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Lifts of Derivations to the Semitangent Bundle

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Abstract

The main purpose of this paper is to investigate the complete lifts of derivations for semitangent bundle and to discuss relations between these and lifts already known.

Key Words: Vector, Field, Derivation, Complete lift, Semitangent bundle.

1. Semitangent bundle

Let M_n be an n-dimensional differentiable manifold of class C^∞ and $\pi : M_n \to B_m$ the differentiable bundle determined by a submersion π. Suppose that $(x^a, x^\alpha), a, b, ... = 1, ..., n-m; \alpha, \beta, ... = n-m+1, ..., n; i = 1, 2, ..., n$ is a system of local coordinates adapted to the bundle $\pi : M_n \to B_m$ where x^α are coordinates in B_m, x^a are fibre coordinates of the bundle (see [1, p. 190]). If $(x^{a'}, x^{\alpha'})$ is another system of local coordinates in the bundle, then we have

$$\begin{cases} x^{a'} = x^{a'}(x^a, x^\alpha), \\ x^{\alpha'} = x^{\alpha'}(x^\alpha). \end{cases} \tag{1}$$

The Jacobian of (1) is given by the matrix

$$(A_i^{i'}) = \left(\frac{\partial x^{i'}}{\partial x^i}\right) = \begin{pmatrix} A_a^{a'} & A_\alpha^{a'} \\ 0 & A_\alpha^{\alpha'} \end{pmatrix}.$$

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Let $T_p(B_m)$ $(p = \pi(\widetilde{p}), \widetilde{p} = (x^a, x^\alpha) \in M_n)$ be the tangent space at a point p of B_m. If $X^\alpha = dx^\alpha(X)$ are components of X in tangent space $T_p(B_m)$ with respect to the natural base $\{\partial_\alpha\}$ $(\partial_\alpha = \frac{\partial}{\partial x^\alpha})$, then we have the set of all points $(x^a, x^\alpha, x^{\overline{\alpha}})$, $x^{\overline{\alpha}} = X^\alpha, \overline{\alpha} = \alpha + m$ is by definition, the semitangent bundle $t(M_n)$ over the manifold M_n (see [1], [2]), $dim t(M_n) = n + m$. In the special case $n = m, t(M_n)$ is a tangent bundle $T(M_n)$.

To a transformation of local coordinates of M_n (see (1)), there corresponds in $t(M_n)$ the coordinate transformation

$$\begin{cases} x^{a'} = x^{a'}(x^a, x^\alpha), \\ x^{\alpha'} = x^{\alpha'}(x^\alpha), \\ x^{\overline{\alpha}'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha} x^{\overline{\alpha}}. \end{cases} \tag{2}$$

The Jacobian of (2) is given by

$$\overline{A} = \begin{pmatrix} A_a^{a'} & A_\alpha^{a'} & 0 \\ 0 & A_\alpha^{\alpha'} & 0 \\ 0 & A_{\alpha\sigma}^{\alpha'} x^{\overline{\sigma}} & A_\alpha^{\alpha'} \end{pmatrix}, \tag{3}$$

where

$$A_{\alpha\sigma}^{\alpha'} = \frac{\partial^2 x^{\alpha'}}{\partial x^\alpha \partial x^\sigma}.$$

We denote by $\mathcal{T}_q^p(M_n)$ the module over $F(M_n)$ of all tensor fields of class C^∞ and of type (p, q) in M_n, where $F(M_n)$ denotes the ring of real-valued C^∞ functions on M_n. Let $v \in \mathcal{T}_0^1(M_n)$ be a projectable vector field, that is,

$$v = (v^i) = \begin{pmatrix} v^a(x^b, x^\beta) \\ v^\alpha(x^\beta) \end{pmatrix}.$$

Taking account of (3), we easily see that ${}^c v' = \overline{A}\,{}^c v$, where

$${}^c v' = \begin{pmatrix} v^{a'} \\ v^{\alpha'} \\ x^{\overline{\sigma}'} \frac{\partial v^{\alpha'}}{\partial x^{\sigma'}} \end{pmatrix}, {}^c v = \begin{pmatrix} v^a \\ v^\alpha \\ x^{\overline{\sigma}} \frac{\partial v^\alpha}{\partial x^\sigma} \end{pmatrix}, \tag{4}$$

that is ${}^c v \in \mathcal{T}_0^1(t(M_n))$. The vector field ${}^c v$ is called the complete lift of v to the semitangent bundle $t(M_n)$ [1, p. 194].

