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Conjugacy Structure Type and Degree Structure Type in Finite $p-$groups

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Abstract

Let G be a finite $p-$group, and denote by $k(G)$ number of conjugacy classes in G. The aim of this paper is to introduce the conjugacy structure type and degree structure type for $p-$groups, and determine these parameters for $p-$groups of order p^5, and calculate $k(G)$ for them.

Key Words: breadth, conjugacy structure type, degree structure type.

1. Introduction

Let G be a finite $p-$group, and denote by $k(G)$ number of conjugacy classes of G. We remind the reader that an element g of $p-$group G is said to have *breadth* $b_G(g)\big($b(g) if no ambiguity is possible$\big)$ if $p^{b_G(g)}$ is the size of conjugacy class of g in G. The breadth $b(G)$ of G will be maximum of breadths of its elements. We have,
$b(G) = 1$ if and only if $|G'| = p$ (see [4]),
$b(G) = 2$ if and only if $|G'| = p^2$ or $|G : Z(G)| = p^3$ and $|G'| = p^3$ (see [7]).

Definition 1. *Let s_i be the number of conjugacy classes of size p^i in G. Let m be a non-negative integer such that $s_m \neq 0$, and $s_i = 0$ for $i > m$. Then $|G| = \sum_{i=0}^{m} s_i p^i$, and $k(G) = \sum_{i=0}^{m} s_i$. We define the tuple $(s_0, s_1, \ldots, s_m)$,* Conjugacy Structure Type *of G, and denote by $T_c(G)$. It is clear that G is abelian if and only if $m = 0$.*

Defintion 2. *Let α_i be the number of irreducible characters of G of order p^i. Let h be a non-negative integer such that $\alpha_h \neq 0$, and $\alpha_i = 0$ for $i > h$. Then $|G| = \sum_{i=0}^{h} \alpha_i p^{2i}$, and $k(G) = \sum_{i=0}^{h} \alpha_i$. We define the tuple $(\alpha_0, \alpha_1, \ldots, \alpha_h)$, Degree Structure Type of G, and denote by $T_d(G)$.*

We know that $b(G)$ is the maximum index of i such that s_i is nonzero, that means $b(G) = m$. We denote by $\beta(G)$ the maximum index of i such that α_i is nonzero that is $\beta(G) = h$.

Burnside's Formula. Let G be a finite $p-$group and M be a maximal subgroup in G. If s and t are the number respectively of invariant and fused conjugacy classes of M then $k(G) = ps + \frac{t}{p} = s(p - \frac{1}{p}) + \frac{k(M)}{p}$.

Proof. See [1,p.472].

The main theorem is:

Theorem A. *Let G be a nonabelian finite $p-$group of order p^5. Then one of the following occurs:*

(i) $k(G) = p^4 + p^3 - p^2$, $T_d(G) = (p^4, p^3 - p^2)$,

(ii) $k(G) = p^4 + p - 1$, $T_d(G) = (p^4, 0, p - 1)$,

(iii) $k(G) = p^3 + p^2 - 1$, $T_d(G) = (p^2, p^3 - 1)$ or $(p^3, p^2 - p, p - 1)$,

(iv) $k(G) = 2p^3 - p$, $T_d(G) = (p^3, p^3 - p)$,

(v) $k(G) = 2p^2 + p - 2$, $T_d(G) = (p^2, p^2 - 1, p - 1)$.

2. Elementary Lemmas

Throughout this section, G denote a $p-$group of order p^n. To proof the main theorem we need some lemmas:

Lemma 1. *(i) Let G be a nonabelian finite $p-$group. If $b(G) \geq k$, then $|G : Z(G)| \geq p^{k+1}$.*

(ii) Let G be a nonabelian finite $p-$group. If $\beta(G) \geq 2$, then $|G : Z(G)| \geq p^4$.

Proof. (i) Suppose that $g \in G$ such that $|G : C_G(g)| \geq p^k$. Since $Z(G) \subset C_G(g)$ we have

$$|G : Z(G)| > |G : C_G(g)| \geq p^k$$

Therefore $|G : Z(G)| \geq p^{k+1}$.

(ii) It is clear from the fact that for any irreducible character χ of G, $\chi(1)^2 \leq |G : Z(G)|$. □

Lemma 2. *Let G be a finite $p-$group with $b(G) = 1$ and $\beta(G) = \beta$. Then*

(i) $G/Z(G)$ is an elementary abelian subgroup of order $p^{2\beta}$,

(ii) Every character of G has degree 1 or p^{β},

(iii) $k(G) = p^{n-1} + p^{n-2\beta} - p^{n-2\beta-1}$,
$T_d(G) = (p^{n-1}, p^{n-2\beta} - p^{n-2\beta-1})$, $T_c(G) = (p^{n-2\beta}, p^{n-1} - p^{n-2\beta-1})$.

