N^*}$ *then it is a multiplier from* ℓ_M *into* ℓ_N*. Moreover, the following generalization of the Hölder inequality holds:*

$$\|\lambda x\|_N \leq 2\|\lambda\|_{M_N^*} \|x\|_M \qquad \forall \lambda \in \ell_{M_N^*},\, \forall x \in \ell_M. \tag{3}$$

Proof. First, observe that if S is an Orlicz function then

$$\|(y_i)\|_S > 1 \;\Rightarrow\; \|(y_i)\|_S \le \sum_i S(|y_i|). \tag{4}$$

Indeed, since S is a convex function and $S(0) = 0$ we have for every $\beta > 1$ that $S(\beta^{-1}t) = S(\beta^{-1}t + (1-\beta^{-1})\cdot 0) \le \beta^{-1}S(t)$. Therefore, from the definition of the norm $\|(y_i)\|_S$, it follows that for every β such that $1 < \beta < \|(y_i)\|_S$ we have

$$1 < \sum_i S(|y_i|/\beta) \le \beta^{-1} \sum_i S(|y_i|).$$

So, letting $\beta \to \|(y_i)\|_S$ we obtain $\|(y_i)\|_S \le \sum_i S(|y_i|)$.

Fix $\lambda = (\lambda_i) \in \ell_{M_N^*}$ and $x = (x_i) \in \ell_M$, and let

$$\rho > \|\lambda\|_{M_N^*}, \quad r > \|x\|_M.$$

Consider the sequences $\tilde{\lambda} = \lambda/\rho$ and $\tilde{x} = x/r$.

Then

$$\sum_i N(|\tilde{\lambda}_i \tilde{x}_i|) \le \sum_i M(|\tilde{x}_i|) + \sum_i M_N^*(|\tilde{\lambda}_i|) \le 2,$$

and from (4) it follows that $\|\tilde{\lambda}\tilde{x}\|_N \le 2$, thus $\|\lambda x\|_N \le 2\rho r$. Letting $\rho \uparrow \|\lambda\|_{M_N^*}$ and $r \uparrow \|x\|_M$ we obtain the claim.

Theorem 3 *For every pair of Orlicz functions M, N the sequence spaces $D(\ell_M, \ell_N)$ and $\ell_{M_N^*}$ coincide as sets, and moreover, they are isomorphic as Banach spaces.*

Proof. First, observe that if S is an Orlicz function then

$$\|(y_i)\|_S < 1 \;\Rightarrow\; \sum_i S(|y_i|) \le \|(y_i)\|_S. \tag{5}$$

Indeed, since S is a convex function and $S(0) = 0$ we have for $\alpha \in (0,1)$ that $S(\alpha t) \le \alpha S(t)$. Therefore for every α such that $\|(y_i)\|_S < \alpha < 1$

$$\|(y_i)\|_S < \alpha \;\Rightarrow\; \sum_i S(|y_i|/\alpha) \le 1 \;\Rightarrow\; \sum_i S(|y_i|) \le \alpha \sum S(|y_i|/\alpha) \le \alpha,$$

so letting $\alpha \to \|(y_i)\|_S$ we obtain $\sum N(|y_i|) \le \|(y_i)\|_S$.

Consider, in the space of multipliers $D(\ell_M, \ell_N)$, the operator norm

$$\|\lambda\|_0 = \sup\{\|\lambda x\|_N : \|x\|_M = 1\}.$$

From Proposition 2 it follows immediately that

$$D(\ell_M, \ell_N) \supset \ell_{M_N^*}$$

and

$$\|\mu\|_0 \le 2\|\mu\|_{M_N^*} \qquad \forall \mu \in \ell_{M_N^*}.$$

Further we show that

$$D(\ell_M, \ell_N) \subset \ell_{M_N^*}$$

and

$$\|\mu\|_{M_N^*} \le 2\|\mu\|_0 \qquad \forall \mu \in D(\ell_M, \ell_N).$$

We may assume without loss of generality that $M(1) = 1$ and $N(1) = 1$. Then we have

$$\forall i \quad \|e_i\|_M = 1, \qquad \|e_i\|_N = 1.$$

Fix a multiplier $\lambda = (\lambda_i) \in D(\ell_M, \ell_N)$ such that $\|\lambda\|_0 = 1/2$. Then $|\lambda_i| = \|\lambda e_i\|_N \le 1/2\,\|e_i\|_M = 1/2$.

Since M and N are Orlicz functions they are continuous. Thus for every $i = 1, 2, \dots$ there exists an $x_i \in [0, 1]$ such that

$$M_N^*(|\lambda_i|) = N(|\lambda_i| x_i) - M(x_i),$$

that is

$$N(|\lambda_i| x_i) = M(x_i) + M_N^*(|\lambda_i|). \tag{6}$$

Consider the sequence $(x_i)_1^\infty$. Since by our assumption $\|\lambda\|_0 = 1/2$, we have by (5)

$$\forall i \quad N(|\lambda_i| x_i) \le \|\lambda_i x_i e_i\|_N \le 1/2\,\|x_i e_i\|_M \le 1/2,$$

therefore

$$M(x_i) = N(|\lambda_i| x_i) - M_N^*(x_i) < 1/2, \quad i = 1, 2, \dots. \tag{7}$$

We shall prove by induction that $\sum_1^n M(x_i) \le 1/2$. It was shown that the statement is true for $n = 1$.