If $f = f(x^a, x^\alpha)$ is a function in M_n, we write ${}^c f$ for the function in $t(M_n)$ defined by

$$ {}^c f = \imath(df) = x^{\overline{\beta}} \partial_\beta f $$

and call of the complete lift of f to $t(M_n)$.

Lemma: Let $\tilde{v}$ and $\tilde{w}$ be vector fields in $t(M_n)$ such that $\tilde{v}\,{}^c f = \tilde{w}\,{}^c f$ for any $f \in \mathcal{T}_0^0(M_n)$. Then $\tilde{v} = \tilde{w}$.

Proof: It is sufficient to show that if ${}'\tilde{v}\,{}^c f = (\tilde{v} - \tilde{w})^c f = 0$ for any $f \in \mathcal{T}_0^0(M_n)$, then ${}'\tilde{v} = 0$. If $\begin{pmatrix} {}'\tilde{v}^a \\ {}'\tilde{v}^\alpha \\ {}'\tilde{v}^{\overline{\alpha}} \end{pmatrix}$ are components of ${}'\tilde{v}$ with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ in $t(M_n)$, then we have from ${}'\tilde{v}\,{}^c f = 0$

$$ {}'\tilde{v}^a x^{\overline{\beta}} \partial_a \partial_\beta f +{}' \tilde{v}^\alpha x^{\overline{\beta}} \partial_\alpha \partial_\beta f +{}' \tilde{v}^{\overline{\alpha}} \partial_\alpha f = 0. $$

If this holds for any $f \in \mathcal{T}_0^0$, $\partial_\alpha f$, $\partial_\alpha \partial_\beta f$ and $\partial_a \partial_\beta f$ taking any preassigned values at a fixed point, we have

$$ {}'\tilde{v}^\alpha x^{\overline{\beta}} +{}' \tilde{v}^\beta x^{\overline{\alpha}} = 0, \quad {}'\tilde{v}^a x^{\overline{\beta}} = 0, \quad {}'\tilde{v}^{\overline{\alpha}} = 0. \tag{5} $$

Suppose $x^{\overline{\alpha}} \neq 0$ and assume that $x^{\overline{1}} \neq 0$. Then from ${}'\tilde{v}^a x^{\overline{\beta}} = 0$ we have ${}'\tilde{v}^a = 0$. Putting $\alpha = 1$ in the first equation of (5), we have ${}'\tilde{v}^\beta x^{\overline{1}} +{}' \tilde{v}^1 x^{\overline{\beta}} = 0$, from which ${}'\tilde{v}^\beta = \lambda x^{\overline{\beta}}$ for a certain function $\lambda = -\frac{{}'\tilde{v}^1}{x^{\overline{1}}}$. Substituting this into the first equation of (5), we find $2\lambda x^{\overline{\alpha}} x^{\overline{\beta}} = 0$, from which, putting $\alpha = \beta = 1$, we have $\lambda = 0$, i.e. ${}'\tilde{v}^\alpha = 0$. Thus we see that the vector field ${}'\tilde{v}$ is zero at a point such that $y^i \neq 0$, that is, in $t(M_n) - M_n$. But the vector field ${}'\tilde{v}$ is continuous at every point of $t(M_n)$. So, we have ${}'\tilde{v} = 0$ in $t(M_n)$.

Thus we have the following.

Remark: Any element $\tilde{v}$ of $\mathcal{T}_0^1(t(M_n))$ is completely determined by its action on functions of the type ${}^c f \in \mathcal{T}_0^0(t(M_n))$.

2. Lifts of Derivations to $t(M_n)$

Let $\mathcal{T}(M_n)$ be the direct sum $\sum_{p,q} \mathcal{T}_q^p(M_n)$. A map $\mathcal{D} : \mathcal{T}(M_n) \to \mathcal{T}(M_n)$ is a derivation on M_n if [3]

a) $\mathcal{D}$ is linear with respect to constant coefficients,

b) for all p, q, $\mathcal{D}\mathcal{T}_q^p(M_n) \subset \mathcal{T}_q^p(M_n)$,

c) for all C^∞ tensor fields T_1 and T_2 on M_n

$$\mathcal{D}(T_1 \otimes T_2) = (\mathcal{D}T_1) \otimes T_2 + T_1 \otimes \mathcal{D}T_2,$$

d) $\mathcal{D}$ commutes with contraction.