Proof. We have $|G'| = p$. Hence $G' \subseteq Z(G)$ and $G/Z(G)$ is abelian. We know that exponent of $G/Z(G)$ is p(see [5]). Therefore $G/Z(G)$ is elementary abelian. If χ is a nonlinear irreducible character of G, then

$$\chi(1)^2 = |G : Z(G)|$$

by exercise 2.13 of [3]. Hence $\chi(1) = p^{\beta}$ for any nonlinear irreducible character χ of G. So by character degrees formula,

$$k(G) = p^{n-1} + p^{n-2\beta} - p^{n-2\beta-1}, T_d(G) = (p^{n-1}, p^{n-2\beta} - p^{n-2\beta-1}).$$

since $p^n = p^z + s_1 p$, where $|Z(G)| = p^z$. We have

$$T_c(G) = (p^{n-2\beta}, p^{n-1} - p^{n-2\beta-1}).$$

□

Corollary 1. *Let G be a nonabelian $p-$group of order p^3. Then $k(G) = p^2 + p - 1$ and $T_d(G) = T_c(G) = (p, p^2 - 1)$.*

Proof. It is clear by $\beta(G) = 1$. □

Lemma 3. *Let G be a finite $p-$group and $b(G) \geq 2$, If $|G : G'| = p^k$, then $2 \leq k \leq n - 2$.*

Proof. By lemma. 1(ii) of [2], $|G'| \geq p^2$ and by character degrees formula proof is trivial. □

Corollary 2. *Let G be a nonabelian $p-$group of order p^4. Then one of the following occurs:*

(i) $k(G) = p^3 + p^2 - p$, $T_c(G) = (p^2, p^3 - p)$, *and* $T_d(G) = (p^3, p^2 - p)$,

(ii) $k(G) = 2p^2 - 1$, $T_c(G) = (p, p^2 - 1, p^2 - p)$, *and* $T_d(G) = (p^2, p^2 - 1)$.

Proof. It is clear by lemmas 2 and 3. □

Example 1. Let $G = E(p^3) =< x, y | x^p = y^p = [x, y]^p = 1, [x, y] \in Z(G) >$. We know that all of conjugacy classes of order 1 are in $Z(G)$, and has form $\{[x, y]^i\}$ for some $i = 1, 2, \ldots, p$.

Other classes of G are:

- Classes of the form $\{x^i[x, y]^j | 0 \leq j \leq p - 1\}$ where $i = 1, 2, \ldots, p - 1$.
- Classes of the form $\{y^i[x, y]^j | 0 \leq j \leq p - 1\}$ where $i = 1, 2, \ldots, p - 1$.
- Classes of the form $\{x^i y^j [x, y]^k | 0 \leq k \leq p - 1\}$ where $i, j = 1, 2, \ldots, p - 1$.

Hence $T_c(E(p^3)) = (p, p^2 - 1)$.

Lemma 4. *Let G be a finite $p-$group and M be an abelian maximal subgroup of G. Then $k(G) = p^{n-2} + p^{z+1} + p^{z-1}$, where $|Z(G)| = p^z$.*

Proof. We know that $Z(G) \subseteq M$, otherwise $M' = G'$, which is a contradiction. Then the Burnside's formula completes the proof. □

3. Proof of Theorem A

In this section we proof theorem A and present some other information about conjugacy structure type:

Proof. We consider three possible casses

Case 1. Let $b(G) = 1$. Then $|G'| = p$. By lemma 2, for $|G : Z(G)| = p^2$ or p^4 we have,
$k(G) = p^4 + p^3 - p^2$, $T_d(G) = (p^4, p^3 - p^2)$, and $T_c(G) = (p^3, p^4 - p^2)$, or
$k(G) = p^4 + p - 1$, $T_d(G) = (p^4, 0, p - 1)$, and $T_c(G) = (p, p^4 - 1)$.

Case 2. Let $b(G) = 2$. Then $|G'| = p^2$ or $|G'| = p^3$ and $|G : Z(G)| = p^3$.
First suppose $|G : Z(G)| = p^3$, then by lemma1(i). For $|G'| = p^2$ or p^3, we have
$k(G) = 2p^3 - p$, $T_d(G) = (p^3, p^3 - p)$, and $T_c(G) = (p^2, P^3 - p, p^3 - p^2)$, or
$k(G) = p^3 + p^2 - 1, T_d(G) = (p^2, p^3 - 1)$, and $T_c(G) = (p^2, 0, p^3 - 1)$.
Now suppose $|G : Z(G)| = p^4$. Then$|G'| = p^2$ and $k = 3$. Since $\alpha_i p^{2i}$ is divided by $(p-1)p^k$ (see corollary 11 of [6]), then by character degrees formula,

$$p^5 = p^3 + p(p-1)t_1 p^2 + (p-1)t_2 p^4$$

for some non-negative integer t_1 and t_2. Hence $t_1 = t_2 = 1$ and
$k(G) = p^3 + p^2 - 1, T_d(G) = (p^3, p^2 - p, p - 1), T_c(G) = (p, p^2 - 1, p^3 - p)$.

Case 3. Let $b(G) = 3$. Then $|G : Z(G)| = p^4$ and $|G'| = p^3$, by lemma 1. If G has an abelian maximal subgroup then $k(G) = p^3 + p^2 - 1$(by lemma 4), and $T_d(G) = (p^2, p^3 - 1)$.
If $\beta(G) = 2$, then By character degrees formula,
$p^5 = p^2 + \alpha_1 p^2 + \alpha_2 p^4$, which implies that $1 + \alpha_1 = hp^2$ for some non-negative integer

h. Hence $\alpha_2 = p - h$. Since α_2 is nonzero and divided by $p - 1$(by corollary 11 of [6]), $h = 1$. Therefore $k(G) = 2p^2 + p - 2$ and
$T_d(G) = (p^2, p^2 - 1, p - 1)$.

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