Consider the sequences $\xi^{(n)} = \sum_1^n x_i e_i,\ n = 1, 2, \dots$. Assume that the claim is true for some n. Then

$$\sum_1^{n+1} M(x_i) = \sum_1^{n} M(x_i) + M(x_{n+1}) \le 1/2 + 1/2 \le 1,$$

so $\|\xi^{n+1}\|_M \le 1$. Therefore we obtain by (6) and (5)

$$\sum_1^{n+1} M(x_i) \le \sum_1^{n+1} N(|\lambda_i| x_i) \le \|\lambda \xi^{n+1}\|_N \le 1/2,$$

which proves the claim.

Since $\sum_1^n M(x_i) < 1/2$ for every n we have $\sum_1^\infty M(x_i) \le 1/2$, thus $x \in \ell_M$ and $\|x\|_M < 1$. Now from (6) and (5) it follows

$$\sum_1^{\infty} M_N^*(\lambda_i) \le \sum_1^{\infty} N(|\lambda_i| x_i) \le \|\lambda x\|_N \le 1/2\, \|x\|_M \le 1/2,$$

hence $\lambda \in \ell_{M_N^*}$ and $\|\lambda\|_{M_N^*} \le 1$.

Suppose $\mu \in D(\ell_M, \ell_N)$ is an arbitrary multiplier. Consider the sequence $\lambda = \mu/\rho$, where $\rho = 2\|\mu\|_0$. Then we have $\lambda \in \ell_{M_N^*}$ and $\|\lambda\|_{M_N^*} = \|\mu/\rho\|_{M_N^*} \le 1$, hence $\mu \in \ell_{M_N^*}$ and

$$\|\mu\|_{M_N^*} \le 2\|\mu\|_0.$$

The theorem is proved.

Remark 1. An Orlicz function S is called degenerate, if $S(t) = 0$ for some $t > 0$; then the corresponding Orlicz sequence space ℓ_S coincides with ℓ_∞. In view of the theorem $D(\ell_M, \ell_N) = \ell_\infty$ if and only if the Orlicz function M_N^* is degenerate.

Example. It is well known that for $p, q \ge 1$

$$D(\ell_p, \ell_q) = \begin{cases} \ell_r, \quad 1/r = 1/q - 1/p, & \text{if } p > q; \\ \ell_\infty, & \text{if } p \le q. \end{cases}$$

Let us see how this result follows from Theorem 2. Consider $M(t) = t^p/p$ and $N(t) = t^q/q$. If $p > q$ then it is easy to see that for each fixed $s \in (0,1)$ the expression $N(st) - M(t) = (st)^q/q - t^p/p$ attains its maximum for $t \in [0,1]$ at $t = s^{q/(p-q)}$. Thus for $s \in [0,1]$

$$M_N^*(s) = (1/q - 1/p)s^{pq/(p-q)} = s^r/r$$

with $1/r = 1/q - 1/p$, hence $D(\ell_q, \ell_p) = \ell_r$. In the case $p \le q$, if $s^q \le q/p$, then

$$N(st) - M(t) = (st)^q/q - t^p/p \le 0, \quad t \in [0,1].$$

Therefore $M_N^*(s) = 0$ for $s \le (q/p)^{1/q}$, that is M_N^* is a degenerate Orlicz function, hence $D(\ell_p, \ell_q) = \ell_\infty$.

Remark 2. Let $D_c(\ell_M, \ell_N)$ be the space of all compact multipliers between the spaces ℓ_M and ℓ_N. It is easy to see by Proposition 1 that each multiplier from the subspace $h_{M_N^*}$ is compact (as limit of finitely-supported multipliers), thus

$$h_{M_N^*} \subset D_c(\ell_M, \ell_N).$$

In particular, if the function M_N^* satisfies the Δ_2-condition near zero, then each multiplier between the spaces ℓ_M and ℓ_N is compact, that is

$$D(\ell_M, \ell_N) = D_c(\ell_M, \ell_N).$$

Up to our knowledge the following question is open:

Question. Is it true that every compact multiplier between the spaces ℓ_M and ℓ_N is a limit of finitely-supported multipliers ?

Obviously, a positive answer to that question would imply that

$$D_c(\ell_M, \ell_N) = h_{M_N^*},$$

so we would have

$$D(\ell_M, \ell_N) = D_c(\ell_M, \ell_N).$$

if and only if the function M_N^* satisfies the Δ_2-condition.

References

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