From definition of Derivations we obtain $\mathcal{D}I = 0$, where I denotes the identity tensor field of type (1,1) in M_n.

For a derivation $\mathcal{D}$ in M_n, there exists a vector field P in M_n such that

$$Pf = \mathcal{D}f \tag{6}$$

for any $f \in \mathcal{T}_0^0(M_n)$. If we put

$$\mathcal{D}\frac{\partial}{\partial x^i} = Q_i^h \frac{\partial}{\partial x^h} \tag{7}$$

in each coordinate neighborhood U of M_n, we have in U

$$\mathcal{D}(dx^k) = -Q_i^h dx^i. \tag{8}$$

Let $X \in \mathcal{T}_0^1(M_n)$ and $w \in \mathcal{T}_1^0(M_n)$. From (7) and (8), we see that $\mathcal{D}X$ and $\mathcal{D}w$ have, respectively, components of the form

$$\mathcal{D}X : (P^i \partial_i X^h + Q_i^h X^i); \quad \mathcal{D}w : (P^j \partial_j w_i - Q_i^h w_h), \tag{9}$$

where P^i are components of P given by (6). The pair (P^h, Q_i^h) is called components of the Derivation $\mathcal{D}$ in U [4, p. 26].

We define vector fields ${}^c\mathcal{D}$ in $t(M_n)$ by (see Remark in Section 1)

$${}^c\mathcal{D}\,{}^c f = \iota(\mathcal{D}df), \qquad f \in \mathcal{T}_0^0(M_n) \tag{10}$$

and call ${}^c\mathcal{D}$ the complete lift of $\mathcal{D}$ to $t(M_n)$, where ι is the operator defined by

$$\iota S = x^{\overline{\beta}} S_\beta \tag{11}$$

$S = S_b dx^b + S_\beta dx^{\overline{\beta}}$ being an arbitrary covector field in M_n.

If $\begin{pmatrix} {}^c\mathcal{D}^a \\ {}^c\mathcal{D}^\alpha \\ {}^c\mathcal{D}^{\overline{\alpha}} \end{pmatrix}$ are components of ${}^c\mathcal{D}$ with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$, then taking account of the argument similar to that used in the proof of Lemma in Section 1 we have, from (9)-(11)

$$\begin{aligned} {}^c\mathcal{D}^a x^{\overline{\beta}}\partial_a\partial_\beta f +{}^c \mathcal{D}^\alpha x^{\overline{\beta}}\partial_\alpha\partial_\beta f +{}^c \mathcal{D}^{\overline{\alpha}}\partial_\alpha f &= x^{\overline{\beta}}(P^i\partial_i\partial_\beta f - Q^i_\beta \partial_i f) \\ &= x^{\overline{\beta}} P^a\partial_a\partial_\beta f + x^{\overline{\beta}} P^\alpha \partial_\alpha\partial_\beta f - \\ &\quad - x^{\overline{\beta}} Q^a_\beta \partial_a f - x^{\overline{\beta}} Q^\alpha_\beta \partial_\alpha f \end{aligned}$$

and ${}^c\mathcal{D}^a = P^a$, ${}^c\mathcal{D}^\alpha = P^\alpha$, ${}^c\mathcal{D}^{\overline{\alpha}} = -x^{\overline{\beta}} Q^\alpha_\beta$. Thus ${}^c\mathcal{D}$ has components

$${}^c\mathcal{D} = \begin{pmatrix} P^a \\ P^\alpha \\ -x^{\overline{\beta}} Q^\alpha_\beta \end{pmatrix} \tag{12}$$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$.

2.1. The Lifts of Lie Derivations

Let $v \in \mathcal{T}^1_0(M_n)$ is a projectable vector field and L_v denote Lie derivation with respect to v:

$$L_v f = vf, \quad L_v w = [v, w] : (v^i\partial_i w^j - w^i \partial_i v^j).$$

Then L_v is a derivation in M_n having components

$$L_v : (v^h, -\partial_i v^h), \quad Q^h_i = -\partial_i v^h \tag{13}$$

Using (4), (12) and (13), we have

$${}^c(L_v) = \begin{pmatrix} v^a \\ v^\alpha \\ x^{\overline{\beta}}\partial_\beta v^\alpha \end{pmatrix} ={}^c v.$$

2.2. The Lifts of covariant differentiations

Suppose now that ∇ is a projectable linear connection in M_n (see [1, p.198]). Let $v \in \mathcal{T}_0^1(M_n)$ is a projectable vector field and ∇_v denote covariant differentiation with respect to v:

$$\nabla_v f = vf, \quad \nabla_v w : (v^j \partial_j w^h + v^j \Gamma_{jm}^h w^m),$$

where Γ_{ji}^h are local components of ∇ in M_n. Then ∇_v is a derivation in M_n having components

$$\nabla_v : (v^h, v^j \Gamma_{ji}^h), \quad Q_i^h = v^j \Gamma_{ji}^h. \tag{14}$$

Using (12) and (14), we have

$$^c(\nabla_v) = \begin{pmatrix} v^a \\ v^\alpha \\ -v^\gamma x^{\overline{\beta}} \Gamma_{\gamma\beta}^\alpha \end{pmatrix}. \tag{15}$$

We introduce now a projectable linear connection $\overset{\mathrm{v}}{\nabla}$ in M_n by

$$\begin{aligned} \overset{\mathrm{v}}{\nabla}_v w &= \nabla_v w - S(v, w) = \nabla_v w - (\nabla_v w - \nabla_w v - [v, w]) \\ &= \nabla_w v + [v, w], \end{aligned} \tag{16}$$

v and w being arbitrary projectable vector fields in M_n, where S is the torsion tensor of the given connection ∇. Denoting by Γ_{ij}^h the components of the given connection, we have from (16)

$$\overset{\mathrm{v}}{\Gamma}{}_{ji}^h = \Gamma_{ij}^h, \tag{17}$$

where $\overset{\mathrm{v}}{\Gamma}{}_{ji}^h$ are components of the new connection $\overset{\mathrm{v}}{\nabla}$.

We define vector fields $\gamma(\overset{\mathrm{v}}{\nabla} v)$ in $t(M_n)$ by

$$\gamma(\overset{\mathrm{v}}{\nabla} v) = (x^{\overline{\beta}} \overset{\mathrm{v}}{\nabla}_\beta v^\alpha) \frac{\partial}{\partial x^{\overline{\alpha}}}, \tag{18}$$

where $\overset{\mathrm{v}}{\nabla}_\beta v^\alpha = \partial_\beta v^\alpha + \overset{\mathrm{v}}{\Gamma}{}^{\alpha}_{\beta\gamma} v^\gamma$. Using (3), we can easily verify that the vector field $\gamma(\overset{\mathrm{v}}{\nabla} v)$ defined in each coordinate neighborhood in $t(M_n)$ determine global vector field in $t(M_n)$. From (18), we see that $\gamma(\overset{\mathrm{v}}{\nabla} v)$ have components

$$\gamma(\overset{\mathrm{v}}{\nabla} v) = \begin{pmatrix} 0 \\ 0 \\ x^{\overline{\beta}} \overset{\mathrm{v}}{\nabla}_\beta v^\alpha \end{pmatrix}$$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$. Using (4), (15) and (17), we have

$$\begin{aligned} {}^c v - {}^c (\nabla_v) &= \begin{pmatrix} v^a \\ v^\alpha \\ x^{\overline{\beta}} \partial_\beta v^\alpha \end{pmatrix} - \begin{pmatrix} v^a \\ v^\alpha \\ -v^\gamma x^{\overline{\beta}} \Gamma^\alpha_{\gamma\beta} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ x^{\overline{\beta}} (\partial_\beta v^\alpha + \Gamma^\alpha_{\gamma\beta} v^\gamma) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ x^{\overline{\beta}} (\partial_\beta v^\alpha + \overset{\mathrm{v}}{\Gamma}{}^{\alpha}_{\beta\gamma} v^\gamma) \end{pmatrix} \\ &= \gamma(\overset{\mathrm{v}}{\nabla} v). \end{aligned}$$

Thus, we find the formula

$${}^c (\nabla_v) = {}^c v - \gamma(\overset{\mathrm{v}}{\nabla} v). \tag{19}$$

From (19), we see that

$${}^c (\nabla_v) = {}^c v$$

if and only if

$$\overset{\mathrm{v}}{\nabla} v = 0.$$

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